Exact solutions of the Kudryashov–Sinelshchikov equation and nonlinear telegraph equation via the first integral method

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Abstract. In this article we find the exact traveling wave solutions of the Kudryashov–Sinelshchikov equation and nonlinear telegraph equation by using the first integral method. This method is based on the theory of commutative algebra. This method can be applied to nonintegrable equations as well as to integrable ones.

Keywords: first integral method, Kudryashov–Sinelshchikov equation, nonlinear telegraph equation.

1 Introduction

Nonlinear evolution equations are widely used to describe complex phenomena in various sciences such as fluid physics, condensed matter, biophysics, plasma physics, nonlinear optics, quantum field theory and particle physics, etc. In recent years, various powerful methods have been presented for finding exact solutions of the nonlinear evolution equations in mathematical physics, such as, tanh method [1–3], multiple exp-function method [4], transformed rational function method [5], Hirotas direct method [6, 7], extended tanh-function method [8] and so on.

The first integral method, which is based on the ring theory of commutative algebra, was first proposed by Feng [9]. This method was further developed by the same author in [10–13].

The aim of this work is to find new exact solutions of Kudryashov–Sinelshchikov equation by using the first integral method.

The rest of this paper is organized as follows. In Section 2, we give the description of the first integral method. In Sections 3 and 4, we apply this method to nonlinear telegraph equation and Kudryashov–Sinelshchikov equation.

2 The first integral method

Consider a general nonlinear partial differential equation (PDE) in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0. \tag{1}$$

Using traveling wave

$$u(x,t) = U(\xi), \quad \xi = x - ct,$$

from Eq. (1), we obtain the ordinary differential equation (ODE):

$$G(U, U', U'', \dots) = 0, \tag{2}$$

where prime denotes the derivative with respect to the same variable ξ.

Suppose that the solution of ODE (2) can be written as follows:

$$u(x,t) = U(\xi) = f(\xi).$$

Next, we introduce a new independent variable

$$X(\xi) = f(\xi), \quad Y(\xi) = f'(\xi), \tag{3}$$

which leads a system of nonlinear ordinary differential equations

$$\begin{aligned} X'(\xi) &= Y(\xi), \\ Y'(\xi) &= F_1\bigl(X(\xi), Y(\xi)\bigr). \end{aligned} \tag{4}$$

By the qualitative theory of ordinary differential equations [14], if we can find the integrals to Eq. (4) under the same conditions, then the general solutions to Eq. (4) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the division theorem to obtain one first integral to Eq. (4), which reduces Eq. (2) to a first order integrable ordinary differential equation. An exact solution to Eq. (1) is then obtained by solving this equation. Now, let us recall the division theorem:

Division theorem. *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $\mathcal{C}[w,z]$ and $P(w,z)$ is irreducible in $\mathcal{C}[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $\mathcal{C}[w,z]$ such that*

$$Q(w,z) = P(w,z)G(w,z).$$

3 Nonlinear telegraph equation

Let us consider the nonlinear telegraph equation [15]

$$u_{tt} - u_{xx} + u_t + \alpha u + \beta u^3 = 0. \tag{5}$$

Using traveling wave $u(x,t) = U(\xi),\ \ \xi = x - ct$ carries (5) into an ODE as follows:

$$(c^2-1)U'' - cU' + \alpha U + \beta U^3 = 0, \tag{6}$$

where prime denotes the derivative with respect to the same variable ξ.

Using (3) and (4), we get

$$\begin{aligned} X'(\xi) &= Y(\xi), \\ Y'(\xi) &= \frac{c}{c^2-1}Y(\xi) - \frac{\alpha}{c^2-1}X(\xi) - \frac{\beta}{c^2-1}X^3(\xi). \end{aligned} \tag{7}$$

Now, we apply the above division theorem to look for the first integral of system (7). Suppose that $X = X(\xi)$ and $Y = Y(\xi)$ are nontrivial solutions to (7), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in $\mathcal{C}[X,Y]$ such that

$$q\big(X(\xi), Y(\xi)\big) = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{8}$$

where $a_i(X)$ $(i = 0,1,\dots,m)$ are polynomials of X and $a_m(X) \neq 0$. Eq. (8) is a first integral of Eq. (7). We note that $\mathrm{d}q/\mathrm{d}\xi$ is a polynomial of X and Y, and $q(X(\xi), Y(\xi)) = 0$ implies that $\mathrm{d}q/\mathrm{d}\xi|_{(7)} = 0$. According to the division theorem, there exists a polynomial [9] $T(X,Y) = g(X) + h(X)Y$ in $\mathcal{C}[X,Y]$ such that

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = \frac{\mathrm{d}q}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}\xi} + \frac{\mathrm{d}q}{\mathrm{d}Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi} = \big(g(X) + h(X)Y\big)\sum_{i=0}^{m} a_i(X)Y^i. \tag{9}$$

In this example, we assume that $m = 2$ in Eq. (8). Taking Eq. (7) and (9) into account, we get

$$\begin{aligned} &\sum_{i=0}^{2} a_i'(X)Y^{i+1} + \left(\frac{c}{c^2-1}Y - \frac{\alpha}{c^2-1}X - \frac{\beta}{c^2-1}X^3\right)\sum_{i=0}^{2} i a_i(X)Y^{i-1} \\ &\quad = \big(g(X) + h(X)Y\big)\sum_{i=0}^{2} a_i(X)Y^i. \end{aligned} \tag{10}$$

Equating the coefficients of Y^i $(i = 3,2,1,0)$ in Eq. (10) leads to the system

$$a_2'(X) = h(X)a_2(X), \tag{11}$$

$$a_1'(X) = \left(g(X) - \frac{2c}{c^2-1}\right)a_2(X) + h(X)a_1(X), \tag{12}$$

$$\begin{aligned} a_0'(X) &= 2a_2(X)\left[\frac{\alpha}{c^2-1}X + \frac{\beta}{c^2-1}X^3\right] \\ &\quad + \left(g(X) - \frac{c}{c^2-1}\right)a_1(X) + h(X)a_0(X), \end{aligned} \tag{13}$$

$$a_1(X)\left[-\frac{\alpha}{c^2-1}X - \frac{\beta}{c^2-1}X^3\right] = g(X)a_0(X). \tag{14}$$

Since $a_i(X)$ $(i = 0,1,2)$ are polynomials, then, from (11), we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Then Eqs. (12) and (13) reduce to the following equations:

$$a_1'(X) = g(X) - \frac{2c}{c^2-1}, \tag{15}$$

$$a_0'(X) = 2\left(\frac{\alpha}{c^2-1}X + \frac{\beta}{c^2-1}X^3\right) + \left(g(X) - \frac{c}{c^2-1}\right)a_1(X). \tag{16}$$

Balancing the degrees of $a_0(X)$, $a_1(X)$ and $g(X)$, we conclude that $\deg(g(X)) = 0$ and $\deg(a_1(X)) = 1$ or $\deg(g(X)) = 1$ and $\deg(a_1(X)) = 2$. If $\deg(g(X)) = 0$ and $\deg(a_1(X)) = 1$, assuming $g(X) = A_1$ $(A_1 \neq 0)$ and $a_1(X) = B_0 + B_1 X$ $(B_1 \neq 0)$ in Eq. (15), we get $A_1 = B_1 + 2c/(c^2-1)$. Thus, from Eq. (16), we have

$$\begin{aligned} a_0(X) = {} & d + \left(B_0 B_1 + \frac{c}{c^2-1}B_0\right)X + \frac{1}{2}\left(B_1^2 + \frac{c}{c^2-1}B_1 + \frac{2\alpha}{c^2-1}\right)X^2 \\ & + \frac{\beta}{2(c^2-1)}X^4, \end{aligned} \tag{17}$$

where d denotes an integration constant. By substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into Eq. (14) and equating the coefficients of X^i $(i = 4,3,2,1)$ to zero, we obtain the following system of nonlinear algebraic equations:

$$X^4\colon \quad \frac{3\beta}{2(c^2-1)}B_1 + \frac{c\beta}{(c^2-1)^2} = 0, \tag{18}$$

$$X^3\colon \quad \frac{\beta}{c^2-1}B_0 = 0, \tag{19}$$

$$X^2\colon \quad \frac{1}{2}B_1^3 + \frac{3c}{2(c^2-1)}B_1^2 + \left(\frac{c^2}{(c^2-1)^2} + \frac{2\alpha}{c^2-1}\right)B_1 + \frac{2c\alpha}{(c^2-1)^2} = 0, \tag{20}$$

$$X^1\colon \quad B_0 B_1^2 + \frac{3c}{c^2-1}B_0 B_1 + \frac{2c^2}{(c^2-1)^2}B_0 + \frac{\alpha}{c^2-1}B_0 = 0, \tag{21}$$

$$X^0\colon \quad \left(B_1 + \frac{2c}{c^2-1}\right)d = 0. \tag{22}$$

Solving the system (18)–(22) simultaneously, we get the solutions set

$$B_0 = 0, \quad B_1 = \sqrt{\alpha(9\alpha-2)}, \quad c = -3\sqrt{\frac{\alpha}{9\alpha-2}}, \quad d = 0. \tag{23}$$

$$B_0 = 0, \quad B_1 = -\sqrt{\alpha(9\alpha-2)}, \quad c = 3\sqrt{\frac{\alpha}{9\alpha-2}}, \quad d = 0. \tag{24}$$

Now, taking the solution set (23) into account, Eq. (8) becomes

$$\frac{\beta(9\alpha-2)}{4}X^4 + \frac{\alpha(9\alpha-2)}{4}X^2 + \sqrt{\alpha(9\alpha-2)}XY + Y^2 = 0, \tag{25}$$

which is a first integral of Eq. (7). Solving Eq. (25), we get

$$Y(\xi) = \pm\frac{\sqrt{9\alpha-2}}{2}\big(\sqrt{-\beta}X^2(\xi) \mp \sqrt{\alpha}X(\xi)\big). \tag{26}$$

Combining (26) with (7), we obtain the exact solution to Eq. (6) and then exact solutions to nonlinear telegraph equation can be written as

$$u(x,t) = -\sqrt{\alpha}\frac{\exp[-\frac{\sqrt{\alpha(9\alpha-2)}}{2}(x+3\sqrt{\frac{\alpha}{9\alpha-2}}t+\xi_0)]}{1-\sqrt{-\beta}\exp[-\frac{\sqrt{\alpha(9\alpha-2)}}{2}(x+3\sqrt{\frac{\alpha}{9\alpha-2}}t+\xi_0)]}, \quad \alpha\neq 0, \frac{2}{9}, \tag{27}$$

$$u(x,t) = \sqrt{\alpha}\frac{\exp[-\frac{\sqrt{\alpha(9\alpha-2)}}{2}(x+3\sqrt{\frac{\alpha}{9\alpha-2}}t+\xi_0)]}{1-\sqrt{-\beta}\exp[-\frac{\sqrt{\alpha(9\alpha-2)}}{2}(x+3\sqrt{\frac{\alpha}{9\alpha-2}}t+\xi_0)]}, \quad \alpha\neq 0, \frac{2}{9}, \tag{28}$$

where ξ_0 is an arbitrary constant.

Similarly, in the case of (24), from (8), we obtain

$$Y(\xi) = \frac{\sqrt{9\alpha-2}}{2}\big(\sqrt{-\beta}X^2(\xi) + \sqrt{\alpha}X(\xi)\big), \tag{29}$$

and then the exact solution of nonlinear telegraph equation can be written as

$$u(x,t) = \sqrt{\alpha}\frac{\exp[\frac{\sqrt{\alpha(9\alpha-2)}}{2}(x-3\sqrt{\frac{\alpha}{9\alpha-2}}t+\xi_0)]}{1-\sqrt{-\beta}\exp[\frac{\sqrt{\alpha(9\alpha-2)}}{2}(x-3\sqrt{\frac{\alpha}{9\alpha-2}}t+\xi_0)]}, \quad \alpha\neq 0, \frac{2}{9}, \tag{30}$$

where ξ_0 is an arbitrary constant.

Comparing our results with Wang's results [15] then it can be seen that the results are same.

Remark. If $\alpha = 0$, we have

$$u(x,t) = \pm\frac{\sqrt{2}}{\sqrt{\beta}(x+\xi_0)}.$$

If $\alpha = 2/9$, we have

$$u(x,t) = \pm\frac{\sqrt{2}}{3\sqrt{\beta}}\tan\left(\frac{1}{3}(x+\xi_0)\right).$$

4 Kudryashov–Sinelshchikov equation

Consider the Kudryashov–Sinelshchikov equation [16, 17]:

$$u_t + \gamma u u_x + u_{xxx} - \varepsilon(u u_{xx})_x - \kappa u_x u_{xx} - \nu u_{xx} - \delta(u u_x)_x = 0, \tag{31}$$

where γ, ε, κ, ν and δ are real parameters.

Eq. (31) describes the pressure waves in the liquid with gas bubbles taking into account the heat transfer and viscosity. More details are presented [18, 19].

Now, applying the transformation $u(x,t) = u(\xi)$, $\xi = x - ct$ to Eq. (31) and integrating the resultant equation once, we get

$$-cu + \frac{\gamma}{2}u^2 + u'' - \varepsilon uu'' - \frac{\kappa}{2}(u')^2 - \nu u' - \delta uu' = 0, \tag{32}$$

where integration constant is taken to zero and the primes denote derivative with respect to ξ.

Using (3) and (4), we get

$$\begin{aligned} &\dot{X}(\xi) = Y(\xi),\\ &\dot{Y}(\xi) = \frac{1}{1-\varepsilon X(\xi)}\biggl[cX(\xi) - \frac{\gamma}{2}X^2(\xi) + \frac{\kappa}{2}Y^2(\xi) + \bigl(\nu + \delta X(\xi)\bigr)Y(\xi)\biggr]. \end{aligned} \tag{33}$$

Now, we make the transformation $\mathrm{d}\xi = (1 - \varepsilon X)\,\mathrm{d}\eta$ in Eq. (33) to avoid the singular line $\varepsilon X = 1$ temporarily. Thus, system (33) becomes

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\eta} &= (1-\varepsilon X)Y,\\ \frac{\mathrm{d}Y}{\mathrm{d}\eta} &= cX - \frac{\gamma}{2}X^2 + \frac{\kappa}{2}Y^2 + (\nu + \delta X)Y. \end{aligned} \tag{34}$$

In this example, we assume that $m = 1$ in Eq. (8). From now on, we shall omit some details because the procedure is the same. Then, by equating the coefficients of Y^i ($i = 2, 1, 0$) on both sides of Eq. (9), we have

$$(1-\varepsilon X)\dot{a}_1(X) = \biggl(h(X) - \frac{k}{2}\biggr)a_1(X), \tag{35}$$

$$(1-\varepsilon X)\dot{a}_0(X) = \bigl(g(X) - \nu - \delta X\bigr)a_1(X) + h(X)a_0(X), \tag{36}$$

$$g(X)a_0(X) = a_1(X)\biggl(cX - \frac{\gamma}{2}X^2\biggr). \tag{37}$$

As $a_1(X)$ and $h(X)$ are polynomials, from Eq. (35), we deduce that $h(X) = k/2$ and $a_1(X)$ must be a constant. For simplicity, we can take $a_1(X) = 1$. Then Eq. (36) indicates that $\deg(g(X)) \le \deg(a_0(X))$. Thus, from Eq. (37), we conclude that $\deg(g(X)) \le \deg(a_0(X)) = 1$. Assuming $g(X) = A_0 + A_1X$ ($A_1 \ne 0$) and $a_0(X) = B_0 + B_1X$ ($B_1 \ne 0$) in Eq. (36), we get $A_0 = B_1 - (\kappa/2)B_0 + \nu$ and $A_1 = \delta - (\varepsilon + \kappa/2)B_1$. Substituting $g(X)$ and $a_0(X)$ into Eq. (37) and setting the coefficients of X^i ($i = 2, 1, 0$) to be zero, we derive a system of nonlinear algebraic equations B_0, B_1 and c. Solving the resultant system simultaneously, we get the solution set

$$\begin{aligned} &B_0 = \frac{2(2\nu\varepsilon + \nu\kappa + \delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa})}{\kappa(2\varepsilon + \kappa)}, \quad B_1 = \frac{\delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa}}{2\varepsilon + \kappa},\\ &c = -\frac{\gamma - \nu\delta \pm \nu\sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa}}{\kappa}. \end{aligned} \tag{38}$$

Using the condition (38) in (8), we obtain

$$Y = -\frac{2\nu}{\kappa} \mp \frac{2(\delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa})}{\kappa(2\varepsilon + k)} - \frac{\delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa}}{2\varepsilon + \kappa} X. \tag{39}$$

Combining Eq. (33) with Eq. (39) and changing to the original variables, we find exact solutions to Eq. (31) as

$$\begin{aligned} u(x,t) &\\ = -&\frac{2\nu(2\varepsilon + \kappa)}{\kappa(\delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa})} \mp \frac{2}{\kappa} - \frac{2\varepsilon + \kappa}{\delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa}} \\ &\times \exp\left[-\frac{\delta \pm \sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa}}{2\varepsilon + \kappa} \left(x + \left(\frac{\gamma - \nu\delta \pm \nu\sqrt{\delta^2 + 2\gamma\varepsilon + \gamma\kappa}}{\kappa} \right) t + \xi_0 \right) \right], \end{aligned} \tag{40}$$

where ξ_0 is an arbitrary constant.

Comparing our results with Kudryashov's results [16] and Ryabov's results [17] then it can be seen that the results are new.

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