

## Robust exponential stability of nonlinear impulsive switched systems with time-varying delays\*

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**Abstract.** This paper deals with a class of uncertain nonlinear impulsive switched systems with time-varying delays. A novel type of piecewise Lyapunov functionals is constructed to derive the exponential stability. This type of functionals can efficiently overcome the impulsive and switching jump of adjacent Lyapunov functionals at impulsive switching times. Based on this, a delay-independent sufficient condition of exponential stability is presented by minimum dwell time. Finally, an illustrative numerical example is given to show the effectiveness of the obtained theoretical results.

**Keywords:** impulsive systems, switched systems, minimum dwell time, exponential stability, delays.

### 1 Introduction

Switched systems are an important class of hybrid dynamical systems, which are composed of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switch among them. These systems arise when modeling dynamical systems which exhibit switching among several subsystems due to jumping parameters or changing environmental factors. This class of systems has numerous applications in the control of mechanical systems, the automotive industry, aircraft and air traffic control, switching power converters and many other fields. The main concern in the study of switched systems is the issue of stability (see [1–3]) and references therein). Another category of hybrid systems is the system with impulse effects, namely, impulsive systems, which arose in scientific practice in 1950s in order to describe certain evolutionary processes and dynamical control systems that are subjected to sudden and sharp changes of states [4]. Due to the existence of the states jump, this new class of hybrid systems cannot be well described by using pure continuous or pure discrete models [5, 6].

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In some other cases, impulsive effects may arise as a result of switching that leads us to impulsive switched systems. Examples of these systems include biological systems, mechanical systems, automotive industry, aircraft, air traffic control and chaotic based secure communication. Recently, the stability analysis and stabilization design of impulsive switched systems have been received much attention, and many theoretical results and control applications have been reported (see, e.g. [7–15]).

As we know, the stability analysis/stabilization of the dynamic systems, in the control field, is a hot topic and has been attract more and more attention (see e.g., [16–22], and the references therein). When the stability and the stabilization of impulsive switched systems are considered, the piecewise Lyapunov functions or functionals are naturally employed. A key point is, in this case, to deal with the decay properties of the impulsive switching times and each continuous interval. Previous works on such problems have followed two distinct lines of enquiry. First, by piecewise Lyapunov–Razumikhin functions or functionals, a number of authors attempted to derive stability of impulsive switched systems. Results obtained in this direction include the works of [23, 24] and others. An alternative approach is piecewise Lyapunov–Krasovskii functions or functionals. Such an approach is usually more difficult than the Lyapunov–Razumikhin technique. The reason is that, in general, we cannot expect an impulse that occurs at a discrete time to bring the value of a functional down instantaneously, whereas, in the Lyapunov–Razumikhin method, the value of a function can subside simultaneously as the impulse occurs. Therefore, the stability results by Lyapunov–Krasovskii functions or functionals are not abundant (see, for example [8–14]).

Noticing that the related works of stability and stabilization for impulsive switched systems mainly established on the dwell time (minimum or maximum dwell time). Furthermore, for the delayed systems, the dwell time conditions are always related to the upper or lower bounds of delays. Namely, the dwell time is delay dependent (see, e.g. [8, 9, 13, 23]). However, two problems arise immediately. The first refers to the maximum dwell time. More precisely, if we restrict the upper bounds of maximum dwell time, then the stability of the impulsive switched system, in some cases, may be destroyed. Such as the case when the dwell time on a stable subsystem is sufficiently large. In contrast, if the lower bounds of maximum dwell time is fixed, that is, the running time on some subsystems must be sufficiently large. The design for switching law, in this case, will be greatly restricted. Therefore, the maximum dwell time approach is rather conservative. Secondly, when the bounds of delays are unknown or tend to infinity, the dwell time derived in previous works will not be applicable.

To solve the two problems mentioned above, this paper shall consider on a class of uncertain impulsive switched delayed system with nonlinear states and nonlinear impulsive increments. As we shall see, different from the traditional piecewise Lyapunov–Krasovskii functionals, a new class of Lyapunov functionals is constructed to overcome the impulsive increments and switching jump. Some delay independent minimum dwell time criteria in terms of linear matrix inequalities (LMIs) are proposed. The organization is as follows. The preliminaries are stated in Section 2. Section 3 focuses on robust exponential stability. A numerical example based on LMI is presented in Section 4. Finally, Section 5 concludes this paper.

## 2 Preliminaries

Throughout,  $\mathbb{R}$  denotes the space of all real numbers.  $\mathbb{R}^n$  stands for the  $n$ -dimensional real vector space and  $\mathbb{R}^{n \times m}$  is the space of  $n \times m$  matrices with real entries.  $\mathbb{C}$  denotes the space of all real-valued continuous functions. For matrix  $A$  in  $\mathbb{R}^{n \times n}$ ,  $A > 0$  ( $< 0$ ) means that  $A$  is a symmetric positive (negative) definite matrix and  $A \geq 0$  ( $\leq 0$ ) means that  $A$  is a symmetric positive (negative) semi-definite matrix. We use  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  to denote the smallest and largest eigenvalue of  $A$ , respectively.  $\mathbb{N}$  presents the set of all nonnegative integers and  $\|\cdot\|$  denotes the Euclidean norm of vectors.

Consider the uncertain impulsive switched delayed system with nonlinear perturbations and impulsive increments given by

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}(t)x(t) + \bar{A}_{\sigma(t)}(t)x(t-h(t)) \\ &\quad + f_{\sigma(t)}(t, x(t), x(t-h(t))), \quad t \neq t_k, \\ \Delta x(t) &= D_{\sigma(t_k)}x(t) + g_{\sigma(t_k)}(t, x(t)), \quad t = t_k, \\ x(t) &= \varphi(t), \quad t \in [-h, 0], \end{aligned} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\sigma(t) : [0, \infty) \rightarrow \mathbb{M}$  is the switching signal,  $\sigma(t) = i_k \in \mathbb{M}$  for  $t \in [t_k, t_{k+1})$ ,  $\mathbb{M} = \{1, 2, \dots, m\}$ ,  $m, k \in \mathbb{N}$ . Under the control of a switching signal  $\sigma$ , coupling with the impulsive effects, system (1) enters from the  $i_{k-1}$  subsystem to the  $i_k$  subsystem at the point  $t = t_k$ ,  $t_k$  is impulsive switching time point and satisfies  $t_0 < t_1 < \dots < t_k < \dots$  with  $t_0 = 0$  and  $\lim_{k \rightarrow +\infty} t_k = +\infty$ .  $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ ,  $x(t_k^+) = \lim_{\Delta t \rightarrow 0^+} x(t_k + \Delta t)$  and  $x(t_k^-) = \lim_{\Delta t \rightarrow 0^+} x(t_k - \Delta t)$  mean that the solution of the system (1) is left continuous. The time-varying delay  $h(t)$  satisfies  $0 \leq h(t) \leq h$  and  $0 \leq \dot{h}(t) \leq d < 1$ .

For each  $k$ ,  $D_{i_k} \in \mathbb{R}^{n \times n}$  is known matrix.  $A_{i_k}(t)$  and  $\bar{A}_{i_k}(t)$  are assumed to be uncertain and satisfy

$$[A_{i_k}(t) \bar{A}_{i_k}(t)] = [A_{i_k} \bar{A}_{i_k}] + E_{i_k} F_{i_k}(t) [H_{i_k} \bar{H}_{i_k}] \quad (2)$$

with  $A_{i_k}$ ,  $\bar{A}_{i_k}$ ,  $E_{i_k}$ ,  $H_{i_k}$ ,  $\bar{H}_{i_k}$  are known constant matrices and  $F_{i_k}(t)$  is unknown time-varying matrix satisfying  $\|F_{i_k}(t)\| \leq 1$ . Nonlinear perturbation  $f_{i_k}(t) \triangleq f_{i_k}(t, x(t), x(t-h(t))) : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is globally Lipschitz continuous and satisfies

$$f_{i_k}^T(t) f_{i_k}(t) \leq \alpha_{1i_k} x^T(t) \Gamma_{i_k}^T \Gamma_{i_k} x(t) + \alpha_{2i_k} x^T(t - \tau_{i_k}(t)) A_{i_k}^T A_{i_k} x(t - h_{i_k}(t)) \quad (3)$$

with given matrices  $\Gamma_{i_k}$ ,  $A_{i_k}$  and nonnegative scalars  $\alpha_{1i_k}$ ,  $\alpha_{2i_k}$ . Similarly, nonlinear impulsive increment  $g_{i_k}(t) \triangleq g_{i_k}(t, x(t)) : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies

$$g_{i_k}^T(t) g_{i_k}(t) \leq x^T(t) (I + D_{i_k})^T (I + D_{i_k}) x(t). \quad (4)$$

with the identity matrix  $I \in \mathbb{R}^{n \times n}$ . Also,  $f_{i_k}(t, 0, 0) \equiv g_{i_k}(t, 0) \equiv 0$  for all  $t \in [t_0, +\infty)$ . Besides,  $\varphi \in \mathbb{C}([-h, 0], \mathbb{R}^n)$  is the initial function with  $\|\varphi\|_h = \sup_{-h \leq t \leq 0} \|\varphi(t)\|$ .

This paper shall focus on robust exponential stability for system (1) and the definition is as follows.

**Definition 1.** For given a switching signal  $\sigma(t)$ , the system (1) is robustly exponentially stable if there exist positive scalars  $\beta$  and  $\delta$  such that

$$\|x(t)\| \leq \beta \|\varphi\|_h e^{-\delta(t-t_0)}, \quad t \geq t_0, \quad (5)$$

where  $\beta$  and  $\delta$  depend on the choice of the switching signal  $\sigma(t)$ . If  $F_{i_k}^i(t) = 0$ , the system (1) with (5) is called exponentially stable.

For robust exponential stability, we introduce the following assumption.

**Assumption 1.** In system (1), there exists a positive real number  $\tau_D$  such that, for any given switching signal  $\sigma(t)$ ,

$$\inf_{k \in \mathbb{N}} \{t_{k+1} - t_k\} \geq \tau_D.$$

In fact,  $\tau_D$  satisfying the above condition is called *minimum dwell time*. This assumption in the literature can rule out Zeno behavior for all types of switching. Hereafter, Assumption 1 always holds for system (1).

Next, two lemmas that are useful in deriving the principal contribution of this paper are presented.

**Lemma 1.** (See [25].) Let  $E, F$  and  $H$  be real matrices of appropriate dimensions with  $\|F\| \leq 1$ . Then for any scalar  $\varepsilon > 0$ ,

$$EFH + H^T F^T E^T \leq \varepsilon^{-1} EE^T + \varepsilon H^T H.$$

The following result being checked easily establishes a connection between a symmetric matrix and a symmetric positive definite matrix.

**Lemma 2.** Let  $P, U \in \mathbb{R}^{n \times n}$  be symmetric positive definite and symmetric matrices, respectively. Then there is a positive real number  $\gamma \geq 1$  such that for  $x(t) \in \mathbb{R}^n$ ,

$$x^T(t)Ux(t) \leq \gamma x^T(t)Px(t).$$

### 3 Main results

In this section, to derive robustly exponential stability of the impulsive switched system (1), for each mode, a new type of piecewise Lyapunov functional is chosen as the form

$$W(t) = e^{\delta_0 t} \psi(t) x^T(t) P_{i_k} x(t) + e^{\delta_0 t} \int_{t-h(t)}^t x^T(s) R x(s) ds, \quad (6)$$

where  $\delta_0 > 0$  is a given sufficiently small constant,  $P_{i_k}, R \in \mathbb{R}^{n \times n}$  are positive definite matrices, and the function  $\psi(t)$  is defined as following steps.

*Step 1.* Consider matrices  $P_{i_{k-1}}$  and  $(I + D_{i_k})^T P_{i_k} (I + D_{i_k})$ . By Lemma 2 there exist a positive real number  $\gamma_{i_k}^*$  such that

$$x^T(t)(I + D_{i_k})^T P_{i_k} (I + D_{i_k})x(t) \leq \gamma_{i_k}^* x^T(t)P_{i_{k-1}}x(t), \quad (7)$$

where  $\gamma_{i_k}^*$  is no less than one.

*Step 2.* With this  $\gamma_{i_k}^*$ , we define the following function:

$$\psi_k(t) = \frac{c}{(t_{k+1} - t_k)^2} \left(1 - \frac{1}{\gamma_{i_k}}\right) (t - t_k)^2 + \frac{c}{\gamma_{i_k}}, \quad t \in [t_k, t_{k+1}], \quad (8)$$

where  $c > 0$  and

$$\gamma_{i_k} = 2\gamma_{i_k}^* \left(1 + \frac{\lambda_{\max}(P_{i_k})}{\lambda_{\min}(P_{i_k})}\right).$$

*Step 3.* Based on the preparation above, the piecewise continuously function  $\psi(t): [0, +\infty) \rightarrow [0, +\infty)$  is given as the form

$$\psi(t) = \begin{cases} \psi_k(t), & t \in (t_k, t_{k+1}), \\ \psi(t_k^+) = \psi_k(t_k), & t = t_k. \end{cases} \quad (9)$$

**Comment 1.** Notice that, different from condition (7), the previous works (see, e.g., [7, 8]) always require  $x^T(t)(I + D_{i_k})^T P_{i_k} (I + D_{i_k})x(t) \leq x^T(t)P_{i_{k-1}}x(t)$ . Clearly, if  $\gamma_{i_k}^* = 1$ , (7) reduces the above condition. Indeed, under condition (7), the inequality  $x^T(t)(I + D_{i_k})^T P_{i_k} (I + D_{i_k})x(t) \geq x^T(t)P_{i_{k-1}}x(t)$  maybe hold.

Next, we shall see that the piecewise Lyapunov function (6) can effectively eliminate impulsive increments and switching jump phenomena of Lyapunov function at impulsive switching times  $t = t_k$  and further derive our principal result. For brevity, set

$$\mathcal{A}_i = [A_i \bar{A}_i], \quad \mathcal{H}_i = [H_i \bar{H}_i], \quad \mathcal{I}_1 = [I \ 0], \quad \mathcal{I}_2 = [0 \ I], \quad i \in \mathbb{M}.$$

**Theorem 1.** For any  $i \in \mathbb{M}$ , given positive real numbers  $\tau_D$ ,  $\varepsilon_i$ ,  $c$ , and  $\gamma_i \geq 4$ , if there exist positive definite matrices  $P_i, R \in \mathbb{R}^{n \times n}$ , positive semi-definite  $Q_i \in \mathbb{R}^{n \times n}$ , positive semi-definite  $L_i, W_i \in \mathbb{R}^{2n \times 2n}$ , and any matrices  $T_i \in \mathbb{R}^{n \times 2n}$ , such that (9) and the following linear matrix inequalities are satisfied:

$$\Omega_i = \begin{bmatrix} \Omega_{11i} & \Omega_{12i} \\ * & \Omega_{22i} \end{bmatrix} \leq 0, \quad (10)$$

$$\Theta_i = \begin{bmatrix} \Theta_{1i} & \mathcal{I}_1^T P_i E_i & \mathcal{H}_i^T & \mathcal{I}_1^T P_i & \mathcal{I}_1^T \Gamma_i^T \\ * & -\varepsilon_i I & 0 & 0 & 0 \\ * & * & -\varepsilon_i^{-1} I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\alpha_{1i}^{-1} I \end{bmatrix} < 0, \quad (11)$$

where

$$\begin{aligned} \Omega_{11i} &= -Q_i, \quad \Omega_{12i} = -T_i, \quad \Omega_{22i} = cL_i - \frac{c}{\gamma_i}W_i + \mathcal{I}_1^T \Phi_{1i} \mathcal{I}_1 + \mathcal{I}_2^T \Phi_{2i} \mathcal{I}_2, \\ \Phi_{1i} &= \frac{2c}{\tau_D} \left(1 - \frac{1}{\gamma_i}\right) P_i + R - \frac{c^2}{\gamma_i} Q_i, \quad \Phi_{2i} = \alpha_{2i} A_i^T A_i - (1-d)R, \\ \Theta_{1i} &= \mathcal{I}_1^T P_i \mathcal{A}_i + \mathcal{A}_i^T P_i \mathcal{I}_1 - L_i + W_i + \mathcal{I}_1^T \frac{c(1+\gamma_i)}{\gamma_i} Q_i \mathcal{I}_1 + \mathcal{I}_1^T T_i + T_i^T \mathcal{I}_1. \end{aligned}$$

Then system (1) is robustly exponentially stable.

*Proof.* Consider the function  $\psi(t)$  defined as (8), for each interval  $t \in [t_k, t_{k+1})$ , the following facts are easily verified.

- (i)  $\psi(t_k) = c/\gamma_{i_k}$  and  $\psi(t_{k+1}^-) = c$ .
- (ii)  $\psi(t)$  and  $\dot{\psi}(t)$  is monotone and bounded, respectively, i.e.,  $\psi(t_k) \leq \psi(t) \leq \psi(t_{k+1}^-)$  and  $0 \leq \dot{\psi}(t) \leq (2c/\tau_D)(1 - 1/\gamma_{i_k})$ .

The rest of proof is broken into three stages.

*Stage 1.* Firstly, at the impulsive switching time point  $t = t_k$ , we claim that the Lyapunov functional (6) is non-increasing, i.e.,

$$W(t_k^+) \leq W(t_k^-). \tag{12}$$

To this end, note that the Lyapunov functional (6), by shortly calculating one can obtain

$$\begin{aligned} W(t_k^+) &= e^{\delta_0 t_k^+} \psi(t_k^+) x^T(t_k^+) P_{i_k} x(t_k^+) + e^{\delta_0 t_k^+} \int_{t_k^+ - h(t_k^+)}^{t_k^+} x^T(s) R x(s) ds \\ &= e^{\delta_0 t_k^+} \psi(t_k^+) (\Delta x(t_k) + x(t_k^-))^T P_{i_k} (\Delta x(t_k) + x(t_k^-)) \\ &\quad + e^{\delta_0 t_k^+} \int_{t_k - h(t_k)}^{t_k} x^T(s) R x(s) ds \\ &= e^{\delta_0 t_k} \psi(t_k) [(I + D_{i_k})x(t_k) + g_{i_k}(t_k)]^T P_{i_k} [(I + D_{i_k})x(t_k) + g_{i_k}(t_k)] \\ &\quad + e^{\delta_0 t_k} \int_{t_k - h(t_k)}^{t_k} x^T(s) R x(s) ds \end{aligned}$$

By Lemma 1, we further derive from (4)

$$\begin{aligned} W(t_k^+) &\leq e^{\delta_0 t_k} \psi(t_k) [2x^T(t_k)(I + D_{i_k})^T P_{i_k} (I + D_{i_k})x(t_k) + 2g_{i_k}^T(t_k) P_{i_k} g_{i_k}(t_k)] \\ &\quad + e^{\delta_0 t_k} \int_{t_k - h(t_k)}^{t_k} x^T(s) R x(s) ds \end{aligned}$$

$$\begin{aligned} &\leq e^{\delta_0 t_k} \psi(t_k) \left( 2 + 2 \frac{\lambda_{\max}(P_{i_k})}{\lambda_{\min}(P_{i_k})} \right) x^T(t_k) (I + D_{i_k})^T P_{i_k} (I + D_{i_k}) x(t_k) \\ &\quad + e^{\delta_0 t_k} \int_{t_k - h(t_k)}^{t_k} x^T(s) R x(s) ds, \end{aligned}$$

Due to  $\psi(t_k) = c/\gamma_{i_k}$  and  $\psi(t_{k+1}^-) = c$ , then taking (7) into account yields

$$\begin{aligned} W(t_k^+) &\leq e^{\delta_0 t_k} \gamma_{i_k} \psi(t_k) x^T(t_k) P_{i_{k-1}} x(t_k) + e^{\delta_0 t_k} \int_{t_k - h(t_k)}^{t_k} x^T(s) R x(s) ds \\ &= e^{\delta_0 t_k} c x^T(t_k^-) P_{i_{k-1}} x(t_k^-) + e^{\delta_0 t_k} \int_{t_k^- - h(t_k^-)}^{t_k^-} x^T(s) R x(s) ds \\ &= e^{\delta_0 t_k^-} \psi(t_k^-) x^T(t_k^-) P_{i_{k-1}} x(t_k^-) + e^{\delta_0 t_k^-} \int_{t_k^- - h(t_k^-)}^{t_k^-} x^T(s) R x(s) ds \\ &= W(t_k^-). \end{aligned}$$

That is, (12) holds for all impulsive switching time points  $t = t_k$ .

*Stage 2.* We shall prove that the Lyapunov functional  $W(t)$  defined by (6) satisfies the following fact.

$$D^+ W(t) - \delta_0 W(t) < 0, \quad t \in (t_k, t_{k+1}]. \quad (13)$$

First of all, consider the case  $t \in (t_k, t_{k+1})$ , note that the fact (ii), it follows that

$$\begin{aligned} &D^+ W(t) - \delta_0 W(t) \\ &\leq e^{\delta_0 t} [x^T(t) R x(t) - (1-d)x^T(t-h(t)) R x(t-h(t))] \\ &\quad + 2e^{\delta_0 t} \psi(t) x^T(t) P_{i_k} [A_{i_k}(t)x(t) + \bar{A}_{i_k}(t)x(t-h(t)) + f_{i_k}(t)] \\ &\quad + 2e^{\delta_0 t} \frac{c}{\tau_D} \left( 1 - \frac{1}{\gamma_{i_k}} \right) x^T(t) P_{i_k} x(t). \end{aligned}$$

By Lemma 1, combining (2) and (3), we get

$$\begin{aligned} &D^+ W(t) - \delta_0 W(t) \\ &\leq e^{\delta_0 t} \psi(t) x^T(t) (P_{i_k} A_{i_k} + A_{i_k}^T P_{i_k} + 2P_{i_k} E_{i_k} F_{i_k}(t) H_{i_k}) x(t) \\ &\quad + e^{\delta_0 t} x^T(t) \left( \frac{2c}{\tau_D} \left( 1 - \frac{1}{\gamma_{i_k}} \right) P_{i_k} + R \right) x(t) \\ &\quad + e^{\delta_0 t} \psi(t) x^T(t) (P_{i_k}^2 + \alpha_{1i_k} \Gamma_{i_k}^T \Gamma_{i_k}) x(t) \\ &\quad + e^{\delta_0 t} x^T(t-h(t)) (\alpha_{2i_k} A_{i_k}^T A_{i_k} - (1-d)R) x(t-h(t)) \\ &\quad + e^{\delta_0 t} \psi(t) x^T(t) (2P_{i_k} \bar{A}_{i_k} + 2e^{\delta_0 t} P_{i_k} E_{i_k} F_{i_k}(t) \bar{H}_{i_k}) x(t-h(t)). \end{aligned}$$

Now, define  $\xi^T(t) = [x^T(t)x^T(t-h(t))]$ ,  $\zeta^T(t) = [(\psi(t)x(t))^T \xi^T(t)]$ . Then, the above expression can be rewritten as

$$\begin{aligned} & D^+W(t) - \delta_0W(t) \\ & \leq e^{\delta_0 t} x^T(t) \left( \frac{2c}{\tau_D} \left( 1 - \frac{1}{\gamma_{i_k}} \right) P_{i_k} + R \right) x(t) + e^{\delta_0 t} \psi(t) x^T(t) (P_{i_k}^2 + \alpha_{1i_k} \Gamma_{i_k}^T \Gamma_{i_k}) x(t) \\ & \quad + e^{\delta_0 t} x^T(t-h(t)) (\alpha_{2i_k} A_{i_k}^T A_{i_k} - (1-d)R) x(t-h(t)) \\ & \quad + e^{\delta_0 t} \psi(t) \xi^T(t) \\ & \quad \times (\mathcal{I}_1^T P_{i_k} \mathcal{A}_{i_k} + \mathcal{A}_{i_k}^T P_{i_k} \mathcal{I}_1 + \mathcal{I}_1^T P_{i_k} E_{i_k} F_{i_k}(t) \mathcal{H}_{i_k} + \mathcal{H}_{i_k}^T F_{i_k}^T(t) E_{i_k}^T P_{i_k} \mathcal{I}_1) \xi(t). \end{aligned}$$

Now according to Lemma 1, for any scalar  $\varepsilon_{i_k} > 0$ ,

$$\begin{aligned} & \mathcal{I}_1^T P_{i_k} E_{i_k} F_{i_k}(t) \mathcal{H}_{i_k} + \mathcal{H}_{i_k}^T F_{i_k}^T(t) E_{i_k}^T P_{i_k} \mathcal{I}_1 \\ & \leq \varepsilon_{i_k}^{-1} \mathcal{I}_1^T P_{i_k} E_{i_k} E_{i_k}^T P_{i_k} \mathcal{I}_1 + \varepsilon_{i_k} \mathcal{H}_{i_k}^T \mathcal{H}_{i_k}. \end{aligned} \tag{14}$$

Note that, for any positive definite  $Q_{i_k} \in \mathbb{R}^{n \times n}$  and positive semi-definite  $L_{i_k}, W_{i_k} \in \mathbb{R}^{2n \times 2n}$ ,

$$(c - \psi(t)) \xi^T(t) L_{i_k} \xi(t) + \left( \psi(t) - \frac{c}{\gamma_{i_k}} \right) \xi^T(t) W_{i_k} \xi(t) \geq 0, \tag{15}$$

$$(c - \psi(t)) \left( \psi(t) - \frac{c}{\gamma_{i_k}} \right) x^T(t) Q_{i_k} x(t) \geq 0 \tag{16}$$

result from the facts (i) and (ii).

Furthermore, combining (14), (15) and (16) yields that

$$\begin{aligned} & D^+W(t) - \delta_0W(t) \\ & \leq e^{\delta_0 t} \psi(t) x^T(t) (-Q_{i_k}) \psi(t) x(t) + e^{\delta_0 t} \xi^T(t) \left( cL_{i_k} - \frac{c}{\gamma_{i_k}} W_{i_k} \right) \xi(t) \\ & \quad + e^{\delta_0 t} \psi(t) \xi^T(t) (\mathcal{I}_1^T P_{i_k} \mathcal{A}_{i_k} + \mathcal{A}_{i_k}^T P_{i_k} \mathcal{I}_1 + \varepsilon_{i_k}^{-1} \mathcal{I}_1^T P_{i_k} E_{i_k} E_{i_k}^T P_{i_k} \mathcal{I}_1 \\ & \quad \quad + \varepsilon_{i_k} \mathcal{H}_{i_k}^T \mathcal{H}_{i_k} - L_{i_k} + W_{i_k}) \xi(t) \\ & \quad + e^{\delta_0 t} \psi(t) x^T(t) \left( P_{i_k}^2 + \alpha_{1i_k} \Gamma_{i_k}^T \Gamma_{i_k} + \frac{c(1 + \gamma_{i_k})}{\gamma_{i_k}} Q_{i_k} \right) x(t) \\ & \quad + e^{\delta_0 t} x^T(t) \left( \frac{2c}{\tau_D} \left( 1 - \frac{1}{\gamma_{i_k}} \right) P_{i_k} + R - \frac{c^2}{\gamma_{i_k}} Q_{i_k} \right) x(t) \\ & \quad + e^{\delta_0 t} x^T(t-h(t)) (\alpha_{2i_k} A_{i_k}^T A_{i_k} - (1-d)R) x(t-h(t)) \\ & = e^{\delta_0 t} \zeta^T(t) \Omega_{i_k} \zeta(t) + e^{\delta_0 t} \psi(t) \xi^T(t) \bar{\Theta}_{i_k} \xi(t), \end{aligned}$$

where

$$\begin{aligned} \bar{\Theta}_{i_k} = & \mathcal{I}_1^T P_{i_k} \mathcal{A}_{i_k} + \mathcal{A}_{i_k}^T P_{i_k} \mathcal{I}_1 + \varepsilon_{i_k}^{-1} \mathcal{I}_1^T P_{i_k} E_{i_k} E_{i_k}^T P_{i_k} \mathcal{I}_1 + \varepsilon_{i_k} \mathcal{H}_{i_k}^T \mathcal{H}_{i_k} - L_{i_k} + W_{i_k} \\ & + \mathcal{I}_1^T \left[ P_{i_k}^2 + \alpha_{1i_k} \Gamma_{i_k}^T \Gamma_{i_k} + \frac{c(1 + \gamma_{i_k})}{\gamma_{i_k}} Q_{i_k} \right] \mathcal{I}_1 + \mathcal{I}_1^T T_{i_k} + T_{i_k}^T \mathcal{I}_1. \end{aligned}$$

Applying Schur complement, for all  $t \in (t_k, t_{k+1})$ , we thus conclude from (10) and (11) that

$$D^+W(t) - \delta_0 W(t) \leq e^{\delta_0 t} \zeta^T(t) \Omega_{i_k} \zeta(t) + e^{\delta_0 t} \psi(t) \xi^T(t) \bar{\Theta}_{i_k} \xi(t) < 0. \quad (17)$$

Moreover, since  $x(t_{k+1}) = x(t_{k+1}^-)$ , then the fact (13) follows immediately.

*Stage 3.* Now it remains to show that the impulsive switched system (1) is robustly exponentially stable.

To begin with, set  $\lambda_1 = \min_{i \in \mathbb{M}} \{\lambda_{\min}(-\Omega_i)\} \geq 0$ ,  $\lambda_2 = \min_{i \in \mathbb{M}} \{\lambda_{\min}(-\bar{\Theta}_i)\} > 0$ ,  $\gamma = \max_{i \in \mathbb{M}} \{\gamma_i\}$ . Note that the fact (i), then it follows from (17) that

$$\begin{aligned} & D^+W(t) - \delta_0 W(t) \\ & \leq -\lambda_1 e^{\delta_0 t} \|\zeta(t)\|^2 - \frac{c\lambda_2}{\gamma} e^{\delta_0 t} \|\xi(t)\|^2 \\ & \leq -\left(\lambda_1 + \frac{c^2\lambda_1}{\gamma^2} + \frac{c\lambda_2}{\gamma}\right) e^{\delta_0 t} \|x(t)\|^2 - \left(\lambda_1 + \frac{c\lambda_2}{\gamma}\right) e^{\delta_0 t} \|x(t-h(t))\|^2 \\ & \leq -\left(\lambda_1 + \frac{c^2\lambda_1}{\gamma^2} + \frac{c\lambda_2}{\gamma}\right) e^{\delta_0 t} \|x(t)\|^2, \quad t \in (t_k, t_{k+1}]. \end{aligned} \quad (18)$$

Also by the fact (i), then according to the expression of  $W(t)$  in (6), there exist positive scalars  $\kappa_j$  ( $j = 0, 1, 2$ ) such that, for  $t \in [0, +\infty)$ ,

$$\kappa_0 e^{\delta_0 t} \|x(t)\|^2 \leq W(t) \leq \kappa_1 e^{\delta_0 t} \|x(t)\|^2 + \kappa_2 e^{\delta_0 t} \int_{t-h}^t \|x(s)\|^2 ds. \quad (19)$$

For simplicity, write  $\bar{\lambda} = \lambda_1 + c^2\lambda_1/\gamma^2 + c\lambda_2/\gamma$ . We further choose sufficiently small  $\delta_0$  such that

$$\bar{\lambda} \geq \delta_0 (\kappa_1 + \kappa_2 h e^{\delta_0 h}). \quad (20)$$

Then, combining (18), (19), and (20), it follows for  $t \in (t_k, t_{k+1}]$  that

$$D^+W(t) \leq e^{\delta_0 t} \left[ (\delta_0 \kappa_1 - \bar{\lambda}) \|x(t)\|^2 + \delta_0 \kappa_2 \int_{t-h}^t \|x(s)\|^2 ds \right]. \quad (21)$$

Now integrating both sides of (21) from  $t_k^+$  to  $t$  gives

$$W(t) \leq W(t_k^+) + \int_{t_k^+}^t e^{\delta_0 s} \left[ (\delta_0 \kappa_1 - \bar{\lambda}) \|x(s)\|^2 + \delta_0 \kappa_2 \int_{s-h}^s \|x(\theta)\|^2 d\theta \right] ds.$$

Applying (12) and (19), by directly deducing we further get, for  $t \in [0, \infty)$ ,

$$\begin{aligned} & \kappa_0 e^{\delta_0 t} \|x(t)\|^2 \\ & \leq W(t) \leq W(0) + \int_0^t e^{\delta_0 s} \left[ (\delta_0 \kappa_1 - \bar{\lambda}) \|x(s)\|^2 + \delta_0 \kappa_2 \int_{s-h}^s \|x(\theta)\|^2 d\theta \right] ds. \end{aligned} \quad (22)$$

Observe that

$$\int_0^t e^{\delta_0 s} ds \int_{s-h}^s \|x(\theta)\|^2 d\theta \leq h e^{\delta_0 h} \int_0^t e^{\delta_0 s} \|x(s)\|^2 ds + h e^{\delta_0 h} \int_{-h}^0 e^{\delta_0 s} \|x(s)\|^2 ds.$$

Therefore, it follows from (20) and (22) that

$$\kappa_0 e^{\delta_0 t} \|x(t)\|^2 \leq W(0) + \delta_0 \kappa_2 h e^{\delta_0 h} \int_{-h}^0 e^{\delta_0 s} \|x(s)\|^2 ds.$$

It turns out that, for  $t \in [0, \infty)$ ,

$$\|x(t)\| \leq \beta \|\varphi\|_h e^{-\delta t} \quad \text{with} \quad \delta = \frac{\delta_0}{2}, \quad \beta = \left( \frac{\kappa_1 + \kappa_2 h + \delta_0 \kappa_2 h^2 e^{\delta_0 h}}{\kappa_0} \right)^{1/2}.$$

Hence the impulsive switched system (1) is robustly exponentially stable. This completes the proof.  $\square$

**Comment 2.** It is easy to see from the above result that the impulsive switched systems (1) is delay-independently exponentially stable. Namely, if the upper bound  $h$  of time-varying delays unknown or sufficiently large, Theorem 1 also holds. Note that, for time-invariant delays case, [7] presented the delay-independent criteria of asymptotical stability. However, this result required that the delayed system state  $x(t-h)$  can be located a known subsystem. Therefore, if the upper bound  $h$  time-varying delays tend to infinity, the result in [7] is not applicable. In [8], the authors also considered the impulsive switched systems with time-invariant delays and constructed a class of piecewise Lyapunov functionals subjected to the size of time-delays. It is worth emphasizing that, when the upper bound  $h$  of time-varying delays is unknown, such type of Lyapunov functional becomes invalid.

**Comment 3.** Paper [7] considers the uncertain impulsive switched systems with time-invariant delays and presented the maximum dwell time criteria. Namely, the upper bounds of maximum dwell time are given. In this case, the set of switching signal stabilizing the impulsive switched systems shall be shrunk. Indeed, when the dwell time on some stable subsystems is sufficiently large, the stability of the whole impulsive switched systems is always achieved. However, our result only established on the minimum dwell time. Obviously, this will improve the design of switching signal greatly.

A reduced case is straightforward from Theorem 1.

**Corollary 1.** For any  $i \in \mathbb{M}$ , given  $\tau_D > 0$ , given positive real numbers  $c > 0$ ,  $\gamma_i \geq 4$ , if there exist positive definite matrices  $P_i, R \in \mathbb{R}^{n \times n}$ , positive semi-definite matrices  $Q_i \in \mathbb{R}^{n \times n}$ , positive semi-definite  $L_i, W_i \in \mathbb{R}^{2n \times 2n}$ , and any matrices  $T_i \in \mathbb{R}^{n \times 2n}$ , such that (8) and the following linear matrix inequalities are satisfied:

$$\Omega_i \leq 0, \quad \Theta_i = \begin{bmatrix} \Theta_{1i} & \mathcal{I}_1^T P_i \\ * & -I \end{bmatrix} < 0,$$

where  $\Omega_i$  is defined as (10) and

$$\Theta_{1i} = \mathcal{I}_1^T (P_i A_i + T_i) + (A_i^T P_i + T_i^T) \mathcal{I}_1 - L_i + W_i + \mathcal{I}_1^T \left[ \frac{c(1 + \gamma_i)}{\gamma_i} U_i + \Gamma_i^T \Gamma_i \right] \mathcal{I}_1.$$

Then system (1) with  $[A_i(t) \bar{A}_i(t)] = [A_i \bar{A}_i]$  is exponentially stable.

#### 4 A numerical example

As an illustrative example, consider the impulsive switched system (1) with  $h(t) = 5 + 2 \sin t$  and

$$\begin{aligned} A_1 &= \begin{bmatrix} -9 & 1 \\ 3 & -12 \end{bmatrix}, & A_2 &= \begin{bmatrix} -9 & 4 \\ 0 & -11 \end{bmatrix}, & \bar{A}_1 &= \begin{bmatrix} -1.5 & 1 \\ 2 & 0 \end{bmatrix}, \\ \bar{A}_2 &= \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}, & E_1 &= \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, & E_2 &= \begin{bmatrix} -0.3 & 0.1 \\ -0.2 & -0.1 \end{bmatrix}, \\ H_1 &= \begin{bmatrix} -0.1 & -0.3 \\ 0 & -0.2 \end{bmatrix}, & H_2 &= \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & -0.3 \end{bmatrix}, & \bar{H}_1 &= \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \\ \bar{H}_2 &= \begin{bmatrix} -0.3 & 0.1 \\ 0.1 & 0 \end{bmatrix}, & f_1 &= \begin{bmatrix} \sin x_1 \\ \sin x_2 \end{bmatrix}, & f_2 &= \begin{bmatrix} 0 \\ \sin x_2 \end{bmatrix}, \\ D_1 = D_2 &= \begin{bmatrix} 20 & 0 \\ 0 & 12 \end{bmatrix}, & g_1 = g_2 &= \begin{bmatrix} 0.2 \sin x_1(t_k) \\ 0.5 \sin x_2(t_k) \end{bmatrix}. \end{aligned}$$

Furthermore, choose  $F_1 = F_2 = \cos 10t$ ,  $A_1 = A_2 = \Gamma_1 = \Gamma_2 = I$ ,  $c = 8$ ,  $\gamma_1 = 4$ ,  $\gamma_2 = 4$ ,  $\tau_D = 0.5$ , and  $\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{22} = \varepsilon_1 = \varepsilon_2 = 1$ . Solving the LMIs (10) and (11) in  $P_1, P_2$  and  $R$ , we can obtain the following feasible solutions

$$\begin{aligned} P_1 &= \begin{pmatrix} 1.6278 & -0.0487 \\ -0.0487 & 1.1973 \end{pmatrix}, & P_2 &= \begin{pmatrix} 1.3211 & 0.2623 \\ 0.2623 & 2.1331 \end{pmatrix}, \\ R &= \begin{pmatrix} 7.5836 & -5.4892 \\ -5.4892 & 13.5217 \end{pmatrix}. \end{aligned}$$

With this, then according to Theorem 1, the impulsive switched system is robustly exponentially stable. See Figs. 1 and 2. Figure 1 depicts the state variables  $x_1$  and  $x_2$ ,  $[x_1 x_2]^T$  starts from  $[0.04 \ -0.01]^T$ , and then rapidly approaches the equilibrium point  $[0 \ 0]^T$ .

Figure 2 shows that the trajectory of the whole impulsive system, where the dash line indicates the impulsive jump phenomenon.

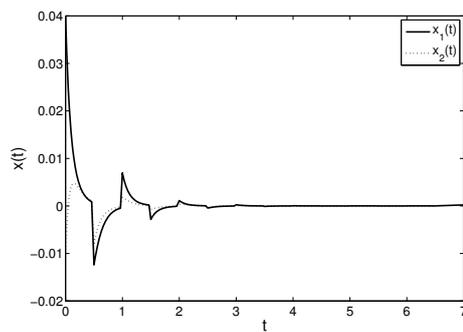


Fig. 1. The state variables of the impulsive switched system in example.

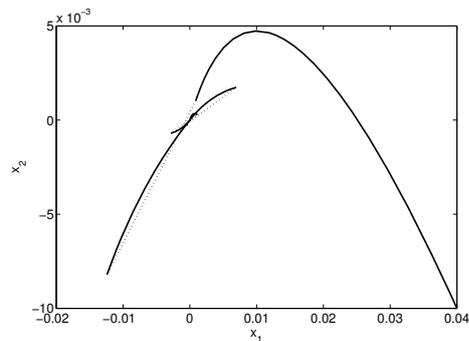


Fig. 2. The trajectory of the impulsive switched system in example.

## 5 Conclusions

The robust exponential stability problem for nonlinear impulsive switched delayed systems is studied in this paper. By constructing a new piecewise Lyapunov functional, it is successful to eliminate the jump phenomenon at impulsive switching points and shown that the each subsystem is exponential decay. Then the robust exponential stability of the whole systems is guaranteed and formulated in terms of LMI conditions. Such conditions only depend on the minimum dwell time, irrespective the size of time-varying delays.

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