The first integral method and traveling wave solutions to Davey–Stewartson equation

Hossein Jafari[a1], **Atefe Sooraki**[a], **Yahya Talebi**[a], **Anjan Biswas**[b]

[a]Department of Mathematics and Computer Science
University of Mazandaran
P.O. Box 47416-95447, Babolsar, Iran
jafari@umz.ac.ir; atefe_sooraki@umz.ac.ir; talebi@umz.ac.ir

[b]Department of Mathematical Sciences
Delaware State University
Dover, DE 19901-2277, USA
biswas.anjan@gmail.com

Received: 24 October 2011 / **Revised:** 17 February 2012 / **Published online:** 21 May 2012

Abstract. In this paper, the first integral method will be applied to integrate the Davey–Stewartson's equation. Using this method, a few exact solutions will be obtained using ideas from the theory of commutative algebra. Finally, soliton solution will also be obtained using the traveling wave hypothesis.

Keywords: first integral method, Davey–Stewartson equations, exact solutions.

1 Introduction

The first integral method was first proposed for solving Burger-KdV equation [1] which is based on the ring theory of commutative algebra. This method was further developed by the same author [2, 3] and some other mathematicians [2, 4, 5]. The present paper investigates, for the first time, the applicability and effectiveness of the first integral method on the Davey–Stewartson equations. We consider the Davey–Stewartson (DS) equations [6, 7]:

$$\begin{aligned} &\mathrm{i}q_t + \frac{1}{2}\sigma^2\big(q_{xx} + \sigma^2 q_{yy}\big) + \lambda|q|^2 q - \phi_x q = 0,\\ &\phi_{xx} - \sigma^2\phi_{yy} - 2\lambda\big(|q|^2\big)_x = 0. \end{aligned} \tag{1}$$

The case $\sigma = 1$ is called the DS-I equation, while $\sigma = \mathrm{i}$ is the DS-II equation. The parameter λ characterizes the focusing or defocusing case. The Davey–Stewartson I and II are two well-known examples of integrable equations in two space dimensions, which arise

[1]Corresponding author.

as higher dimensional generalizations of the nonlinear Schrodinger (NLS) equation [8]. They appear in many applications, for example in the description of gravity-capillarity surface wave packets in the limit of the shallow water. Davey and Stewartson first derived their model in the context of water waves, from purely physical considerations. In the context, $q(x, y, t)$ is the amplitude of a surface wave packet, while $\phi(x, y)$ reperesents the velocity potential of the mean flow interacting with the surface wave [8].

The remaining portion of this article is organized as follows: Section 2 is a brief introduction to the first integral method. In Section 3, the first integral method will be implemented and some new exact solutions for Davey–Stewartson equation will be reported. Additionally, in this section, the traveling wave solution will also be obtained to retrieve soliton solution. A conclusion and future directions for research will be summarized in the last section.

2 The first integral method

Consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_y, u_{xx}, u_{tt}, u_{yy}, u_{xt}, u_{xy}, u_{yt}, u_{xxx}, \dots) = 0.$$

Initially, we consider case $\sigma = 1$ (DS-I). So using the wave variable $\eta = x - 2\alpha y + \alpha t$ leads into the following ordinary differential equation (ODE):

$$Q\big(U, U', U'', U''', \dots\big) = 0, \tag{2}$$

where prime denotes the derivative with respect to the same variable η.

For case $\sigma = \mathrm{i}$ (DS-II), we use the wave variable $\eta = x + 2\alpha y - \alpha t$ into Eq. (2).

Next, we introduce new independent variables $x = u$, $y = u_\eta$ which change to a dynamical system of the following type:

$$\begin{aligned} x' &= y, \\ y' &= f(x, y). \end{aligned} \tag{3}$$

According to the qualitative theory of differential equations [1,9], if one can find two first integrals to Eqs. (3) under the same conditions, then analytic solutions to Eqs. (3) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given autonomous system in two spatial dimensions, there does not exist any general theory that allows us to extract its first integrals in a systematic way. A key idea of our approach here, to find first integral, is to utilize the Division theorem. For convenience, first let us recall the Division theorem for two variables in the complex domain $\mathbf{C}$ [10].

Division theorem. (*See* [11].) *Suppose $P(x, y)$ and $Q(x, y)$ are polynomials of two variables x and y in $\mathbf{C}[x, y]$ and $P(x, y)$ is irreducible in $\mathbf{C}[x, y]$. If $Q(x, y)$ vanishes at all zero points of $P(x, y)$, then there exists a polynomial $G(x, y)$ in $\mathbf{C}[x, y]$ such that $Q(x, y) = P(x, y)G(x, y)$.*

3 Exact solutions of Davey–Stewartson equation

In order to seek exact solutions of Eqs. (1), we first consider case $\sigma = 1$. So Eqs. (1) reduces to

$$\begin{aligned} &\mathrm{i}q_t + \frac{1}{2}(q_{xx} + q_{yy}) + \lambda|q|^2 q - \phi_x q = 0, \\ &\phi_{xx} - \phi_{yy} - 2\lambda\big(|q|^2\big)_x = 0. \end{aligned} \tag{4}$$

Now, in order to seek exact solutions of Eqs. (4), we assume

$$q(x, y, t) = u(x, y, t)\exp\big[\mathrm{i}(\alpha x + \beta y + kt + l)\big], \tag{5}$$

where α, β and k are constants to be determined later, l is an arbitrary constant. We assume $\beta = 1$. Substitute Eq. (5) into Eqs. (4) to yield

$$\begin{aligned} &\mathrm{i}(u_t + \alpha u_x + u_y) + \frac{1}{2}(u_{xx} + u_{yy}) - \frac{1}{2}\big(\alpha^2 + 2k + 1\big)u + \lambda u^3 - \phi_x u = 0, \\ &\phi_{xx} - \phi_{yy} - 2\lambda\big(u^2\big)_x = 0. \end{aligned} \tag{6}$$

Using the transformation

$$u = u(\eta), \quad \phi = \phi(\eta), \quad \eta = x - 2\alpha y + \alpha t,$$

where α is a constant, Eqs. (6) further reduces to

$$\begin{aligned} &\frac{1}{2}\big(1 + 4\alpha^2\big)u'' - \frac{1}{2}\big(\alpha^2 + 2k + 1\big)u + \lambda u^3 - \phi' u = 0, \\ &\phi'' - 4\alpha^2\phi'' - 2\lambda\big(u^2\big)' = 0, \end{aligned} \tag{7}$$

where prime denotes the differential with respect to η. Integrating the second part of Eq. (7) with respect to η and taking the integration constant as zero yields

$$\phi' = \frac{2\lambda}{1 - 4\alpha^2}u^2. \tag{8}$$

Substituting Eq. (8) into the first part of (7) yields

$$\frac{1}{2}\big(1 + 4\alpha^2\big)u'' - \frac{1}{2}\big(\alpha^2 + 2k + 1\big)u + \left(\lambda - \frac{2\lambda}{1 - 4\alpha^2}\right)u^3 = 0. \tag{9}$$

3.1 Application of Division theorem

In order to apply the Division theorem, we introduce new independent variables $x = u$, $y = u_\eta$ which change Eq. (9) to the dynamical system given by

$$\begin{aligned} &x' = y, \\ &y' = \frac{\alpha^2 + 2k + 1}{1 + 4\alpha^2}x + \frac{2\lambda}{1 - 4\alpha^2}x^3. \end{aligned} \tag{10}$$

Now, we are going to apply the Division theorem to seek the first integral to (10). Suppose that $x = x(\eta)$ and $y = y(\eta)$ are the nontrivial solutions to (10), and

$$P(x, y) = \sum_{i=0}^{m} a_i(x) y^i$$

is an irreducible polynomial in $\mathbf{C}[x, y]$ such that

$$P\big(x(\eta), y(\eta)\big) = \sum_{i=0}^{m} a_i\big(x(\eta)\big) y(\eta)^i = 0, \tag{11}$$

where $a_i(x)$ $(i = 0, 1, \dots, m)$ are polynomials in x and all relatively prime in $\mathbf{C}[x, y]$, $a_m(x) \neq 0$. Equation (11) is also called the first integral to (10). We start our study by assuming $m = 1$ in (11). Note that $\frac{\mathrm{d}P}{\mathrm{d}\eta}$ is a polynomial in x and y, and $P[x(\eta), y(\eta)] = 0$ implies $\frac{\mathrm{d}P}{\mathrm{d}\eta} = 0$. By the Division theorem, there exists a polynomial $H(x, y) = h(x) + g(x)y$ in $\mathbf{C}[x, y]$ such

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\eta} &= \left[\frac{\partial P}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial \eta}\right]_{(12)} \\ &= \sum_{i=0}^{1} a_i'(x) y^{i+1} + \sum_{i=0}^{1} i a_i(x) y^{i-1}\left[\frac{\alpha^2 + 2k + 1}{1 + 4\alpha^2}x + \frac{2\lambda}{1 - 4\alpha^2}x^3\right] \\ &= \big(h(x) + g(x)y\big)\left(\sum_{i=0}^{1} a_i(x) y^i\right), \end{aligned} \tag{12}$$

where prime denotes differentiating with respect to the variable x. On equating the coefficients of y^i $(i = 2, 1, 0)$ on both sides of (12), we have

$$a_1'(x) = g(x) a_1(x), \tag{13}$$

$$a_0'(x) = h(x) a_1(x) + g(x) a_0(x), \tag{14}$$

$$a_1(x)\left[\frac{\alpha^2 + 2k + 1}{1 + 4\alpha^2}x + \frac{2\lambda}{1 - 4\alpha^2}x^3\right] = h(x) a_0(x). \tag{15}$$

Since, $a_1(x)$ is a polynomial in x, from (13) we conclude that $a_1(x)$ is a constant and $g(x) = 0$. For simplicity, we take $a_1(x) = 1$, and balancing the degrees of $h(x)$ and $a_0(x)$ we conclude that $\deg h(x) = 1$. Now suppose that $h(x) = Ax + B$, then from (14), we find

$$a_0(x) = \frac{1}{2}Ax^2 + Bx + D,$$

where D is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$ and $h(x)$ in (15) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A = 2\sqrt{\frac{\lambda}{1 - 4\alpha^2}}, \quad B = 0, \quad D = \frac{\alpha^2 + 2k + 1}{1 + 4\alpha^2}\,\frac{\sqrt{1 - 4\alpha^2}}{2\sqrt{\lambda}}, \tag{16}$$

and

$$A = -2\sqrt{\frac{\lambda}{1-4\alpha^2}}, \quad B = 0, \quad D = -\frac{\alpha^2+2k+1}{1+4\alpha^2}\,\frac{\sqrt{1-4\alpha^2}}{2\sqrt{\lambda}}. \tag{17}$$

Using (16) and (17) in (11), we obtain

$$y + \sqrt{\frac{\lambda}{1-4\alpha^2}}x^2 + \frac{\alpha^2+2k+1}{1+4\alpha^2}\,\frac{\sqrt{1-4\alpha^2}}{2\sqrt{\lambda}} = 0$$

and

$$y - \sqrt{\frac{\lambda}{1-4\alpha^2}}x^2 - \frac{\alpha^2+2k+1}{1+4\alpha^2}\,\frac{\sqrt{1-4\alpha^2}}{2\sqrt{\lambda}} = 0,$$

respectively, where $\lambda \neq 0$. Combining this equations with (10), where we obtain the exact solutions of Eq. (10) as follows:

$$u_1(\eta) = A_1 \tan\bigl(-B_1\eta - \sqrt{2}A_1c_1\bigr)$$

and

$$u_2(\eta) = A_1 \tan\bigl(B_1\eta - \sqrt{2}A_1c_1\bigr),$$

where

$$A_1 = \sqrt{\frac{(\alpha^2+2k+1)(1-4\alpha^2)}{2\lambda(1+4\alpha^2)}}, \quad B_1 = \frac{1}{2}\sqrt{\frac{2(\alpha^2+2k+1)}{1+4\alpha^2}},$$

where $\lambda \neq 0$ and c_1 is an arbitrary constant. Therefore, the exact solutions to (10) can be written as

$$u_1(x,y,t) = A_1 \tan\bigl[-B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\bigr]$$

and

$$u_2(x,y,t) = A_1 \tan\bigl[B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\bigr],$$

where $\lambda \neq 0$.

Then exact solutions for Eqs. (4) are

$$q_1 = \mathrm{e}^{\mathrm{i}(\alpha x+y+kt+l)}A_1 \tan\bigl[-B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\bigr],$$
$$\phi_1 = \frac{2\lambda}{3(1-4\alpha^2)}A_1^{3/2}\tan^3\bigl[-B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\bigr]$$

and

$$q_2 = \mathrm{e}^{\mathrm{i}(\alpha x+y+kt+l)}A_1 \tan\bigl[B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\bigr],$$
$$\phi_1 = \frac{2\lambda}{3(1-4\alpha^2)}A_1^{3/2}\tan^3\bigl[B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\bigr],$$

where $\lambda \neq 0$.

Notice that integrating of Eq. (8) with respect to η and taking the integration constant as zero yields

$$\phi = \frac{2\lambda}{3(1-4\alpha^2)}u^3.$$

Now we assume that $m = 2$ in (11). By the Division theorem, there exists a polynomial $H(x,y) = h(x) + g(x)y$ in $\mathbf{C}[x,y]$ such that

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\eta} &= \left[\frac{\partial P}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial \eta}\right]_{(12)} \\ &= \sum_{i=0}^{2} a_i'(x)y^{i+1} + \sum_{i=0}^{2} i a_i(x)y^{i-1}\left[\frac{\alpha^2+2k+1}{1+4\alpha^2}x + \frac{2\lambda}{1-4\alpha^2}x^3\right] \\ &= \bigl(h(x)+g(x)y\bigr)\left(\sum_{i=0}^{2} a_i(x)y^i\right), \end{aligned} \tag{18}$$

On equating the coefficients of y^i ($i = 3, 2, 1, 0$) from both sides of (18), we have

$$a_2'(x) = g(x)a_2(x), \tag{19}$$

$$a_1'(x) = h(x)a_2(x) + g(x)a_1(x), \tag{20}$$

$$a_0'(x) = -2a_2(x)\left(\frac{\alpha^2+2k+1}{1+4\alpha^2}x + \frac{2\lambda}{1-4\alpha^2}x^3\right) + h(x)a_1(x) + g(x)a_0(x), \tag{21}$$

$$a_1(x)\left[\frac{\alpha^2+2k+1}{1+4\alpha^2}x + \frac{2\lambda}{1-4\alpha^2}x^3\right] = h(x)a_0(x). \tag{22}$$

Since, $a_2(x)$ is a polynomial of x, from (19) we conclude that $a_2(x)$ is a constant and $g(x) = 0$. For simplicity, we take $a_2(x) = 1$, and balancing the degrees of $h(x)$, $a_0(x)$ and $a_1(x)$ we conclude that deg $h(x) = 1$ or 0, therefore we have two cases:

Case 1. Suppose that deg $h(x) = 1$ and $h(x) = Ax + B$, then from (20) we find

$$a_1(x) = \frac{1}{2}Ax^2 + Bx + D,$$

where D is an arbitrary integration constant. From (21) we find

$$a_0(x) = \left[\frac{A^2}{8} - \frac{\lambda}{1-4\alpha^2}\right]x^4 + \frac{AB}{2}x^3 + \left[\frac{1}{2}\bigl(B^2+AD\bigr) - \frac{\alpha^2+2k+1}{1+4\alpha^2}\right]x^2 + BDx + E,$$

where E is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$ and $h(x)$ in (22) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain:

$$\begin{aligned} &A = \frac{4\sqrt{\lambda}}{\sqrt{1-4\alpha^2}}, \quad B = 0, \quad D = \sqrt{\frac{1-4\alpha^2}{\lambda}}\,\frac{\alpha^2+2k+1}{1+4\alpha^2}, \\ &E = \frac{1-4\alpha^2}{4\lambda}\left(\frac{\alpha^2+2k+1}{1+4\alpha^2}\right)^2 \end{aligned} \tag{23}$$

and

$$A = -\frac{4\sqrt{\lambda}}{\sqrt{1-4\alpha^2}}, \quad B = 0, \quad D = -\sqrt{\frac{1-4\alpha^2}{\lambda}}\,\frac{\alpha^2+2k+1}{1+4\alpha^2},$$
$$E = \frac{1-4\alpha^2}{4\lambda}\left(\frac{\alpha^2+2k+1}{1+4\alpha^2}\right)^2. \tag{24}$$

Using (23) and (24) in (11), we obtain

$$y + \sqrt{\frac{\lambda}{1-4\alpha^2}}x^2 + \frac{\alpha^2+2k+1}{1+4\alpha^2}\,\frac{\sqrt{1-4\alpha^2}}{2\sqrt{\lambda}} = 0$$

and

$$y - \sqrt{\frac{\lambda}{1-4\alpha^2}}x^2 - \frac{\alpha^2+2k+1}{1+4\alpha^2}\,\frac{\sqrt{1-4\alpha^2}}{2\sqrt{\lambda}} = 0,$$

respectively, where $\lambda \neq 0$. Combining this equations with (10), we obtain two exact solutions to Eq. (10) which was obtained in case $m = 1$.

Case 2. In this case suppose that $\deg h(x) = 0$ and $h(x) = A$, then from (20) we find $a_1(x) = Ax + B$, where B is an arbitrary integration constant. From (21) we find

$$a_0(x) = -\frac{\lambda}{1-4\alpha^2}x^4 + \left[\frac{A^2}{2} - \frac{\alpha^2+2k+1}{1+4\alpha^2}\right]x^2 + ABx + D,$$

where D is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$ and $h(x)$ in (22) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain:

$$A = 0, \quad B = 0, \quad D = 0. \tag{25}$$

Using (25) in (11), we obtain

$$y^2 - \frac{\alpha^2+2k+1}{1+4\alpha^2}x^2 - \frac{\lambda}{1-4\alpha^2}x^4 = 0.$$

Combining this equations with (10), we obtain the exact solutions to Eq. (10) as follows:

$$u_3(\eta) = \frac{A_2 \mathrm{e}^{B_2(\eta + A_2/4c_1)}}{\mathrm{e}^{2B_2} + 4\lambda(1+4\alpha^2)\mathrm{e}^{A_2/2}}$$

and

$$u_4(\eta) = \frac{A_2 \mathrm{e}^{B_2(\eta + A_2/4c_1)}}{1 + 4\lambda(1+4\alpha^2)\mathrm{e}^{2B_2(\eta + A_2/2c_1)}},$$

where

$$A_2 = 4\sqrt{\left(4\alpha^2-1\right)\left(\alpha^2+2k+1\right)}, \quad B_2 = \sqrt{\frac{(4\alpha^2-1)(\alpha^2+2k+1)}{(4\alpha^2+1)(16\alpha^2-1)}}$$

and $\lambda \neq 0$ while c_1 is an arbitrary constant. Then the exact solutions to (10) can be written as:

$$u_3(x,y,t) = \frac{A_2 e^{B_2(x-2\alpha y-\alpha t)+A_2/4c_1}}{e^{A_2/2(x-2\alpha y+\alpha t)} + 4\lambda(1+\alpha^2)e^{A_2/2c_1}}$$

and

$$u_4(x,y,t) = \frac{A_2 e^{B_2(x-2\alpha y+\alpha t)+A_2/4c_1}}{1 + e^{2B_2(x-2\alpha y+\alpha t)+A_2/4c_1}}.$$

Then solutions of Eqs. (4) are

$$\begin{aligned}
q_3 &= u_3(x,y,t)e^{i(\alpha x+y+kt+l)},\\
\phi_3 &= \frac{2\lambda}{(1-4\alpha^2)} \frac{A_2^3 e^{3B_2(x-2\alpha y+\alpha t)+3/4A_2c_1}}{e^{6B_2(x-2\alpha y+\alpha t)} + 64\lambda^3(1+4\alpha^2)^3 e^{3/2A_2c_1}},\\
q_4 &= u_4(x,y,t)e^{i(\alpha x+y+kt+l)},\\
\phi_4 &= \frac{2\lambda}{(1-4\alpha^2)} \frac{A_2^3 e^{3B_2(x-2\alpha y+\alpha t)+3/4A_2c_1}}{1 + A_2^3 e^{6B_2(x-2\alpha y+\alpha t)+3/2A_2c_1}},
\end{aligned}$$

where $\lambda \neq 0$.

Now, we consider the case when $\sigma = i$, where Eqs. (1) transforms the following:

$$\begin{aligned}
&iq_t - \frac{1}{2}(q_{xx} - q_{yy}) + \lambda|q|^2 q - \phi_x q = 0,\\
&\phi_{xx} + \phi_{yy} - 2\lambda\bigl(|q|^2\bigr)_x = 0.
\end{aligned} \tag{26}$$

We will solve equation (26) similarly as in the case where $\sigma = 1$, with the only difference being $\eta = x + 2\alpha y - \alpha t$, $\phi' = \frac{2\lambda}{1+4\alpha^2}u^2$ and

$$\begin{aligned}
x' &= y,\\
y' &= \frac{\alpha^2 - 2k - 1}{1 - 4\alpha^2}x + \frac{2\lambda}{-1-4\alpha^2}x^3.
\end{aligned}$$

By applying the Division theorem as in the case where $\sigma = 1$, and by assuming $m = 1$, the exact solutions of Eqs. (10) as follows:

$$u_1(\eta) = A_3 \tan\bigl(-B_3\eta - \sqrt{2}A_3c_1\bigr)$$

and

$$u_2(\eta) = A_3 \tan\bigl(B_3\eta - \sqrt{2}A_3c_1\bigr),$$

where

$$A_3 = \sqrt{\frac{(\alpha^2 - 2\alpha - 1)(-1-4\alpha^2)}{2\lambda(1-4\alpha^2)}}, \quad B_3 = \frac{1}{2}\sqrt{\frac{2(\alpha^2 - 2k + 1)}{1 - 4\alpha^2}}$$

and $\lambda \neq 0$.

Then the exact solutions of (26) are

$$q_1 = \mathrm{e}^{\mathrm{i}(\alpha x+y+kt+l)} A_1 \tan\left[-B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\right],$$
$$\phi_1 = \frac{2\lambda}{3(1-4\alpha^2)} A_1^{3/2} \tan^3\left[-B_1(x-2\alpha y+\alpha t) - \sqrt{2}A_1c_1\right]$$

and

$$q_1 = \mathrm{e}^{\mathrm{i}(\alpha x+y+kt+l)} A_3 \tan\left[B_3(x-2\alpha y+\alpha t) - \sqrt{2}A_3c_1\right],$$
$$\phi_1 = \frac{2\lambda}{3(1-4\alpha^2)} A_3^{3/2} \tan^3\left[B_3(x-2\alpha y+\alpha t) - \sqrt{2}A_3c_1\right],$$

where $\lambda \neq 0$ and for case $m = 2$ in Case 1, we obtain two exact solutions to Eqs. (10) which was obtained in case $m = 1$.

As for case $m = 2$ in Case 2, we obtain exact solutions for Eqs. (10) as follows:

$$u_3(\eta) = \frac{A_4 \mathrm{e}^{B_4(\eta+A_4/4c_1)}}{\mathrm{e}^{2B_4} + 4\lambda(1+4\alpha^2)\mathrm{e}^{A_4/2}}$$

and

$$u_4(\eta) = \frac{A_4 \mathrm{e}^{B_4(\eta+A_4/4c_1)}}{1 + 4\lambda(1+4\alpha^2)\mathrm{e}^{2B_4(\eta+A_4/2c_1)}},$$

where

$$A_4 = 4\sqrt{\left(1-4\alpha^2\right)\left(\alpha^2-2k-1\right)}, \quad B_4 = \sqrt{\frac{(1-4\alpha^2)(\alpha^2-2k-1)}{(-4\alpha^2-1)(1-16\alpha^2)}},$$

where $\lambda \neq 0$ and c_1 is an arbitrary constant. Hence the exact solutions to (26) are

$$q_3 = \frac{A_4 \mathrm{e}^{B_2(x-2\alpha y-\alpha t)+A_4/4c_1}}{\mathrm{e}^{A_4/2(x-2\alpha y+\alpha t)} + 4\lambda(1+\alpha^2)\mathrm{e}^{A_4/2c_1}} \mathrm{e}^{\mathrm{i}(\alpha x+y+kt+l)},$$
$$\phi_3 = \frac{2\lambda}{(1-4\alpha^2)} \frac{A_4^3 \mathrm{e}^{3B_4(x-2\alpha y+\alpha t)+3/4A_4c_1}}{\mathrm{e}^{6B_2(x-2\alpha y+\alpha t)} + 64\lambda^3(1+4\alpha^2)^3\mathrm{e}^{3/2A_4c_1}},$$
$$q_4 = \frac{A_4 \mathrm{e}^{B_4(x-2\alpha y+\alpha t)+A_4/4c_1}}{1 + \mathrm{e}^{2B_4(x-2\alpha y+\alpha t)+A_4/4c_1}} \mathrm{e}^{\mathrm{i}(\alpha x+y+kt+l)},$$
$$\phi_4 = \frac{2\lambda}{(1-4\alpha^2)} \frac{A_4^3 \mathrm{e}^{3B_4(x-2\alpha y+\alpha t)+3/4A_4c_1}}{1 + A_4^3\mathrm{e}^{6B_4(x-2\alpha y+\alpha t)+3/2A_4c_1}}.$$

3.2 Traveling wave solutions

The traveling wave hypothesis will be used to obtain the 1-soliton solution to the Davey–Stewartson equation (1). It needs to be noted that this equation was already studied by traveling wave hypothesis in 2011 [12] where the power law nonlinearity was considered. Additionally, the ansatz method was used to extract the exact 1-soliton solution to the Davey–Stewartson equation in 2011 and that too was studied with power law nonlinearity [13].

In this subsection, the starting point is going to be Eq. (9) which can now be re-written as

$$u'' = \frac{\alpha^2 + 2k + 1}{4\alpha^2 + 1} u + \frac{2\lambda}{1 - 4\alpha^2} u^3. \tag{27}$$

Now, multiplying both sides of (27) by u' and integrating while choosing the integration constant to be zero, since the search is for soliton solutions, yields

$$\left(u'\right)^2 = bu^2 - au^4, \tag{28}$$

where

$$a = \frac{\lambda}{4\alpha^2 - 1} \quad \text{and} \quad b = \frac{\alpha^2 + 2k + 1}{4\alpha^2 + 1}.$$

Separating variables in (28) and integrating gives

$$x - 2\alpha y + \alpha t = -\frac{1}{\sqrt{b}} \operatorname{sech}^{-1} \left| u \sqrt{\frac{a}{b}} \right|$$

that yields the 1-soliton solution as

$$u(x - 2\alpha y + \alpha t) = A \operatorname{sech}\bigl[B(x - 2\alpha y + \alpha t)\bigr],$$

where the amplitude A of the soliton is given by

$$A = \sqrt{\frac{b}{a}} = \left[\frac{(4\alpha^2 - 1)(\alpha^2 + 2k + 1)}{\lambda(4\alpha^2 + 1)}\right]^{1/2} \quad \text{and} \quad B = \sqrt{b} = \sqrt{\frac{\alpha^2 + 2k + 1}{4\alpha^2 + 1}}$$

and this leads to the constraints

$$\alpha^2 + 2k + 1 > 0$$

and

$$\lambda\bigl(4\alpha^2 - 1\bigr)\bigl(\alpha^2 + 2k + 1\bigr) > 0.$$

Hence by virtue of (5),

$$q(x, y, t) = A \operatorname{sech}\bigl[B(x - 2\alpha y + \alpha t)\bigr] e^{i(\alpha x + \beta y + kt + l)}. \tag{29}$$

Finally from (8), the topological 1-soliton solution is given by

$$\phi(x, y, t) = -2B \tanh\bigl[B(x - 2\alpha y + \alpha t)\bigr]. \tag{30}$$

Thus, (29) and (30) together constitute the 1-soliton solution of the Davey–Stewartson equation given by (1).

4 Conclusions

We described the first integral method for finding some new exact solutions for the Davey–Stewartson equation. We have obtained four exact solutions to the Davey–Stewartson equation. The solutions obtained are expressed in terms of trigonometric and exponential functions. In addition, the traveling wave hypothesis is used to obtain the 1-soliton solution of the equation where a topological and non-topological soliton pair is retrieved. These new solutions may be important for the explanation of some practical problems.

One context where this equation is studied from a practical standpoint is in the study of water waves with finite depth which moves in one direction [14]. However, in a two-dimensional scenario, this equation models both short waves and long waves. Many explicit solutions are given in this reference which are all applicable to the study of finite depth water waves. This paper also gives a substantial set of solution set that is also meaningful. Addditionally, this equation can also be generalized to a fractional derivative case that has been recently touched (15).

References

1. Z. Feng, On explicit exact solutions to the compound Burgers-KdV equation, *Phys. Lett., A*, **293**(1–2), pp. 57–66, 2002.

2. X. Deng, Traveling wave solutions for the generalized Burgers–Huxley equation, *Appl. Math. Comput.*, **204**(2), pp. 733–737, 2008.

3. B. Lu, H. Zhang and F. Xie, Traveling wave solutions of nonlinear partial equations by using the first integral method, *Appl. Math. Comput.*, **216**(4), pp. 1329–1336, 2010.

4. A. Maccari, The Kadomtsev–Petviashvili equation as a source of integrable model equations, *J. Math. Phys.*, **37**, pp. 5897–6590, 1996.

5. L.F. Tascan, A. Bekir, M. Koparan, Traveling wave solutions of nonlinear evolution equations by using the first integral method, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(5), pp. 1810–1815, 2009.

6. M. McConnell, A.S. Fokas, B. Pelloni, Localised coherent solutions of the DSI and DSII equations – a numerical study, *Math. Comput. Simul.*, **64**, pp. 424–438, 2005.

7. H.A. Zedan, S.Sh. Tantawy, Solutions of Davey–Stewartson equations by homotopy perturbation method, *Comput. Math. Phys.*, **49**, pp. 1382–1388, 2009.

8. A. Davey, K. Stewartson, On three-dimensional packets of surfaces waves, *Proc. R. Soc. Lond., Ser. A*, **338**, pp. 101–110, 1974.

9. T.R. Ding, C.Z. Li, *Ordinary Differential Equations*, Peking Univ. Press, Peking, China, 1996.

10. Z. Feng, Traveling wave behavior for a generalized fisher equation, *Chaos Solitons Fractals*, **38**(2), pp. 481–488, 2008.

11. T.S. Zhang, Exp-function for solving Maccari's system, *Phys. Lett., A*, **371**(1–2), pp. 65–71, 2007.

12. G. Ebadi, E.V. Krishnan, M. Labidi, E. Zerrad, A. Biswas, Analytical and numerical solutions for Davey–Stewartson equation with power law nonlinearity, *Waves Random Complex Media*, **21**(4), pp. 559–590, 2011.

13. G. Ebadi, A. Biswas, The G'/G method and 1-soliton solution of the Davey–Stewartson equation, *Math. Comput. Modelling*, **53**(5–6), pp. 694–698, 2011.

14. R.-J. Wang, Y.-C. Huang, Exact solutions and excitations for the Davey–Stewartson equations with nonlinear and gain terms, *Eur. Phys. J. D*, **57**, pp. 395–401, 2010.

15. H. Jafari, A. Kadem, D. Baleanu, T. Yilmaz, Solutions of the fractional Davey–Stewartson equations with variational iteration method, *Rom. Rep. Phys.*, **64**(2), 2012 (in press).

16. D. Rostamy, F. Zabihi, K. Karimi, S. Khalehoghli, The first integral method for solving Maccari's system, *Appl. Math.*, **2**(2), pp. 258–263, 2011.

17. H. Jafari, M. Alipour, Solutions of Davey–Stewartson equation using homotopy analysis method, *Nonlinear Anal. Model. Control*, **15**(4), pp. 423–433, 2010.