Exact solutions of the $(2+1)$-dimensional Camassa–Holm Kadomtsev–Petviashvili equation

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Abstract. This paper studies the $(2+1)$-dimensional Camassa–Holm Kadomtsev–Petviashvili equation. There are a few methods that will be utilized to carry out the integration of this equation. Those are the G'/G method as well as the exponential function method. Subsequently, the ansatz method will be applied to obtain the topological soliton solution of this equation. The constraint conditions, for the existence of solitons, will also fall out of these.

Keywords: solitons, exp-function method, G'/G method.

1 Introduction

It is well known that nonlinear evolution equations (NLEEs) are often presented to describe the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamics, plasma physics, nonlinear optics, etc. [1–15]. The investigation of exact solutions of NLEEs is therefore an important area of research in this area. There are various other NLEEs that are studied in this context. One such equation that will be studied in this paper is the $(2+1)$-dimensional Camassa–Holm Kadomtsev–Petviashvili (CH–KP) equation.

There are several tools of integration that was developed in the past few years [1–15]. They are the Adomian decomposition method, Fan's F-expansion method, variational iteration method, semi-inverse variational principle, homotopy perturbation method and several others. One of the major drawbacks of these methods is that it is not possible to extract an analytical structure of the soliton radiation. The Inverse Scattering Transform method that was studied earlier can however formulate the soliton solution as well as the radiation. In this paper, however, the G'/G method [2–4] and the exponential function method [3] will be used to study the CH–KP equation. Finally, the ansatz method [1, 6] will obtain topological soliton solution or the shock wave solutions to this equation.

2 Governing equation

The CH–KP equation that will be studied in this paper is given by [1, 5, 9, 15]:

$$\left[u_t + 2ku_x - u_{xxt} - au^n u_x\right]_x + u_{yy} = 0 \tag{1}$$

and

$$\left[u_t + 2ku_x - u_{xxt} + au^n\left(u^n\right)_x\right]_x + u_{yy} = 0. \tag{2}$$

Equations (1) and (2) were studied by Wazwaz in 2005 [7]. Later, Lai and Xu [8] studied the generalized forms of CH–KP (gCH–KP) equation, which are

$$\left[u_t + 2ku_x - \left(u^m\right)_{xxt} - au^n u_x\right]_x + u_{yy} = 0 \tag{3}$$

and

$$\left[u_t + 2ku_x - \left(u^m\right)_{xxt} + au^n\left(u^n\right)_x\right]_x + u_{yy} = 0. \tag{4}$$

Here, in Eqs. (1)–(4), the dependent variable is $u(x, y, t)$ and t is the temporal variable while x and y are the spatial variables. Also, n is the strength of the nonlinearity, and $a > 0$, $k \in \mathbb{R}$.

In the next two sections the exponential function method and the G'/G method are going to be applied to extract some solutions to Eqs. (1) and (2). Finally, in the next section, the topological 1-soliton solution will be obtained to the gCH–KP equation using the soliton ansatz method.

3 Exact solutions by exp-function method

In this section, initially a detailed overview of the exp-function method is presented. Subsequently, this method will be applied to (1) and (2) to obtain a few solutions.

3.1 Details of the method

We now present briefly the main steps of the exp-function method [3]. A traveling wave hypothesis $u = u(\xi)$ for $\xi = x + ct + ly$ converts a partial differential equation

$$\Psi(u, u_t, u_x, u_y, u_{xx}, u_{yy}, u_{tx}, \ldots) = 0$$

into an ordinary differential equation

$$\Phi\left(u, cu', u', lu', u'', l^2u'', cu'', \ldots\right) = 0. \tag{5}$$

The $\exp$-function method is based on the assumption that traveling wave solutions can be expressed in the following form

$$u(\xi) = \frac{\sum_{n=-w}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{6}$$

where w, d, p and q are positive integers to be determined further, a_n and b_m are unknown constants. To determine the values of d and q, we balance the linear term of highest order in Eq. (5) with the highest order nonlinear term. Similarly, to determine the values of w and p, we balance the linear term of lowest order in Eq. (5) with the lowest order nonlinear term.

3.2 Application to the $(2+1)$-dimensional CH–KP equations

In this section, we use $\exp$-function method to obtain new and more general exact solutions of the $(2+1)$-dimensional CH–KP equations. We consider Eqs. (1) and (2) and use the transformation

$$u(x, y, t) = u(\xi), \quad \text{where } \xi = x + ly + ct, \tag{7}$$

where l and c are constants to be determined later. Substituting (7) into Eqs. (1) and (2) yields two ordinary differential equations for $u(\xi)$. Integrating twice and taking integration constant to zero, gives

$$\left(c + 2k + l^2\right)u - cu'' - \frac{au^{n+1}}{n+1} = 0, \tag{8}$$

$$\left(c + 2k + l^2\right)u - cu'' + \frac{1}{2}au^{2n} = 0. \tag{9}$$

We make transformations

$$u = W^{2/n} \quad \text{and} \quad u = W^{2/(2n-1)}$$

respectively for Eqs. (8) and (9). Thus, Eqs. (8) and (9) is transformed respectively into the following ordinary differential equations:

$$\begin{aligned} &-n^3W^2(\xi)c - n^2W^2(\xi)c - 2n^3W^2(\xi)k - 2n^2W^2(\xi)k - n^3W^2(\xi)l^2 \\ &\quad -n^2W(\xi)^2l^2 + 2c(W')^2(\xi)n + 4c(W')^2(\xi) + 2cW''(\xi)n^2W(\xi) \\ &\quad +2cW''(\xi)nW(\xi) - 2c(W')^2(\xi)n^2 + aW^4(\xi)n^2 = 0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} &8W^2(\xi)cn^2 - 8W^2(\xi)cn + 2W^2(\xi)c + 16W^2(\xi)kn^2 - 16W^2(\xi)kn \\ &\quad +4W^2(\xi)k + 8W^2(\xi)l^2n^2 - 8W^2(\xi)l^2n + 2W^2(\xi)l^2 - 12c(W')^2(\xi) \\ &\quad -8cW''(\xi)W(\xi)n + 4cW''(\xi)W(\xi) + 8c(W')^2(\xi)n + 4aW^4(\xi)n^2 \\ &\quad -4aW^4(\xi)n + aW^4(\xi) = 0. \end{aligned} \tag{11}$$

According to the exp-function method, we assume that the solution of Eqs. (10) and (11) can be expressed in the form of (6). In order to determine values of w and p, in (6) we balance the linear term of highest order with the highest order nonlinear term in Eqs. (10) and (11). By simple calculation for both Eqs. (10) and (11), we have

$$W''W = \frac{c_1 \exp[(2w+3p)\xi] + \cdots}{c_2 \exp[5p\xi] + \cdots}$$

and

$$W^4 = \frac{c_3 \exp[(4w+p)\xi] + \cdots}{c_4 \exp[5p\xi] + \cdots},$$

where c_i are determined coefficients. Balancing highest order of exp-function in them gets

$$2w + 3p = 4w + p,$$

which leads to

$$p = w.$$

Similarly to determine values of d and q, we balance the linear term of lowest order with the lowest order nonlinear term in Eqs. (10) and (11). By simple calculation for both Eqs. (10) and (11), we have

$$W''W = \frac{\cdots + d_1 \exp[-(3q+2d)\xi]}{\cdots + d_2 \exp[-5q\xi]} \tag{12}$$

and

$$W^4 = \frac{\cdots + d_3 \exp[-(4d+q)\xi]}{\cdots + d_4 \exp[-5q\xi]}, \tag{13}$$

where d_i are determined coefficients. From (12) and (13), we obtain

$$-(3q + 2d) = -(4d + q),$$

which yields

$$q = d.$$

We can freely choose the values of w and d, but the final solution does not strongly depend upon the choice of values of w and d. So for simplicity, we set $p = w = 1$ and $d = q = 1$, then we have

$$W(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{14}$$

Substituting Eq. (14) into Eqs. (10) and (11), and equating to zero the coefficients of all powers of $\exp(n\xi)$ yields a set of algebraic equations for a_0, b_0, a_1, a_{-1}, b_{-1}, b_1, c and l. First consider Eq. (10). Solving the system of algebraic equations by the help of Maple, we obtain two cases.

Case 1.

$$a_1 = 0, \quad a_{-1} = 0, \quad b_0 = 0, \quad b_{-1} = -\frac{1}{8}\frac{aa_0^2(n-2)}{b_1(n+1)(2k+l^2)}, \tag{15}$$

$$c = -\frac{n^2(2k+l^2)}{(n^2-4)}. \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (14) yields

$$W(\xi) = \frac{a_0}{b_1 \mathrm{e}^{\xi} - \frac{1}{8}\frac{aa_0^2(n-2)\mathrm{e}^{-\xi}}{b_1(n+1)(2k+l^2)}},$$

so

$$u(x,y,t) = \left(\frac{a_0}{b_1 \mathrm{e}^{x+ly+ct} - \frac{1}{8}\frac{aa_0^2(n-2)\mathrm{e}^{-(x+ly+ct)}}{b_1(n+1)(2k+l^2)}}\right)^{2/n},$$

where a_0, b_1, l are arbitrary constants.

Case 2.

$$a_1 = 0, \quad b_1 = 0, \quad a_0 = \frac{a_{-1}b_0}{b_{-1}}, \tag{17}$$

$$c = \frac{-nb_{-1}^2l^2 - 2nb_{-1}^2k + aa_{-1}^2 - b_{-1}^2l^2 - 2b_{-1}^2k}{b_{-1}^2(n+1)}. \tag{18}$$

Substituting Eq. (17) and (18) into Eq. (14) yields

$$W(\xi) = \frac{\frac{a_{-1}b_0}{b_{-1}} + a_{-1}\mathrm{e}^{-\xi}}{b_0 + b_{-1}\mathrm{e}^{-\xi}},$$

so

$$u(x,y,t) = \left(\frac{\frac{a_{-1}b_0}{b_{-1}} + a_{-1}\mathrm{e}^{-\xi}}{b_0 + b_{-1}\mathrm{e}^{-\xi}}\right)^{2/n},$$

where a_{-1}, b_{-1}, b_0, l are arbitrary constants.

Now, we consider Eq. (11).

Case 1.

$$a_1 = 0, \quad a_{-1} = 0, \quad b_0 = 0, \quad b_1 = \frac{1}{16}\frac{aa_0^2(2n-3)}{b_{-1}(2k+l^2)}, \tag{19}$$

$$c = -\frac{2k + l^2 - 4l^2n - 8kn + 4l^2n^2 + 8kn^2}{-4n + 4n^2 - 3}. \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (14) yields

$$W(\xi) = \frac{a_0}{\frac{1}{16}\frac{aa_0^2(2n-3)\mathrm{e}^{\xi}}{b_{-1}(2k+l^2)} + b_{-1}\mathrm{e}^{-\xi}},$$

so

$$u(x,y,t) = \left(\frac{a_0}{\frac{1}{16}\frac{a a_0^2(2n-3)\mathrm{e}^{\xi}}{b_{-1}(2k+l^2)} + b_{-1}\mathrm{e}^{-\xi}} \right)^{2/(2n-1)}, \tag{21}$$

where a_0, b_{-1}, l are arbitrary constants.

Case 2.

$$a_1 = 0, \quad b_1 = 0, \quad a_{-1} = \frac{a_0 b_{-1}}{b_0}, \tag{22}$$

$$c = -\frac{1}{2}\frac{a a_0^2 + 2l^2 b_0^2 + 4k b_0^2}{b_0^2}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (14) yields

$$W(\xi) = \frac{a_0 + \frac{a_0 b_{-1}\mathrm{e}^{-\xi}}{b_0}}{b_0 + b_{-1}\mathrm{e}^{-\xi}},$$

so

$$u(x,y,t) = \left(\frac{a_0 + \frac{a_0 b_{-1}\mathrm{e}^{-\xi}}{b_0}}{b_0 + b_{-1}\mathrm{e}^{-\xi}} \right)^{2/(2n-1)}, \tag{24}$$

where a_0, b_{-1}, b_0, l are arbitrary constants.

Case 3.

$$a_{-1} = 0, \quad b_{-1} = 0, \quad b_1 = \frac{a_1 b_0}{a_0}, \tag{25}$$

$$c = -\frac{1}{2}\frac{a a_0^2 + 2l^2 b_0^2 + 4k b_0^2}{b_0^2}. \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (14) yields

$$W(\xi) = \frac{a_0 + a_1\mathrm{e}^{\xi}}{\frac{a_1 b_0 \mathrm{e}^{\xi}}{a_0} + b_0},$$

so

$$u(x,y,t) = \left(\frac{a_0 + a_1\mathrm{e}^{\xi}}{\frac{a_1 b_0 \mathrm{e}^{\xi}}{a_0} + b_0} \right)^{2/(2n-1)}, \tag{27}$$

where a_0, a_1, b_0, l are arbitrary constants and in (21), (24) and (27), $\xi = x + ct + ly$.

4 Exact solutions by G'/G method

In this section, we first describe the G'/G-expansion method, which will then be applied to construct the traveling wave solutions for the $(2+1)$-dimensional CH–KP equations.

4.1 Details of the method

Suppose that a nonlinear partial differential equation is given by

$$F(u, u_t, u_x, u_y u_{tt}, u_{xt}, u_{xx}, u_{yy}, \ldots) = 0, \tag{28}$$

where $u(x, y, t)$ is an unknown function, F is a polynomial in $u = u(x, y, t)$ and its partial derivatives, in which are involved the highest order derivatives and nonlinear terms. In the following, we give the main steps of the G'/G-expansion method [2–4].

Step 1. The traveling wave variable

$$u(x, y, t) = u(\xi), \quad \xi = x + ct + ly,$$

where l and c are constants, permits us reducing Eq. (28) to an ODE for $u = u(\xi)$ in the form

$$F(u, u', u'', \ldots) = 0. \tag{29}$$

Step 2. Suppose that the solution of (29) can be expressed by a polynomial in G'/G as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{30}$$

where $G = G(\xi)$ satisfies the second order linear differential equation in the form:

$$G'' + \lambda G' + \mu G = 0, \tag{31}$$

where a_i, c, l, λ and μ are constants to be determined later, $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (29).

Step 3. Substituting (30) into (29) and using (31), collecting all terms with the same power of G'/G together, and then equating each coefficient of the $(G'/G)^i$ to zero, yields a set of algebraic equations for a_i, c, l, λ and μ.

Step 4. Since the general solutions of (31) is well known for us, then substituting a_i, c, l, λ, μ and the general solutions of (31) into (30) we have more traveling wave solutions of the nonlinear partial differential Eq. (28).

Step 5. We solve the system with the aid of Maple. Depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (31) are well known to us. So, as a final step, we can obtain exact solutions of the given Eq. (28).

4.2 Application to the $(2 + 1)$-dimensional CH–KP equations

In this section, we use G'/G-expansion method to obtain new and more general exact solutions of the $(2 + 1)$-dimensional CH–KP equations. To apply the G'/G-expansion method to the Eqs. (3), (4), we consider Eqs. (8) and (9) as the converted from of them.

Suppose that the solutions of the Eqs. (8) and (9) can be expressed by a polynomial in G'/G as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i,$$

where a_i is arbitrary constant, while $G(\xi)$ satisfies the following second order linear ODE:

$$G'' + \lambda G' + \mu G = 0, \tag{32}$$

where λ and μ are unknown constants. Eq. (32) can be changed into

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{G'}{G}\right) = -\left(\frac{G'}{G}\right)^2 - \lambda\left(\frac{G'}{G}\right) - \mu. \tag{33}$$

Using Eq. (33) and balancing $\mathrm{d}^2 u(\xi)/\mathrm{d}\xi$ with $(u(\xi))^{n+1}$ in Eq. (8) gives

$$\left(\frac{G'}{G}\right)^{m+2} = \left(\frac{G'}{G}\right)^{(n+1)m},$$

so that

$$m = \frac{2}{n},$$

thus we make transformation

$$u = W^{2/n}.$$

Again using Eq. (33) and balancing $\mathrm{d}^2 u(\xi)/\mathrm{d}\xi^2$ with $(u(\xi))^{2n}$ in (9) gives

$$\left(\frac{G'}{G}\right)^{m+2} = \left(\frac{G'}{G}\right)^{(n+1)m},$$

so that

$$m = \frac{2}{2n-1}, \tag{34}$$

thus we make transformation

$$u = W^{2/(2n-1)}. \tag{35}$$

By transformations (34) and (35), Eqs. (8) and (9) are converted respectively into the following ordinary differential equations:

$$\begin{aligned} &-n^3W^2(\xi)c - n^2W^2(\xi)c - 2n^3W^2(\xi)k - 2n^2W^2(\xi)k - n^3W^2(\xi)l^2 \\ &\quad -n^2W(\xi)^2l^2 + 2c(W')^2(\xi)n + 4c(W')^2(\xi) + 2cW''(\xi)n^2W(\xi) \\ &\quad +2cW''(\xi)nW(\xi) - 2c(W')^2(\xi)n^2 + aW^4(\xi)n^2 = 0 \end{aligned} \tag{36}$$

and

$$\begin{aligned} &8W^2(\xi)cn^2 - 8W^2(\xi)cn + 2W^2(\xi)c + 16W^2(\xi)kn^2 - 16W^2(\xi)kn \\ &\quad +4W^2(\xi)k + 8W^2(\xi)l^2n^2 - 8W^2(\xi)l^2n + 2W^2(\xi)l^2 - 12c(W')^2(\xi) \\ &\quad -8cW''(\xi)W(\xi)n + 4cW''(\xi)W(\xi) + 8c(W')^2(\xi)n + 4aW^4(\xi)n^2 \\ &\quad -4aW^4(\xi)n + aW^4(\xi) = 0. \end{aligned} \tag{37}$$

Suppose that the solutions of Eqs. (36) and (37) can be expressed by a polynomial in G'/G as follows:

$$W = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{38}$$

where $G = G(\xi)$ satisfies Eq. (32). Balancing W^4 with $W''W$ in both of Eqs. (36) and (37) gives

$$\left(\frac{G'}{G}\right)^{4m} = \left(\frac{G'}{G}\right)^{2m+2},$$

so

$$m = 1.$$

Thus we can write Eq. (38) as

$$W = a_0 + a_1 \left(\frac{G'}{G}\right), \tag{39}$$

where a_0 and a_1 are constants to be determined later.

With the help of the symbolic software Maple, substitution of Eq. (39) with Eq. (32) into Eqs. (36) and (37), setting the coefficients of $(G'/G)^i$ $(i = 0, \dots, 4)$ to zero, obtains the following system of algebraic equations for a_0, a_1, c, l, λ, μ:

First for (36), we obtain

$$\begin{aligned}
\left(\frac{G'}{G}\right)^0 :\quad & 2cna_1^2\mu^2 - 2n^2ca_1^2\mu^2 - n^3ca_0^2 - n^2ca_0^2 - 2n^2ka_0^2 \\
& -n^3l^2a_0^2 - n^2l^2a_0^2 + 4ca_1^2\mu^2 + an^2a_0^4 \\
& +2n^2ca_1\lambda\mu a_0 + 2cna_1\lambda\mu a_0 - 2n^3ka_0^2 = 0, \\
\left(\frac{G'}{G}\right)^1 :\quad & 2a_1\big(3a_1cn\lambda\mu + 4\mu\lambda a_1c - a_1n^2c\lambda\mu - n^3ca_0 - n^2ca_0 \\
& -2n^3ka_0 - 2n^2ka_0 - n^3l^2a_0 - n^2l^2a_0 + 2n^2c\mu a_0 \\
& +cn\lambda^2a_0 + n^2c\lambda^2a_0 + 2an^2a_0^3 + 2cn\mu a_0\big) = 0, \\
\left(\frac{G'}{G}\right)^2 :\quad & a_1\big(-a_1n^3c - a_1n^2c - a_1n^3l^2 - a_1n^2l^2 - 2a_1n^3k \\
& -2a_1n^2k + 4a_1\lambda^2c + 8a_1\mu c + 4a_1cn\lambda^2 + 8a_1cn\mu \\
& +6a_1an^2a_0^2 + 6n^2c\lambda a_0 + 6cn\lambda a_0\big) = 0, \\
\left(\frac{G'}{G}\right)^3 :\quad & 2a_1\big(2an^2a_0a_1^2 + 4a_1\lambda c + n^2ca_1\lambda + 5cna_1\lambda + 2cna_0 \\
& +2n^2ca_0\big) = 0, \\
\left(\frac{G'}{G}\right)^4 :\quad & a_1^2\big(an^2a_1^2 + 4c + 6cn + 2n^2c\big) = 0.
\end{aligned} \tag{40}$$

And for (37) we obtain

$$\begin{aligned}
\left(\frac{G'}{G}\right)^0:\quad & 8cna_1^2\mu^2+2ca_0^2+8cn^2a_0^2-8cna_0^2+16kn^2a_0^2-16kna_0^2\\
&+8l^2n^2a_0^2-8l^2na_0^2-12ca_1^2\mu^2+4ka_0^2+2l^2a_0^2+aa_0^4\\
&+4an^2a_0^4-4ana_0^4-8cna_1\lambda\mu a_0+4ca_1\lambda\mu a_0=0,\\
\left(\frac{G'}{G}\right)^1:\quad & 4a_1\big(2a_1cn\lambda\mu-5ca_1\lambda\mu+4cn^2a_0-4cna_0+ca_0+8kn^2a_0\\
&-8kna_0+2ka_0+4l^2n^2a_0-4l^2na_0+l^2a_0+c\lambda^2a_0\\
&-2cn\lambda^2a_0+4an^2a_0^3-4cn\mu a_0+aa_0^3+2c\mu a_0-4ana_0^3\big)=0,\\
\left(\frac{G'}{G}\right)^2:\quad & 2a_1\big(2a_1k+a_1l^2+a_1c+8a_1kn^2-8a_1kn+4a_1cn^2-4a_1cn\\
&-4a_1\lambda^2c-8a_1\mu c+4a_1l^2n^2-4a_1l^2n+12a_1an^2a_0^2\\
&-12a_1ana_0^2+3a_1aa_0^2+6c\lambda a_0-12cn\lambda a_0\big)=0,\\
\left(\frac{G'}{G}\right)^3:\quad & 4a_1\big(4a_1^2an^2a_0-4a_1^2ana_0+a_1^2aa_0-3ca_1\lambda-2cna_1\lambda\\
&-4cna_0+2ca_0\big)=0,\\
\left(\frac{G'}{G}\right)^4:\quad & a_1^2\big(a_1^2a+4a_1^2an^2-4a_1^2an-4c-8cn\big)=0.
\end{aligned}\tag{41}$$

Solving the system (40) with the aid of Maple, gives the following solution:

$$\begin{aligned}
a_0&=\pm\frac{1}{2}\lambda\sqrt{2}\frac{\sqrt{a(2n^2k+n^2l^2+2l^2+6nk+3nl^2+4k)}}{an},\quad \mu=\frac{1}{4}\lambda^2,\\
a_1&=\pm\sqrt{2}\frac{\sqrt{a(2n^2k+n^2l^2+2l^2+6nk+3nl^2+4k)}}{an},\quad c=-2k-l^2.
\end{aligned}\tag{42}$$

where l is an arbitrary constant. Now, solving the system (41) gives

$$\begin{aligned}
a_0&=\frac{1}{2}a_1\lambda,\quad \mu=\frac{1}{4}\lambda^2,\quad c=\frac{1}{4}\frac{aa_1^2(1+4n^2-4n)}{1+2n},\\
l&=\pm\frac{1}{2}\frac{\sqrt{-(1+2n)(8k+aa_1^2+16kn+4a_1^2an^2-4a_1^2an)}}{1+2n},
\end{aligned}\tag{43}$$

where a_1 is an arbitrary constant.

Substituting Eq. (42) into Eq. (39) yields

$$\begin{aligned}
W=&\pm\frac{1}{2}\lambda\sqrt{2}\frac{\sqrt{a(2n^2k+n^2l^2+2l^2+6nk+3nl^2+4k)}}{an}\\
&\pm\sqrt{2}\frac{\sqrt{a(2n^2k+n^2l^2+2l^2+6nk+3nl^2+4k)}}{an}\left(\frac{G'}{G}\right).
\end{aligned}$$

Also with substituting Eq. (43) into Eq. (39), we have

$$W = \frac{1}{2}a_1\lambda + a_1\left(\frac{G'}{G}\right).$$

Since in both of the (42) and (43), $\Delta = \lambda^2 - 4\mu = 0$ so for (42) we obtain the following rational function:

$$W(\xi) = \pm\frac{\sqrt{2}\sqrt{a(n+2)(1+n)(l^2+2k)C_2}}{an(C_1+C_2\xi)}.$$

And for (43), we obtain

$$W(\xi) = \frac{a_1 C_2}{C_1+C_2\xi}.$$

The Eqs. (3) and (4) respectively with any-order nonlinear terms have the solutions

$$u(x,y,t) = \left\{\pm\frac{\sqrt{2}\sqrt{a(n+2)(1+n)(l^2+2k)C_2}}{an(C_1+C_2(x+ly+ct))}\right\}^{2/n},$$

where l, c_1, c_2 are free parameters, and

$$u(x,y,t) = \left\{\frac{a_1 C_2}{C_1+C_2(x+ly+ct)}\right\}^{2/(2n-1)},$$

where a_1, c_1, c_2 are free parameters.

5 Exact solutions by ansatz method

In this section, the gCH–KP equation will be studied. They are Eqs. (3) and (4), respectively. These two equations will be respectively rewritten with arbitrary coefficients as

$$\left[q_t + a_1 q_x + b_1\left(q^m\right)_{xxt} + c_1 q^n q_x\right]_x + d_1 u_{yy} = 0 \tag{44}$$

and

$$\left[q_t + a_2 q_x + b_2\left(q^m\right)_{xxt} + c_2 q^n\left(q^n\right)_x\right]_x + d_2 u_{yy} = 0, \tag{45}$$

respectively. They will also be referred to as Forms-I and II, respectively, in the rest of this section. The ansatz method will be applied to retrieve the topological 1-soliton solutions to these two forms. It needs to be noted that the non-topological soliton solutions are already obtained in 2011 [1]. Here, for (44) and (45) $a_{1,2}, b_{1,2}, c_{1,2}, d_{1,2} \in \mathbb{R}$ are constants while $m, n \in \mathbb{Z}^+$.

To start off, the hypothesis is given by [1, 6]

$$q(x,y,t) = A\tanh^p(B_1 x + B_2 y - vt) \tag{46}$$

is picked, where

$$\tau = B_1 x + B_2 y - vt.$$

Here A, B_1 and B_2 are free parameters, while v represents the velocity of the soliton. The value of the unknown exponent p will be determined as a function of m and n later during the derivation of the soliton solution. From the ansatz (46), we get

$$q_{tx} = -pvAB_1\big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\big\}, \tag{47}$$

$$q_{xx} = pAB_1^2\big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\big\}, \tag{48}$$

$$\begin{aligned}\big(q^m\big)_{xxxt} = -vpmA^mB_1^3\big\{&(pm-1)(pm-2)(pm-3)\tanh^{pm-4}\tau\\ &+(pm+1)(pm+2)(pm+3)\tanh^{pm+4}\tau\\ &-2(pm-1)\big\{p^2m^2+(pm-2)^2\big\}\tanh^{pm-2}\tau\\ &-2(pm+1)\big\{p^2m^2+(pm+2)^2\big\}\tanh^{pm+2}\tau\\ &+\big[(pm-1)^2(pm-2)+(pm+1)^2(pm+2)\\ &\quad+4p^3m^3\big]\tanh^{pm}\tau\big\},\end{aligned} \tag{49}$$

$$\begin{aligned}\big(q^nq_x\big)_x = pA^{n+1}B_1^2\big\{&\big[p(n+1)-1\big]\tanh^{p(n+1)-2}\tau - 2p(n+1)\tanh^{p(n+1)}\tau\\ &+\big[p(n+1)+1\big]\tanh^{p(n+1)+2}\tau\big\},\end{aligned} \tag{50}$$

$$\begin{aligned}\big\{q^n\big(q^n\big)_x\big\}_x = pnA^{2n}B_1^2\big\{&(2pn-1)\tanh^{2pn-2}\tau - 4pn\tanh^{2pn}\tau\\ &+(2pn+1)\tanh^{2pn+2}\tau\big\}\end{aligned} \tag{51}$$

and

$$q_{yy} = pAB_2^2\big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\big\}. \tag{52}$$

5.1 Form-I

Substituting (47)–(52) into (44), gives

$$\begin{aligned}&-pvAB_1\big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\big\}\\ &\quad+a_1pAB_1^2\big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\big\}\\ &\quad-b_1vpmA^mB_1^3\big\{(pm-1)(pm-2)(pm-3)\tanh^{pm-4}\tau\\ &\qquad+(pm+1)(pm+2)(pm+3)\tanh^{pm+4}\tau\\ &\qquad-2(pm-1)\big\{p^2m^2+(pm-2)^2\big\}\tanh^{pm-2}\tau\\ &\qquad-2(pm+1)\big\{p^2m^2+(pm+2)^2\big\}\tanh^{pm+2}\tau\\ &\qquad+\big[(pm-1)^2(pm-2)+(pm+1)^2(pm+2)+4p^3m^3\big]\tanh^{pm}\tau\big\}\end{aligned}$$

$$+c_1pA^{n+1}B_1^2\big\{\big[p(n+1)-1\big]\tanh^{p(n+1)-2}\tau-2p(n+1)\tanh^{p(n+1)}\tau$$
$$+\big[p(n+1)+1\big]\tanh^{p(n+1)+2}\tau\big\}$$
$$+d_1pAB_2^2\big\{(p-1)\tanh^{p-2}\tau-2p\tanh^{p}\tau+(p+1)\tanh^{p+2}\tau\big\}=0. \tag{53}$$

From (53), equating the exponents $p(n+1)+2$ and $pm+4$ gives

$$p(n+1)+2=pm+4,$$

so that

$$p=\frac{2}{n-m-1}, \tag{54}$$

which is also obtained by equating the exponents pair $p(n+1)$ and $pm+2$, and the exponents $p(n+1)-2$ and pm, respectively.

Now, noting that the functions $\tanh^{p+j}\tau$, for $j=-2,0,2$ are linearly independent, setting their respective coefficients to zero, yields a unique expression of the soliton velocity:

$$v=\frac{a_1B_1^2+d_1B_2^2}{B_1}. \tag{55}$$

Setting the coefficient of the stand-alone linearly independent functions $\tanh^{pm-2}\tau$ and $\tanh^{pm-4}\tau$ in (53) to zero, respectively, yields

$$b_1vpmA^mB_1^3(pm-1)(pm-2)(pm-3)=0,$$
$$2b_1vpmA^mB_1^3(pm-1)\big\{p^2m^2+(pm-2)^2\big\}=0,$$

which can be solved together for the case $pm-1=0$, and therefore

$$p=\frac{1}{m}. \tag{56}$$

Now, equating the two values of the exponent p from (54) and (56) gives

$$n=3m+1, \tag{57}$$

which serves as a constraint relation between the integers n and m.

Also, from (53), setting the coefficients of the linearly independent functions are $\tanh^{p(n+1)+j}\tau$ for $j=-2,0,2$ to zero yields the following algebraic equations:

$$-b_1vpmA^mB_1^3(pm+1)(pm+2)(pm+3)+c_1pA^{n+1}B_1^2\big[p(n+1)+1\big]=0, \tag{58}$$
$$b_1vpmA^mB_1^3(pm+1)\big\{p^2m^2+(pm+2)^2\big\}-c_1pA^{n+1}B_1^2p(n+1)=0, \tag{59}$$
$$-b_1vpmA^mB_1^3\big[(pm-1)^2(pm-2)+(pm+1)^2(pm+2)+4p^3m^3\big]$$
$$+c_1pA^{n+1}B_1^2\big[p(n+1)-1\big]=0. \tag{60}$$

By using the relation (56), the expressions (58)–(60) reduces to

$$-24b_1vB_1 + 4c_1pA^{n-m+1} = 0, \tag{61}$$

$$20b_1vB_1 - 3c_1pA^{n-m+1} = 0, \tag{62}$$

$$-16b_1vB_1 + 2c_1pA^{n-m+1} = 0. \tag{63}$$

Combining any two equations from (61)–(63) yields the the same value of v given by

$$v = \frac{c_1pA^{n-m+1}}{4b_1B_1} = \frac{c_1pA^{2m+2}}{4b_1B_1}, \tag{64}$$

which shows the consistency of the method.

Equating the two values of the velocity v from (55) and (64) gives the free parameter A of the soliton pulse as

$$A = \left[\frac{4mb_1(a_1B_1^2 + d_1B_2^2)}{c_1}\right]^{1/(n-m+1)} = \left[\frac{4mb_1(a_1B_1^2 + d_1B_2^2)}{c_1}\right]^{1/(2m+2)}, \tag{65}$$

which forces the constraint relation

$$b_1c_1\left(a_1B_1^2 + d_1B_2^2\right) > 0. \tag{66}$$

Thus, finally, the topological 1-soliton solution to the gCH–KP equation (Form-I) is given by

$$\begin{aligned} q(x,y,t) &= A\tanh^{2/(n-m-1)}(B_1x + B_2y - vt) \\ &= A\tanh^{1/(m+1)}(B_1x + B_2y - vt), \end{aligned} \tag{67}$$

where the relation between the free parameters A, B_1 and B_2 is given by (65), and the velocity of the soliton is given by (55) or (64). Finally the conditions for the soliton solution (67) to exist are displayed in (57) and (66).

5.2 Form-II

In this case, substituting (47)–(52) into (45) gives

$$\begin{aligned} &-pvAB_1\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\} \\ &\quad+a_2pAB_1^2\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\} \\ &\quad-b_2vpmA^mB_1^3\left\{(pm-1)(pm-2)(pm-3)\tanh^{pm-4}\tau\right. \\ &\qquad\qquad+(pm+1)(pm+2)(pm+3)\tanh^{pm+4}\tau \\ &\qquad\qquad-2(pm-1)\{p^2m^2 + (pm-2)^2\}\tanh^{pm-2}\tau \\ &\qquad\qquad-2(pm+1)\{p^2m^2 + (pm+2)^2\}\tanh^{pm+2}\tau \\ &\qquad\qquad\left.+\left[(pm-1)^2(pm-2) + (pm+1)^2(pm+2) + 4p^3m^3\right]\tanh^{pm}\tau\right\} \\ &\quad+c_2pnA^{2n}B_1^2\left\{(2pn-1)\tanh^{2pn-2}\tau - 4pn\tanh^{2pn}\tau + (2pn+1)\tanh^{2pn+2}\tau\right\} \\ &\quad+d_2pAB_2^2\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\} = 0. \end{aligned} \tag{68}$$

From (68), matching the exponents of $\tanh^{2pn+2}\tau$ and $\tanh^{pm+4}\tau$ functions gives

$$2pn+2=pm+4,$$

so that

$$p=\frac{2}{2n-m}. \tag{69}$$

It needs to be noted that the same value of p is yielded when the exponents pair $2pn$ and $pm+2$, and the exponents $2pn-2$ and pm, respectively, are equated with each other.

Further, setting the coefficients of the linearly independent functions $\tanh^{p+j}\tau$, for $j=-2,0,2$ in (68), to zero yields a unique value of the velocity v as follows:

$$v=\frac{a_2B_1^2+d_2B_2^2}{B_1}. \tag{70}$$

Setting the coefficient of the stand-alone linearly independent functions $\tanh^{pm-2}\tau$ and $\tanh^{pm-4}\tau$ in (68) to zero, respectively, yields

$$b_2vpmA^mB_1^3(pm-1)(pm-2)(pm-3)=0,$$
$$b_2vpmA^mB_1^3(pm-1)\{p^2m^2+(pm-2)^2\}=0,$$

which can be solved together for the case $pm-1=0$, and therefore

$$p=\frac{1}{m}. \tag{71}$$

Now, equating the two values of the exponent p from (69) and (71) gives

$$2n=3m, \tag{72}$$

which serves as a constraint relation between the integers n and m.

Also, from (68), setting the coefficients of the linearly independent functions are $\tanh^{2pn+j}\tau$ for $j=-2,0,2$ to zero yields

$$-b_2vpmA^mB_1^3(pm+1)(pm+2)(pm+3)+c_2pnA^{2n}B_1^2(2pn+1)=0, \tag{73}$$
$$2b_2vpmA^mB_1^3(pm+1)\{p^2m^2+(pm+2)^2\}-4c_2p^2n^2A^{2n}B_1^2=0, \tag{74}$$
$$-b_2vpmA^mB_1^3[(pm-1)^2(pm-2)+(pm+1)^2(pm+2)+4p^3m^3] +c_2pnA^{2n}B_1^2(2pn-1)=0. \tag{75}$$

Substituting (71) into (73)–(75), one gets the following algebraic equations:

$$-24b_2vB_1^3+4c_2pnA^{2n-m}B_1^2=0, \tag{76}$$
$$20b_2vB_1^3-3c_2pnA^{2n-m}B_1^2=0, \tag{77}$$
$$-16b_2vB_1^3+2c_2pnA^{2n-m}B_1^2=0. \tag{78}$$

Combining any two equations from (76)–(78) gives the the same value of v namely

$$v = \frac{c_2 pnA^{2n-m}}{4b_2 B_1} = \frac{c_2 pnA^{2m}}{4b_2 B_1}, \tag{79}$$

which proves again the effectiveness and convenience of the soliton ansatz method for constructing exact soliton solutions.

Equating the two values of the velocity v from (70) and (79) gives the free parameter A of the topological soliton as

$$A = \left[\frac{4b_2 m(a_2 B_1^2 + d_2 B_2^2)}{c_2 n}\right]^{1/(2n-m)} = \left[\frac{4b_2 m(a_2 B_1^2 + d_2 B_2^2)}{c_2 n}\right]^{1/(2m)}, \tag{80}$$

which forces the constraint relation

$$b_2 c_2 \left(a_2 B_1^2 + d_2 B_2^2\right) > 0. \tag{81}$$

Thus, the topological 1-soliton solution to the gCH–KP equation (Form-II) is given by

$$\begin{aligned} q(x,y,t) &= A \tanh^{2/(2n-m)}(B_1 x + B_2 y - vt) \\ &= A \tanh^{1/m}(B_1 x + B_2 y - vt), \end{aligned} \tag{82}$$

where the relation between the free parameters A, B_1 and B_2 is given by (80), and the velocity of the soliton is given by (70) or (79). Finally, the conditions for the soliton solution (82) to exist are displayed in (72) and (81).

6 Conclusions

In this paper, we have utilized the exp-function and the G'/G-expansion methods to seek exact solutions of the $(2+1)$-dimensional CH–KP equation. These methods were successfully used to establish several solutions. Also, both of the methods are promising and powerful methods for nonlinear evolution equations in mathematical physics. With the aid of Maple, it is confirmed that the solutions are correct since these solutions satisfy the original equation. Additionally, the ansatz method is also applied to extract the topological 1-soliton solution, also known as shock waves, of the gCH–KP equations. In this case, the constraint relations are obtained for these topological soliton solutions to exist.

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