On the existence of solutions to the fractional derivative equations $\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + Au = f$, of relevance to diffusion in complex systems*

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Abstract. Fractional derivative equations account for relaxation and diffusion processes in a large variety of condensed matter systems. For instance, diffusion of position probability density displayed by a random walker in complex systems – such as glassy materials – is often modeled by fractional derivative partial differential equations (e.g. [1]). This paper deals with the existence of solutions to the general fractional derivative equation $\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + Au = f$ for $0 < \alpha < 1$, with A a self-adjoint operator. The results are proved using the von Neumann–Dixmier theorem [2].

Keywords: diffusion in complex systems, fractional derivative evolution equations, separation of variables method, Caputo derivative, integral equations, self-adjoint operators, von Neumann–Dixmier theorem.

1 Introduction

Fractional derivative models and equations are widely used in all science domains, as can be reckoned from [3–7] and the wealth of references cited therein. In particular, in the area of viscoelasticity (for a general presentation of which we refer to e.g. [8–11]), fractional derivative models are of great utility in accurately predicting the rheological behavior of polymer liquids in the glass transition region and beyond (e.g. see [12–21]; for a review of fractional derivative rheological models see [22]), in interpreting experimental measurements of anomalous diffusion processes in glassy materials [1, 23] etc.

Diffusion phenomena in complex systems – a category which includes organic and inorganic glassy materials – are often associated with slower time rates (e.g. monomer diffusion in glassy polymers). When this is the case, it is now routinely to have the ordinary time derivative replaced by a fractional derivative of order $0 < \alpha < 1$.

*Dedicated to Professor Kumbakonam R. Rajagopal, Texas A&M University, College Station, on the occasion of his 60th anniversary.

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In this paper we study the following problem: find $u : \mathbb{R}_+ \mapsto D(A)$ such that

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + Au = f, \quad u(0) = u_0. \tag{1}$$

Here A is a coercive self-adjoint operator with domain $D(A)$, which models – among others – diffusion processes in fluids and solids. A typical example is that of a second order strongly elliptic partial derivative operator, i.e.

$$A := -\sum_{i,j}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + c(x)\,\mathrm{Id}$$

with $a_{ij} \in \mathscr{C}^2(\overline{\Omega} \Subset \mathbb{R}^n)$, $c \in \mathscr{C}^0(\overline{\Omega})$, $c \geq 0$, and for any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$,

$$\sum_{i,j}^{n} a_{ij}(x)\xi_i\xi_j \geq \beta|\xi|^2, \quad \beta > 0.$$

In the above $|\xi|$ stands for the Euclidean norm of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

Equations (1) have been extensively studied by Bazhlekova [24] by means of Da Prato–Iannelli's theorem [25] and very general results within the framework of L^p spaces can be found in [24]. For existence results in the case of systems of equations that generalize (1) see [26]. On the other hand, for some peculiar operators, explicit calculations can be carried out (see e.g. [27–30]; for a review on recent results on fractional boundary value problems see [31]).

In this paper, we restrict to the case of self-adjoint operators and prove that the result obtained by the method of separation of variables converges in $L^2_{\mathrm{loc}}(\mathbb{R}_+, D(A^\theta))$. This is achieved using the von Neumann–Dixmier's spectral theorem.

2 Functional framework

Let H be a separable Hilbert space. Let A be a self-adjoint operator the domain $D(A) \subset H$ of which is such that:

(i) $D(A)$ is dense in H;

(ii) $\exists c > 0$ s.t. for $\forall u \in D(A)$, $(Au|u) \geq c\|u\|^2$.

Using the von Neumann–Dixmier's theorem [2] one has:

Theorem 1. *Let $A : D(A) \subset H \to H$ be compliant with conditions* (i) *and* (ii) *right above. Then there exist a Hilbert integral $\mathscr{H} = \int_{\lambda_0}^{+\infty} \mathscr{H}(\lambda)\,\mathrm{d}\mu(\lambda)$, where $\lambda_0 \in]0, c[$ and μ is a positive, bounded Radon measure, and a surjectif unitary operator $\mathscr{U} : H \to \mathscr{H}$, such that*:

1. $\mathscr{U}(D(A)) = \{f \in \mathscr{H}$ *s.t.* $\lambda f \in \mathscr{H}\}$;
2. *For any $y \in D(A)$, $\mathscr{U}(Ay) = \lambda\mathscr{U}(y)$.*

In many specific cases one may wish to work with non-canonical variants of Theorem 1. For instance, for $A = -\Delta + \mathrm{Id}$, $H = L^2(\mathbb{R}^n)$ and $D(A) = H^2(\mathbb{R}^n)$, one may prefer, instead of part (i) of Theorem 1, the following description (here $\widehat{\ }$ denotes the Fourier transform):

$$\widehat{D(-\Delta+\mathrm{Id})} = \bigl\{f \in L^2\bigl(\mathbb{R}^n_\xi\bigr) \text{ s.t. } \bigl(|\xi|^2+1\bigr)f \in L^2\bigl(\mathbb{R}^n_\xi\bigr)\bigr\}.$$

Nevertheless, in order to get a unified treatment of the different cases (including e.g. the above given operator or that of a self-adjoint compact operator) we have to use Theorem 1. It allows working with a single eigenequation $\frac{\partial^\alpha}{\partial t^\alpha}Z + \lambda Z = g$ (see Propositions 1 and 3) and a fixed functional frame. Of course the convergence result obtained in Theorem 2 below can in the end be stated in more specific (albeit non-canonical) forms.

For sake of clarity we now pause for a few notation explanations and remainders regarding the spaces D_θ used in this paper. These interpolation spaces are very similar to the usual fractional spaces $H^s(\Omega), s \in \mathbb{R}$. In the following, $(\,|\,)_{\mathscr{H}(\lambda)}$ denotes the inner product in $\mathscr{H}(\lambda)$ and $\|\,\|_{\mathscr{H}(\lambda)}$ the corresponding norm.

(a) We denote by $\hat{\ }$ the operator $\mathscr{U}$. Let $A : D(A) \subset H \to H$ satisfy (i) and (ii) above.

Denote by D_θ, $\theta \geq 0$, the space $D(A^\theta) := \{f \in H \mid \lambda^\theta \hat{f} \in \mathscr{H},\ \theta \geq 0\}$. The space D_θ is endowed with the norm

$$\|f\|^2_{D_\theta} := \int\limits_{\lambda_0}^{+\infty} \lambda^{2\theta}\bigl\|\hat{f}(\lambda)\bigr\|^2_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda)$$

for any $f \in D_\theta$. For any $\theta \geq \theta' \geq 0$, the continuous inclusions $D_\theta \hookrightarrow D_{\theta'} \hookrightarrow D_0 = H$ hold true. Likewise, for the topological dual spaces, $H' \hookrightarrow (D_{\theta'})' \hookrightarrow (D_\theta)'$. With the help of the inner product $(\,|\,)_H$ we have an isomorphism $i : H \to H'$. Henceforth $\mathscr{H} \xrightarrow{\mathscr{U}^{-1}} H \xrightarrow{\sim} H' \hookrightarrow (D_\theta)'$.

(b) We now define the Banach spaces $D_{-\theta}$ for $\theta \geq 0$. For $\theta \geq 0$, set $D_{-\theta} = (D_\theta)'$.

For any $\theta \in \mathbb{R}$, let F_θ denote the space of measurable vector fields f s.t. $\lambda^\theta f \in \mathscr{H}$. The space F_θ is endowed with the inner product

$$(f|g)_{F_\theta} := \int\limits_{\lambda_0}^{+\infty} \lambda^{2\theta}\bigl(f(\lambda)\big|g(\lambda)\bigr)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \quad \forall (f,g) \in F_\theta \times F_\theta \tag{2}$$

and the corresponding norm $\|\,\|_{F_\theta}$. For any $\theta \geq 0$, one has $F_{-\theta} \underset{\rho}{\xrightarrow{\sim}} (F_\theta)'$, where ρ is defined by

$$\bigl\langle \rho(\varphi), \psi \bigr\rangle = \int\limits_{\lambda_0}^{+\infty} \bigl(\varphi(\lambda)\big|\psi(\lambda)\bigr)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \quad \forall \varphi \in F_{-\theta},\ \forall \psi \in F_\theta.$$

Since $F_\theta \xrightarrow[\sim]{\mathscr{U}^{-1}|_{F_\theta}} D_\theta$, one also has $F_{-\theta} \overset{G}{\underset{\sim}{\rightarrow}} (D_\theta)'$, with G being given by

$$\langle G(f), g\rangle = \int_{\lambda_0}^{+\infty} \big(f(\lambda)\big|\hat{g}(\lambda)\big)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \quad \forall f \in F_{-\theta},\ \forall g \in F_\theta.$$

In what follows, for $f \in F_{-\theta}$, $\theta \geq 0$, and $h = G(f)$ we (abusively) write $\hat{h} = f$. For $\theta > 0$, the norm $\|\ \|_{D_{-\theta}}$ is defined by (see also Eq. (2))

$$\|h\|_{D_{-\theta}}^2 := \int_{\lambda_0}^{+\infty} \lambda^{-2\theta}\big\|\hat{h}(\lambda)\big\|_{\mathscr{H}(\lambda)}^2\,\mathrm{d}\mu(\lambda) \quad \forall h \in D_{-\theta}.$$

The spaces D_θ and F_θ are complete for any $\theta \in \mathbb{R}$.

The operator A is extended to $D_\theta \to D_{\theta-1}$ ($\theta < 1$) in the following way: for any $u \in D_\theta$, $\widehat{Au} = \lambda\hat{u}$.

(c) We introduce (see below) a function E. This kernel E will allow us to solve the equation

$$\frac{\partial^\alpha \hat{u}}{\partial t^\alpha} + \widehat{Au} = \hat{f}, \quad \hat{u}(0) = 0,$$

where the fractional derivative is formally defined by (see Caputo's definition of it in [32, 33]):

$$\frac{\partial^\alpha h}{\partial t^\alpha}(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{h'(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau.$$

For any $\lambda > 0$, let the functions E and W be given, for any $t > 0$, by:

$$E(\lambda, t) = \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} \frac{r^\alpha \mathrm{e}^{-rt}}{|r^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi} + \lambda|^2}\,\mathrm{d}r,$$

$$W(\lambda, t) = \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} \frac{\lambda r^{\alpha-1} \mathrm{e}^{-rt}}{|r^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi} + \lambda|^2}\,\mathrm{d}r.$$

The functions $E(\lambda,\cdot)$ and $W(\lambda,\cdot)$ are causal functions w.r.t. the variable t, like all t-depending functions considered in this paper.

Proposition 1. *(See* [34].*) Let $\lambda > 0$, $g \in \mathscr{C}^1([0,T])$, $T > 0$. Then, for any $t \in [0,T]$, the function $u_\lambda = E(\lambda) * g$ ($\lambda > 0$) solves*

$$\frac{1}{\Gamma(1-\alpha)}\int_0^T \frac{u_\lambda'(t)}{(t-s)^\alpha}\,\mathrm{d}s = -\lambda u_\lambda(t) + g(t), \quad u_\lambda(0) = 0.$$

The derivative u_λ' is understood in the classical sense.

(d) Finally, for future reference, notice the following estimate:

Proposition 2. *For any* $\lambda > 0$,

$$\int_0^{+\infty} \left|E(\lambda, t)\right| \mathrm{d}t \leq \frac{1}{\lambda}.$$

Proof. Since $\frac{\partial W}{\partial t}(\lambda, t) = -\lambda |E(\lambda, t)|$, it implies that, for any $T > 0$,

$$\int_0^T \left|E(\lambda, t)\right| \mathrm{d}t = -\frac{1}{\lambda} \int_0^T \frac{\partial W}{\partial t}(\lambda, t) \, \mathrm{d}t = \frac{W(\lambda, 0) - W(\lambda, T)}{\lambda} \leq \frac{W(\lambda, 0)}{\lambda}.$$

Since $W(1, 0) = 1$ (see [34]), one gets $W(\lambda, 0) = 1$ as well. Hence $\int_0^T |E(\lambda, t)| \, \mathrm{d}t \leq 1/\lambda$. $\square$

3 Existence of solutions

Let H be a Hilbert space. Let $0 < \alpha < 1$ and $A : D(A) \mapsto H$ be a self-adjoint operator satisfying the conditions (i) and (ii). Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$, $\theta \in \mathbb{R}$. Our goal is to prove an existence result for the equations

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + Au = f, \quad u(0) = u_0. \tag{3}$$

In the above equation, $f : \mathbb{R} \to D_{\theta_1}$, $u_0 \in D_{\theta_2}$, $\theta_1, \theta_2 \in \mathbb{R}$, are given functions. We search for $\theta \in \mathbb{R}$ and functions $u : \mathbb{R}_+ \to D_\theta$.

We first restrict to $u_0 = 0$ as the case $u_0 \neq 0$ can be reduced to

$$\frac{\mathrm{d}^\alpha v}{\mathrm{d}t^\alpha} + Av = f - Au_0, \quad v(0) = 0.$$

Since the system of equations (3) with $u_0 = 0$ is formally equivalent to the below system

$$\frac{\partial^\alpha \hat{u}}{\partial t^\alpha}(\lambda, t) + \lambda \hat{u}(\lambda, t) = \hat{f}(\lambda, t), \quad \hat{u}(\lambda, 0) = 0.$$

We prove the existence in $L^2_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1})$ of the function u formally defined by $\hat{u}(\lambda) = E(\lambda) * \hat{f}(\lambda)$, $\lambda \geq \lambda_0$ (see Proposition 1).

Proposition 3. *Assume* $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$, $\theta \in \mathbb{R}$. *Then there exists a function* $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1})$ *such that* $\hat{u}(\lambda) = E(\lambda) * \hat{f}(\lambda)$, *(*$\mu$ *a.e. in* $\lambda \geq \lambda_0$*). Moreover, for any* $t \geq 0$,

$$\int_0^t \left\|u(s)\right\|^2_{D_{\theta+1}} \mathrm{d}s \leq \int_0^t \|f\|^2_{D_\theta}(s) \, \mathrm{d}s.$$

Proof. Let $t \geq 0$. One has:

$$\begin{aligned}
&\int_{\lambda_0}^{+\infty} \left(\lambda^{2(\theta+1)} \int_0^t \left\| E(\lambda) * \hat{f}(\lambda) \right\|^2_{\mathscr{H}(\lambda)}(s)\, \mathrm{d}s \right) \mathrm{d}\mu(\lambda) \\
&\quad \leq \int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1)} \left(\int_0^t \left\| E(\lambda) * \hat{f}(\lambda) \right\|^2_{\mathscr{H}(\lambda)}(s)\, \mathrm{d}s \right) \mathrm{d}\mu(\lambda) \\
&\quad \leq \int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1)} \left[\int_0^t \left(E(\lambda) * \left\| \hat{f}(\lambda) \right\| \right)^2_{\mathscr{H}(\lambda)}(s)\, \mathrm{d}s \right] \mathrm{d}\mu(\lambda) \\
&\quad \leq \int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1)} \left[\left(\int_0^t E(\lambda, s)\, \mathrm{d}s \right)^2 \left(\int_0^t \left\| \hat{f}(\lambda, s) \right\|^2_{\mathscr{H}(\lambda)}\, \mathrm{d}s \right) \right] \mathrm{d}\mu(\lambda) \\
&\quad = \int_0^t \| f \|^2_{D_\theta}(s)\, \mathrm{d}s,
\end{aligned}$$

where the last equality follows from Proposition 2.

The existence of a function $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1})$ such that $\hat{u}(\lambda) = E(\lambda) * \hat{f}(\lambda)$, ($\mu$ a.e. in $\lambda \geq \lambda_0$) follows from the above inequality. Moreover,

$$\begin{aligned}
\int_0^t \left\| u(s) \right\|^2_{D_{\theta+1}}\, \mathrm{d}s &= \int_{\lambda_0}^{+\infty} \left(\lambda^{2(\theta+1)} \int_0^t \left\| E(\lambda) * \hat{f}(\lambda) \right\|^2_{\mathscr{H}(\lambda)}(s)\, \mathrm{d}s \right) \mathrm{d}\mu(\lambda) \\
&\leq \int_0^t \| f \|^2_{D_\theta}(s)\, \mathrm{d}s. \qquad \square
\end{aligned}$$

It remains to prove that the function u given in Proposition 3 satisfies Eqs. (1). This will be a consequence of the eigenequations solved by $\hat{u}$. Before undertaking this, we need several preliminary results.

We first focus on expressing $\widehat{u'}$ in terms of functions E and f. The expression given in Proposition 4 below is the result obtained by formally differentiating formula $\hat{u}(\lambda) = E(\lambda) * \hat{f}(\lambda)$. Proposition 4 also contains our smoothness results for the function u.

Let g be formally defined by

$$\hat{g}(\lambda, t) = E(\lambda, t)\hat{f}(\lambda, t) + \left[E(\lambda) * \widehat{f'}(\lambda) \right](t).$$

Proposition 4. *Let $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$. Then, for any $\epsilon \in]0,1[$ and $r \in [1,2] \cap [1, \frac{1}{1-\epsilon\alpha}[$,*

$$\widehat{u'}(\lambda) = E(\lambda)\hat{f}(\lambda, 0) + E(\lambda) * \widehat{f'}(\lambda),$$

μ a.e. in $\lambda \geq \lambda_0$. Moreover, $u \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1-\epsilon})$.

Proof. As $W^{1,r}_{\mathrm{loc}}(\mathbb{R}_+, D_\theta) \hookrightarrow W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$ for $r > 1$, we shall consider only the case $r > 1$. For any $T \geq 0$, one has

$$\int_0^T \left\| \frac{u(t+h) - u(t)}{h} - g(t) \right\|^r_{D_{\theta+1-\epsilon}} \mathrm{d}t$$
$$= \int_0^T \left[\int_{\lambda_0}^{+\infty} \left\| \frac{\hat{u}(t+h,\lambda) - \hat{u}(t,\lambda)}{h} - \hat{g}(t,\lambda) \right\|^2_{\mathscr{H}(\lambda)} \lambda^{2(\theta+1-\epsilon)} \,\mathrm{d}\mu(\lambda) \right]^{r/2} \mathrm{d}t := I_h.$$

Let us prove that $I_h \xrightarrow[h \to 0]{} 0$. One has $I_h \leq M(J_h + K_h)$, $M > 0$, where

$$J_h = \int_0^T \left[\int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1-\epsilon)} \left\| \int_t^{t+h} E(\lambda,u) \hat{f}(\lambda, t+h-u) \,\frac{\mathrm{d}u}{h} - E(\lambda,t)\hat{f}(\lambda,0) \right\|^2_{\mathscr{H}(\lambda)} \mathrm{d}\mu(\lambda) \right]^{r/2} \mathrm{d}t,$$

$$K_h = \int_0^T \left\{ \int_{\lambda_0}^{+\infty} \left\| \int_0^t E(\lambda,u) \left[\frac{\hat{f}(\lambda, t+h-u) - \hat{f}(\lambda, t-u)}{h} - \widehat{f'}(t-u) \right] \mathrm{d}u \right\|^2_{\mathscr{H}(\lambda)} \times \lambda^{2(\theta+1-\epsilon)} \,\mathrm{d}\mu(\lambda) \right\}^{r/2} \mathrm{d}t.$$

Observe now that

$$J_h \leq \frac{c}{|h|} \int_0^T \left[\int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1-\epsilon)} \left(\int_t^{t+h} \left| E(\lambda,u) - E(\lambda,t) \right| \times \left\| \hat{f}(\lambda,0) \right\|^2_{\mathscr{H}(\lambda)} \mathrm{d}u \right)^2 \mathrm{d}\mu(\lambda) \right]^{r/2} \mathrm{d}t$$
$$+ \frac{c}{|h|} \int_0^T \left[\int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1-\epsilon)} \left(\int_t^{t+h} \left| E(\lambda,u) \right| \left\| \hat{f}(\lambda, t+h-u) - \hat{f}(\lambda,0) \right\|_{\mathscr{H}(\lambda)} \mathrm{d}u \right)^2 \mathrm{d}\mu(\lambda) \right]^{r/2} \mathrm{d}t$$
$$\equiv A_h + B_h.$$

We now estimate A_h and B_h. On the one hand, for A_h one has

$$\int_t^{t+h} \left| E(\lambda,u) - E(\lambda,t) \right| \mathrm{d}u = \int_t^{t+h} \left[E(\lambda,t) - E(\lambda,u) \right] \mathrm{d}u$$

as $t \le u$. Notice that $|r^\alpha e^{i\alpha\pi} + \lambda|^2 \ge K r^{2q\alpha}\lambda^{2(1-q)}$, $0 \le q \le 1$. Let $q = \frac{1+\epsilon}{2}$. Since $\mathrm{e}^{-rt} - \mathrm{e}^{-ru} \ge 0$ for $t \le u$, one gets, for $\lambda \ge \lambda_0$,

$$
\begin{aligned}
0 \le E(\lambda, t) - E(\lambda, u) &\le \int_0^{+\infty} \frac{K r^\alpha(\mathrm{e}^{-rt} - \mathrm{e}^{-ru})}{r^{(1+\epsilon)\alpha}\lambda^{1-\epsilon}}\,\mathrm{d}r \\
&\le \frac{K}{\lambda^{1-\epsilon}}\biggl(\int_0^{+\infty} \frac{\mathrm{e}^{-\xi}}{\xi^{\alpha\epsilon}}\,\mathrm{d}\xi\biggr)\biggl(\frac{1}{t^{1-\alpha\epsilon}} - \frac{1}{u^{1-\alpha\epsilon}}\biggr). \qquad (4)
\end{aligned}
$$

Therefore,

$$
\int_t^{t+h} \bigl|E(\lambda, u) - E(\lambda, t)\bigr|\,\mathrm{d}u \le \frac{M}{\lambda^{1-\epsilon}}\biggl(\frac{h}{t^{1-\alpha\epsilon}} - \frac{(t+h)^{\alpha\epsilon} - t^{\alpha\epsilon}}{\alpha\epsilon}\biggr)
$$

and

$$
\begin{aligned}
A_h &\le \frac{M}{|h|}\biggl(\int_{\lambda_0}^{+\infty} \lambda^{2\theta}\bigl\|\hat{f}(\lambda, 0)\bigr\|^2_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda)\biggr)^{r/2} \int_0^T \biggl(\frac{h}{t^{1-\alpha\epsilon}} - \frac{(t+h)^{\alpha\epsilon} - t^{\alpha\epsilon}}{\alpha\epsilon}\biggr)^r \mathrm{d}t \\
&\le \frac{M}{|h|}\bigl\|f(0)\bigr\|^r_{D_\theta} K_T \int_0^T \frac{|h|^r}{t^{(1-\alpha\epsilon)r}}\,\mathrm{d}t, \quad (1-\alpha\epsilon)r < 1.
\end{aligned}
$$

Therefore, provided that $1 < r < \frac{1}{1-\alpha\epsilon}$, one has $A_h \xrightarrow[h\to 0]{} 0$.

On the other hand now, using $|E(\lambda, u)| \le \frac{K}{|\lambda|^{1-\epsilon}u^{1-\alpha\epsilon}}$ (and letting $u \to +\infty$ in (4)), one has for B_h the following estimates:

$$
\begin{aligned}
B_h &\le c\int_0^T \biggl[\int_{\lambda_0}^{+\infty} \lambda^{2\theta}\biggl(\int_t^{t+h} \frac{\|\hat{f}(\lambda, t+h-u) - \hat{f}(\lambda, 0)\|_{\mathscr{H}(\lambda)}}{u^{1-\alpha\epsilon}}\,\mathrm{d}u\biggr)^2 \mathrm{d}\mu(\lambda)\biggr]^{r/2} \frac{\mathrm{d}t}{|h|} \\
&\le c\int_0^T \biggl[\int_{\lambda_0}^{+\infty} \frac{\lambda^{2\theta}|h|}{t^{2(1-\alpha\epsilon)}}\biggl(\int_t^{t+h} \bigl\|\hat{f}(\lambda, t+h-u) - \hat{f}(\lambda, 0)\bigr\|^2_{\mathscr{H}(\lambda)}\,\mathrm{d}u\biggr)\mathrm{d}\mu(\lambda)\biggr]^{r/2} \frac{\mathrm{d}t}{|h|} \\
&\le c\int_0^T \frac{1}{t^{r(1-\alpha\epsilon)}}\biggl(\int_{|u-t|\le h} \bigl\|f(t+h-u) - f(0)\bigr\|^2_{D_\theta}\frac{\mathrm{d}u}{|h|}\biggr)^{r/2} \mathrm{d}t|h|^{r-1}.
\end{aligned}
$$

Recall that $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$, and that f is continuous; hence, for $h \to 0$,

$$
\int_{|u-t|\le h} \bigl\|f(t+h-u) - f(0)\bigr\|^2_{D_\theta}\frac{\mathrm{d}u}{|h|} \to 0.
$$

This gives $B_h \xrightarrow[h\to 0]{} 0$.

We now proceed to obtaining estimates for K_h for $r = 2$. Given that:

$$\left\{ \int_0^T \left\{ \int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1-\epsilon)} \left\| \int_0^t E(\lambda,u) \left[\frac{\hat{f}(\lambda,t+h-u) - \hat{f}(\lambda,t-u)}{h} - \widehat{f'}(\lambda,t-u) \right] \mathrm{d}u \right\|_{\mathscr{H}(\lambda)}^2 \mathrm{d}\mu(\lambda) \right\}^{r/2} \mathrm{d}t \right\}^2$$

$$\leq \int_0^T \left\{ \int_{\lambda_0}^{+\infty} T\lambda^{2(\theta+1-\epsilon)} \left\| \int_0^t E(\lambda,u) \left[\frac{\hat{f}(\lambda,t+h-u) - \hat{f}(\lambda,t-u)}{h} - \widehat{f'}(\lambda,t-u) \right] \mathrm{d}u \right\|_{\mathscr{H}(\lambda)}^2 \mathrm{d}\mu(\lambda) \right\}^{r/2} \mathrm{d}t$$

$$\leq T \int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1-\epsilon)} \left\{ \int_0^T \left(\int_0^t |E(\lambda,u)| \left\| \frac{\hat{f}(\lambda,t+h-u) - \hat{f}(\lambda,t-u)}{h} - \widehat{f'}(\lambda,t-u) \right\|_{\mathscr{H}(\lambda)} \mathrm{d}u \right)^2 \mathrm{d}t \right\} \mathrm{d}\mu(\lambda)$$

$$\leq T \int_{\lambda_0}^{+\infty} \lambda^{2(\theta+1-\epsilon)} \left(\int_0^T E(\lambda,u)\,\mathrm{d}u \right)^2 \left(\int_0^T \left\| \frac{\hat{f}(\lambda,t+h-u) - \hat{f}(\lambda,t-u)}{h} - \widehat{f'}(\lambda,t-u) \right\|_{\mathscr{H}(\lambda)}^2 \mathrm{d}u \right) \mathrm{d}\mu(\lambda)$$

and since $|E(\lambda,u)| \leq \frac{K}{\lambda^{1-\epsilon} u^{1-\alpha\epsilon}}$ (see (4)), one finally gets, using the fact that $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$,

$$K_h^2 \leq T \int_{\lambda_0}^{+\infty} \lambda^{2\theta} \left(\int_0^T \frac{\mathrm{d}u}{u^{1-\alpha\epsilon}} \right)^2 \left(\int_0^T \left\| \frac{\hat{f}(\lambda,u+h) - \hat{f}(\lambda,u)}{h} - \widehat{f'}(\lambda,u) \right\|_{\mathscr{H}(\lambda)}^2 \mathrm{d}u \right) \mathrm{d}\mu(\lambda)$$

$$\leq K_T \int_0^T \left\| \frac{f(u+h) - f(u)}{h} - f'(u) \right\|_{D_\theta}^2 \mathrm{d}u \xrightarrow[h\to 0]{} 0. \qquad \square$$

In view of the assumption $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$ made in Proposition 4, we need the below given version of Mainardi's [34] result quoted in our Proposition 1:

Lemma 1. *Let $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$. Then, the function u_λ, $\lambda > 0$, defined below for any $t \geq 0$*

$$u_\lambda(t) = \big(E(\lambda) * f\big)(t) \tag{5}$$

solves the equations

$$\frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{u_\lambda'(s)}{(t-s)^\alpha}\,\mathrm{d}s = -\lambda u_\lambda(t) + f(t), \quad u_\lambda(0) = 0. \tag{6}$$

Proof. Consider the application $H^1_{\mathrm{loc}}(\mathbb{R}_+) \xrightarrow{\Phi} L^1_{\mathrm{loc}}(\mathbb{R}_+)$,

$$f \mapsto \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{u'(s)}{(t-s)^\alpha}\,\mathrm{d}s + \lambda u - f$$

with $u = E(\lambda) * f$. We prove in the following that $\Phi = 0$. Notice first Φ is a properly defined mapping. Indeed, using arguments similar in nature to those presented in Proposition 4 one shows that for $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$, $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$. Moreover, since $E(\lambda) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, one gets $\Phi(f) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

Next, observe that $\forall f \in \mathscr{C}^1(\mathbb{R}_+)$, $\Phi(f) = 0$ (see Proposition 1), and that $\mathscr{C}^1(\mathbb{R}_+)$ is dense in $H^1_{\mathrm{loc}}(\mathbb{R}_+)$. One needs to prove that Φ is continuous. Observe first that $u' = E(\lambda) * f' + E(\lambda)f(0)$ (proof identical to the one given in Proposition 4). One then has, using Proposition 2,

$$\big\|\Phi(f)\big\|_{1,[0,T]} \le K\left\|\frac{1}{s^\alpha}\right\|_{1,[0,T]}\big(\|f'\|_{1,[0,T]} + \big|f(0)\big|\big) + \|f\|_{1,[0,T]}.$$

However, $W^{1,1}([0,T]) \hookrightarrow L^\infty([0,T])$. Hence

$$\big\|\Phi(f)\big\|_{1,[0,T]} \le K\big(\|f'\|_{1,[0,T]} + \|f\|_{W^{1,1}([0,T])}\big) + \|f\|_{1,[0,T]}.$$

It follows that $\Phi = 0$, ending the proof of the first Eq. (6). The proof for $u_\lambda(0) = 0$ can be patterned after the proof of Proposition 1. □

Before proving the existence Theorem 2, we first state the following lemma:

Lemma 2. *Let $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$, $\theta \in \mathbb{R}$. Then, for any $\varphi \in D_{-\theta}$, $(\hat f(\lambda)|\widehat{\varphi}(\lambda))_{\mathscr{H}(\lambda)} \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$, $(\hat f(\lambda)|\widehat{\varphi}(\lambda))'_{\mathscr{H}(\lambda)} = (\widehat{f'}(\lambda) \mid \widehat{\varphi}(\lambda))_{\mathscr{H}(\lambda)}$, for μ almost every $\lambda \ge \lambda_0$.*

Proof. Let $T > 0$. Notice first that

$$\int\limits_0^T \left\|\frac{f(t+h)-f(t)}{h} - f'(t)\right\|^2_{D_\theta}\mathrm{d}t \xrightarrow[h\to 0]{} 0$$

insofar $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$. Therefore,

$$\int\limits_{\lambda_0}^{+\infty}\int\limits_0^T \left\|\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h} - \widehat{f'}(t,\lambda)\right\|^2_{\mathscr{H}(\lambda)} \lambda^{2\theta}\,\mathrm{d}t\,\mathrm{d}\mu(\lambda) \xrightarrow[h\to 0]{} 0. \tag{7}$$

Consequently, $\int_0^T \|\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h}\|^2_{\mathscr{H}(\lambda)}\,\mathrm{d}t \le K(\lambda) < +\infty$ for μ almost every $\lambda \ge \lambda_0$. It follows that, for any $\varphi \in D_{-\theta}$ and $\rho \in \mathscr{D}(\mathbb{R}_+^*)$ such that $\operatorname{supp}\rho \subset [0,T]$, one has

$$\begin{aligned}
&\left[\int_0^T \left(\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h}\Big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \rho(t)\,\mathrm{d}t\right]^2 \\
&\quad\le \left(\int_0^T \left|\left(\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h}\Big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)}\right|^2 \mathrm{d}t\right)\left(\int_0^T \rho^2(t)\,\mathrm{d}t\right) \\
&\quad\le \left(\int_0^T \left\|\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h}\right\|^2_{\mathscr{H}(\lambda)} \|\widehat{\varphi}(\lambda)\|^2_{\mathscr{H}(\lambda)}\,\mathrm{d}t\right)\left(\int_0^T \rho^2(t)\,\mathrm{d}t\right) \\
&\quad\le K(\lambda)\|\widehat{\varphi}(\lambda)\|^2_{\mathscr{H}(\lambda)}\|\rho\|_2^2.
\end{aligned} \tag{8}$$

However, for $h > 0$ small enough,

$$\begin{aligned}
&\int_0^T \left(\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h}\Big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \rho(t)\,\mathrm{d}t \\
&\quad= \int_0^T \left(\widehat{f}(t,\lambda)\big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \frac{\rho(t-h)-\rho(t)}{h}\,\mathrm{d}t \\
&\quad\xrightarrow[h\to 0]{} -\int_0^T \left(\widehat{f}(t,\lambda)\big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \rho'(t)\,\mathrm{d}t.
\end{aligned} \tag{9}$$

Invoking Eqs. (8) and (9) leads to

$$\left|\int_0^T \left(\widehat{f}(t,\lambda)\big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \rho'(t)\,\mathrm{d}t\right| \le M(\lambda)\|\rho\|_2,$$

which implies further that $(\widehat{f}(\lambda)|\widehat{\varphi}(\lambda))' \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ for μ almost every $\lambda \ge \lambda_0$.

From Eq. (7) one obtains

$$\int_{\lambda_0}^{+\infty} \left[\int_0^T \left(\frac{\widehat{f}(t+h,\lambda)-\widehat{f}(t,\lambda)}{h} - \widehat{f'}(t,\lambda)\Big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \rho(t)\,\mathrm{d}t\right] \mathrm{d}\mu(\lambda) \xrightarrow[h\to 0]{} 0.$$

Therefore, there exists $(h_k)_{k\in\mathbb{N}}$, $h_k \xrightarrow[k\to+\infty]{} 0$ such that, for μ almost every $\lambda \ge \lambda_0$,

$$\int_0^T \left(\frac{\widehat{f}(t+h_k,\lambda)-\widehat{f}(t,\lambda)}{h_k} - \widehat{f'}(t,\lambda)\Big|\widehat{\varphi}(\lambda)\right)_{\mathscr{H}(\lambda)} \rho(t)\,\mathrm{d}t \xrightarrow[k\to+\infty]{} 0.$$

However,

$$\int_0^T \Big(\frac{\widehat{f}(t+h_k,\lambda)-\widehat{f}(t,\lambda)}{h_k} \;\Big|\; \widehat{\varphi}(\lambda)\Big)_{\mathscr{H}(\lambda)} \rho(t)\,\mathrm{d}t \xrightarrow[k\to+\infty]{} \int_0^T \big(\widehat{f}(t,\lambda)\big|\widehat{\varphi}(\lambda)\big)'_{\mathscr{H}(\lambda)}\rho(t)\,\mathrm{d}t.$$

Therefore, for μ almost every $\lambda \geq \lambda_0$

$$\int_0^T \big[\big(\widehat{f}(t,\lambda)\big|\widehat{\varphi}(\lambda)\big)'_{\mathscr{H}(\lambda)} - \big(\widehat{f'}(t,\lambda)\big|\widehat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\big]\rho(t)\,\mathrm{d}t = 0$$

and

$$\big(\widehat{f}(t,\lambda)\big|\widehat{\varphi}(\lambda)\big)'_{\mathscr{H}(\lambda)} = \big(\widehat{f'}(t,\lambda)\big|\widehat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)},$$

$t > 0$, which ends the proof. □

In the following we again use the Caputo fractional derivative:

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} := \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^\alpha}\,\mathrm{d}s.$$

Using the eigenequations solved by $\hat{u}$ and the above results, one deduces that u solves Eqs. (1):

Theorem 2. *Let H be a Hilbert space, A a self-adjoint operator with domain $D(A) \subset H$ and satisfying properties* (i), (ii) *of Section* 2. *Let $\theta \in \mathbb{R}$, $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$ and $u_0 \in D_{\theta+1}$. Then the equations*

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} + (Au)(t) = f(t), \quad u(0) = u_0, \tag{10}$$

have a solution u such that $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$, with $\hat{u}(\lambda) = E(\lambda) * [\hat{f}(\lambda) - \widehat{Au_0}(\lambda)]$.

Moreover, for any $\epsilon \in]0,1[$ and $r \in [1,2] \cap [1, \frac{1}{1-\epsilon\alpha}[$, one has $u \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1-\epsilon})$.

Proof. The fact that u given by $\hat{u}(\lambda) = E(\lambda) * [\hat{f}(\lambda) - \widehat{Au_0}(\lambda)]$ satisfies $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, D_\theta) \cap W^{1,r}_{\mathrm{loc}}(\mathbb{R}_+, D_{\theta+1-\epsilon})$ follows from Propositions 3 and 4.

It is sufficient to prove the remaining part of the theorem only for $u_0 = 0$.

Let $\varphi \in D_{-\theta}$. Since $u' \in L^1_{\mathrm{loc}}(\mathbb{R}_+, D_\theta)$ (see Proposition 4), the following calculations are justified. One has:

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\left\langle \int_0^t \frac{u'(s)}{(t-s)^\alpha}\,\mathrm{d}s, \varphi \right\rangle + \langle Au(t), \varphi\rangle - \langle f(t), \varphi\rangle \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \langle u'(s), \varphi\rangle \frac{\mathrm{d}s}{(t-s)^\alpha} + \langle Au(t), \varphi\rangle - \langle f(t), \varphi\rangle \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{\lambda_0}^{+\infty} \left(\int_0^t \big(\widehat{u'}(\lambda, s) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)} \frac{\mathrm{d}s}{(t-s)^\alpha}\right) \mathrm{d}\mu(\lambda) \\
&\quad + \int_{\lambda_0}^{+\infty} \big(\lambda \widehat{u}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) - \int_{\lambda_0}^{+\infty} \big(\widehat{f}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \\
&\overset{\text{Prop. 4}}{=} \frac{1}{\Gamma(1-\alpha)} \int_{\lambda_0}^{+\infty} \left[\int_0^t \big(E(\lambda, s)\hat{f}(\lambda, 0) + \big(E(\lambda) * \widehat{f'}(\lambda)\big)(s) \right. \\
&\qquad\qquad \left. \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)} \frac{\mathrm{d}s}{(t-s)^\alpha} \right] \mathrm{d}\mu(\lambda) \\
&\quad + \int_{\lambda_0}^{+\infty} \big(\lambda \hat{u}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)} - \int_{\lambda_0}^{+\infty} \big(\hat{f}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \\
&\overset{\text{Lemma 2}}{=} \frac{1}{\Gamma(1-\alpha)} \int_{\lambda_0}^{+\infty} \left\{ \int_0^t \big[E(\lambda, s)\big(\hat{f}(\lambda, 0) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)} \right. \\
&\qquad\qquad \left. + E(\lambda) * \big(\hat{f}(\lambda) \big| \hat{\varphi}(\lambda)\big)'_{\mathscr{H}(\lambda)}\big] \frac{\mathrm{d}s}{(t-s)^\alpha} \right\} \mathrm{d}\mu(\lambda) \\
&\quad + \int_{\lambda_0}^{+\infty} \big(\lambda \hat{u}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)} - \int_{\lambda_0}^{+\infty} \big(\hat{f}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{\lambda_0}^{+\infty} \left\{ \int_0^t \big[E(\lambda) * \big(\widehat{f}(\lambda) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\big]'(s) \frac{\mathrm{d}s}{(t-s)^\alpha} \right\} \mathrm{d}\mu(\lambda) \\
&\quad + \int_{\lambda_0}^{+\infty} \lambda E(\lambda) * \big(\widehat{f}(\lambda) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}(t)\,\mathrm{d}\mu(\lambda) \\
&\quad - \int_{\lambda_0}^{+\infty} \big(\widehat{f}(\lambda, t) \big| \hat{\varphi}(\lambda)\big)_{\mathscr{H}(\lambda)}\,\mathrm{d}\mu(\lambda) \overset{\text{Eqs. (5), (6)}}{=} 0.
\end{aligned}
$$

Hence u satisfies Eq. (10). Equation $u(0) = 0$ is a consequence of Eqs. (6). This ends the proof. □

Consider for instance the case of a bounded domain Ω with smooth boundary. When A is a strongly elliptic second order operator as described in the Introduction section, one can choose $H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Notice that $D_0 = H = L^2(\Omega)$ and $D_1 = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, for $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega))$ and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, Theorem 2 ensures that the initial value problem (1) has a strong solution $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, L^2(\Omega))$ given by $\hat{u}(\lambda) = E(\lambda) * [\hat{f}(\lambda) - \widehat{Au_0}(\lambda)]$. The last relationship is equivalent to Eq. (40) in [35].

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References

1. R. Metzler, T.F. Nonnenmacher, Fractional diffusion: Exact representations of spectral functions, *J. Phys. A, Math. Gen.*, **30**, pp. 1089–1093, 1997.

2. R. Dautray, J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques, Vol. 5, Problèmes d'évolution*, INSTN CEA Collection Enseignement, Masson, Paris, 1988.

3. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

4. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models*, Imperial College Press, London, 2010.

5. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.

6. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, New York, 1999.

7. S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon & Breach, New York, 1993.

8. R.G. Larson, *Constitutive Equations for Polymer Melts and Solutions*, Butterworths, Boston, 1988.

9. F.A. Morrison, *Understanding Rheology*, Oxford Univ. Press, Oxford, 2001.

10. S. Cleja-Ţigoiu, V. Ţigoiu, *Rheology and Thermodynamics, Part I. Rheology*, Editura Universităţii din Bucureşti, Bucharest, 1998.

11. C. Truesdell, K.R. Rajagopal, *An Introduction to the Mechanics of Fluids*, Birkhäuser, Boston, 2000.

12. M. Athar, C. Fetecau, M. Kamran, A. Sohail, M. Imran, Exact solutions for unsteady axial Couette flow of a fractional Maxwell fluid due to an accelerated shear, *Nonlinear Anal. Model. Control*, **16**(2), pp. 135–151, 2011.

13. J.F. Douglas, Some applications of fractional calculus to polymer science, in: *Adv. Chem. Phys.*, Vol. 102, John Wiley & Sons, 1997, pp. 121–191.

14. J.F. Douglas, Polymer science applications of path-integration, integral equations, and fractional calculus, in: R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

15. C.S. Drapaca, S. Srivaloganathan, A fractional model of continuum mechanics, *J. Elasticity*, **107**, pp. 105–123, 2012.

16. A. Drozdov, *Viscoelastic Structures: Mechanics of Growth and Aging, Viscoelastic structures, Mechanics of growth and aging*, Academic Press, New York, 1998.

17. C. Fetecau, C. Fetecau, M. Jamil, A. Mahmood, Flow of fractional Maxwell fluid between coaxial cylinders, *Arch. Appl. Mech.*, **81**(8), pp. 1153–1163, 2011.

18. A. Heibig, L.I. Palade, On the rest state stability of an objective fractional derivative viscoelastic fluid model, *J. Math. Phys.*, **49**(4), 043101, 2008.

19. A. Heibig, L.I. Palade, Well posedness of a linearized fractional derivative fluid model, *J. Math. Anal. Appl.*, **380**(1), pp. 188–203, 2011.

20. L.I. Palade, V. Verney, P. Attané, A modified fractional model to describe the entire viscoelastic behavior of polybutadienes from flow to glassy regime, *Rheol. Acta*, **35**(3), pp. 265–273, 1996.

21. L.I. Palade, P. Attané, R.R. Huilgol, B. Mena, Anomalous stability behavior of a properly invariant constitutive equation which generalises fractional derivative models, *Int. J. Eng. Sci.*, **37**(3), pp. 315–329, 1999.

22. Y.A. Rossikhin, M.V. Shitikova, Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, *Appl. Mech. Rev.*, **50**(1), pp. 15–67, 1997.

23. K.L. Ngai, *Relaxation and Diffusion in Complex Systems*, Springer, New York, 2011.

24. E.G. Bajlekova, Fractional Evolution Equations in Banach Spaces, Ph.D. Thesis, Eindhoven University of Technology, 2001.

25. G. Da Prato, M. Iannelli, Linear integro-differential equations in Banach spaces, *Rend. Semin. Univ. Mat. Padova*, **62**, pp. 207–219, 1980.

26. A. Heibig, Existence of solutions for a fractional derivative system of equations, *Integral Equations Oper. Theory*, **72**(4), pp. 483–508, 2012, doi:10.1007/s00020-012-1950-3.

27. R. Figueiredo Camargo, R. Charnet, E. Capelas de Oliveira, On some fractional Green's functions, *J. Math. Phys.*, **50**(4), 043514, 2009.

28. N. Ozdemir, D. Karadeniz, Fractional diffusion-wave problem in cylindrical coordinates, *Phys. Lett., A*, **372**(38), pp. 5968–5972, 2008.

29. N. Ozdemir, O.P. Agrawal, B.B. Iskender, D. Karadeniz, Fractional optimal control of a 2-dimensional distributed system using eigenfunctions, *Nonlinear Dyn.*, **55**(3), pp. 251–260, 2009.

30. N. Ozdemir, D. Karadeniz, Fractional optimal control problem of a distributed system in cylindrical coordinates, *Phys. Lett., A*, **373**(2), pp. 221–226, 2009.

31. R.P. Agarwal, M. Benchora, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.*, **109**, pp. 973–1033, 2010.

32. M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, *Geophys. J. R. Astron. Soc.*, **13**, pp. 529–539, 1967.

33. M. Caputo, *Elasticita e dissipazione*, Zanichelli, Bologna, 1969.

34. A. Carpinteri, F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, New York, 1997.

35. Y. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation, *Comput. Math. Appl.*, **59**(5), pp. 1766–1772, 2010.