Solitary wave solutions of the Vakhnenko–Parkes equation

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Received: 12 October 2011 / **Revised:** 16 January 2012 / **Published online:** 24 February 2012

Abstract. In this paper, two solitary wave solutions are obtained for the Vakhnenko–Parkes equation with power law nonlinearity by the ansatz method. Both topological as well as non-topological solitary wave solutions are obtained. The parameter regimes, for the existence of solitary waves, are identified during the derivation of the solution.

Keywords: nonlinear evolution equation, solitons, integrability.

1 Introduction

The theory of nonlinear evolution equations (NLEEs) is an important area of research in the area of applied mathematics [1–15]. A challenging task is to look for solutions of these NLEEs. There are various types of solutions that are available for these equations. Some of them are soliton solutions, solitary wave solutions, cnoidal and snoidal waves, periodic solutions, shock wave solutions as well as various other types. In this paper there will be one such NLEE that will be studied. This is the Vakhnenko–Parkes (VP) equation with power law nonlinearity. The ansatz method will be used to retrieve the topological as well as non-topological solitary wave solution. The domain restrictions will be revealed during the process of obtaining the solutions.

2 Mathematical analysis

The standard form of the VP equation is given by [11]

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0, \tag{1}$$

where $u(x,t)$ is a real function of the spatial variable x and the temporal variable t.

In this paper, we are interested in the following family of the VP equation with power law nonlinearity:

$$uu_{xxt} + au_x u_{xt} + bu^{2n} u_t = 0, \tag{2}$$

where a and b are nonzero real constants, while $n \in Z^+$. The parameter n indicates the power law nonlinearity parameter. Thus, for $a = -1$ and $b = n = 1$, Eq. (2) collapses to (1).

The purpose of this paper is to calculate the exact topological and non-topological solitary wave solutions for this model, exhibiting power law nonlinearity. The importance of the results presented here is two-fold. First, exact solitary wave solutions to a family of the considered equation and the conditions for their existence are obtained for a general case of power nonlinearity law in a simple way. Importantly, the finding of explicit solutions of a given NLEE with any value of the exponent in the nonlinearity term is very interesting since it offers some knowledge on the general dynamical behavior of the wave propagation so that special cases are truly meaningful both from the physical and mathematical point of view. Second, these results confirm the existence of non-topological solitary waves for any exponent $n > 1/2$, while topological solitary waves exist only in the case when $n = 3/2$. Furthermore, closed form solitary wave solutions exist only for $a = -1$, which corresponds exactly to the value of this coefficient in the standard form of the VP equation (1). To achieve our goal we will use the solitary wave ansatz method which has recently been applied successfully to several NLEEs with constant and variable coefficients.

3 Non-topological solitary wave

In order to solve (2), the starting hypothesis is [7–13]

$$u(x,t) = A\,\mathrm{sech}^p\,\tau, \tag{3}$$

where

$$\tau = B(x - vt) \tag{4}$$

and

$$p > 0 \tag{5}$$

for solitary waves to exist. Here, in (3) and (4), A is the amplitude of the solitary wave while v is the velocity of the solitary wave and B is the inverse width. The exponent p is unknown at this point and its value will fall out in the process of deriving the solution of this equation. From the ansatz (3), one can find that

$$uu_{xxt} = \frac{p^3 A^2 B^3 v \tanh\tau}{\cosh^{2p}\tau} - \frac{p(p+1)(p+2)A^2 B^3 v \tanh\tau}{\cosh^{2p+2}\tau}, \tag{6}$$

$$u_x u_{xt} = \frac{p^3 A^2 B^3 v \tanh\tau}{\cosh^{2p}\tau} - \frac{p^2(p+1)A^2B^3 v \tanh\tau}{\cosh^{2p+2}\tau}, \tag{7}$$

$$u^{2n} u_t = \frac{pvA^{2n+1}B\tanh\tau}{\cosh^{p(2n+1)}\tau}. \tag{8}$$

Inserting the expressions (6)–(8) into (2) yields

$$\frac{(a+1)p^3A^2B^3v)\tanh\tau}{\cosh^{2p}\tau} - \frac{p(p+1)(p+2+ap)A^2B^3v\tanh\tau}{\cosh^{2p+2}\tau} + \frac{bpvA^{2n+1}B\tanh\tau}{\cosh^{p(2n+1)}\tau} = 0. \tag{9}$$

From (9), equating the exponents $2p+2$ and $p(2n+1)$ gives

$$2p+2 = p(2n+1),$$

so that

$$p = \frac{2}{2n-1}. \tag{10}$$

Taking $p > 0$ as a necessary condition for the existence of the solitary wave solution (3) implies that we must have $n > 1/2$ in (10).

Now, from (9), setting the coefficients of the linearly independent functions $\tanh\tau/\cosh^{2p+j}\tau$ to zero, where $j = 0, 2$, gives

$$(a+1)p^3A^2B^3v = 0,$$
$$-p(p+1)(p+2+ap)A^2B^3v + bpvA^{2n+1}B = 0.$$

Solving the above equations yields

$$a = -1, \tag{11}$$

$$B = \left\{\frac{bA^{2n-1}}{2(p+1)}\right\}^{1/2}. \tag{12}$$

The substitution of (10) and (11) into (12) gives the inverse width of the solitary wave as

$$B = \left\{\frac{b(2n-1)A^{2n-1}}{2(2n+1)}\right\}^{1/2} \tag{13}$$

which shows that the solitary waves will exist for

$$b > 0$$

as long as $n > 1/2$ which is guaranteed from (5) and (10). The width of the solitary wave given by (12) or (13) is the same by virtue of (10).

Thus, the solitary wave solution of the VP equation (2) with power law nonlinearity is given by

$$u(x,t) = A\,\mathrm{sech}^{2/(2n-1)}\big[B(x-vt)\big], \tag{14}$$

where the inverse width of the solitary wave B is given by (12) or (13). Finally, we would like to note that the solution (14) exists under the conditions $n > 1/2$, $b > 0$ and $a = -1$. This solitary wave given by (14) is a generalized version of the solitary wave solution that was obtained earlier. In particular when $n = 1$, (14) collapses to Eqs. (3.4) of [11], (2) of [7], (3) of [8] and (2) of [10].

4 Topological solitary wave

In this section, we will calculate the topological solitary wave solution of the family of VP equation (2) with power law nonlinearity, using the solitary wave ansatz. To start off, the hypothesis is taken to be

$$u(x,t) = A\tanh^p \tau, \tag{15}$$

where

$$\tau = B(x - vt) \tag{16}$$

and

$$p > 0$$

for solitary waves to exist. Here, in (1) and (16), A and B are free parameters and v is the velocity of the wave. Also, the unknown exponent p will be determined during the course of the derivation of the solitary wave solution to (2). Therefore from (15),we have

$$\begin{aligned}
uu_{xxt} &= -pA^2B^3v\bigl[(p-1)(p-2)\tanh^{2p-3}\tau \\
&\qquad -\bigl\{2p^2 + (p-1)(p-2)\bigr\}\tanh^{2p-1}\tau \\
&\qquad +\bigl\{2p^2 + (p+1)(p+2)\bigr\}\tanh^{2p+1}\tau \\
&\qquad -(p+1)(p+2)\tanh^{2p+3}\tau\bigr], \qquad (17)\\
u_xu_{xt} &= -p^2A^2B^3v\bigl\{(p-1)\tanh^{2p-3}\tau - (p+1)\tanh^{2p+3}\tau \\
&\qquad +(3p+1)\tanh^{2p+1}\tau - (3p-1)\tanh^{2p-1}\tau\bigr\}, \qquad (18)\\
u^{2n}u_t &= pvA^{2n+1}B\bigl(\tanh^{p(2n+1)+1}\tau - \tanh^{p(2n+1)-1}\tau\bigr). \qquad (19)
\end{aligned}$$

Substituting the expressions (17)–(19) into (2), we obtain

$$\begin{aligned}
&-pA^2B^3v\bigl[(p-1)(p-2)\tanh^{2p-3}\tau - \bigl\{2p^2 + (p-1)(p-2)\bigr\}\tanh^{2p-1}\tau \\
&\qquad \bigl\{2p^2 + (p+1)(p+2)\bigr\}\tanh^{2p+1}\tau - (p+1)(p+2)\tanh^{2p+3}\tau\bigr] \\
&-ap^2A^2B^3v\bigl\{(p-1)\tanh^{2p-3}\tau - (p+1)\tanh^{2p+3}\tau \\
&\qquad +(3p+1)\tanh^{2p+1}\tau - (3p-1)\tanh^{2p-1}\tau\bigr\} \\
&+bpvA^{2n+1}B\bigl(\tanh^{p(2n+1)+1}\tau - \tanh^{p(2n+1)-1}\tau\bigr) = 0. \qquad (20)
\end{aligned}$$

From (20), equating the exponents $2p+3$ and $p(2n+1)+1$ gives

$$2p + 3 = p(2n+1) + 1,$$

so that

$$p = \frac{2}{2n-1}. \tag{21}$$

As a result, the condition $n > 1/2$ for the topological solitary wave solution to exist arises from (21). It needs to be noted that the same value of p is yielded when the exponents $2p + 1$ and $p(2n + 1) - 1$ are equated with each other.

Now from (20) the linearly independent functions are $\tanh^{2p+j}\tau$ for $j = \pm 1, \pm 3$. Hence setting their respective coefficients to zero yields a set of algebraic equations:

$$-pA^2B^3v(p-1)(p-2) - ap^2A^2B^3v(p-1) = 0, \tag{22}$$

$$pA^2B^3v\{2p^2 + (p-1)(p-2)\} + ap^2A^2B^3v(3p-1) = 0, \tag{23}$$

$$-pA^2B^3v\{2p^2 + (p+1)(p+2)\} - ap^2A^2B^3v(3p+1) - bpvA^{2n+1}B = 0, \tag{24}$$

$$pA^2B^3v(p+1)(p+2) + ap^2A^2B^3v(p+1) + bpvA^{2n+1}B = 0. \tag{25}$$

In order to solve (22), we have considered the case:

$$p = 1. \tag{26}$$

By setting $p = 1$ in (23)–(25), we obtain

$$a = -1,$$
$$B = \left\{-\frac{bA^{2n-1}}{4}\right\}^{1/2}. \tag{27}$$

The latter forces the constraint relation

$$b < 0.$$

Now, equating the two values of p from (21) and (26) gives the condition:

$$n = \frac{3}{2}.$$

Thus, for the family of the VP equation (2) with power law nonlinearity, dark solitary waves will exist only when $n = 3/2$. This important observation is being made for the first time in this paper.

Furthermore, the free parameter B in (27) becomes

$$B = \sqrt{-\frac{b}{4}}A.$$

Hence, finally, the topological solitary wave solution to the VP equation (2) with power law nonlinearity is given by

$$u(x,t) = A\tanh\left[\sqrt{-\frac{b}{4}}A(x - vt)\right]$$

which exist provided that $a = -1$ and $b < 0$.

5 Conclusions

In this paper, the topological and non-topologial solitary wave solution to the VP equation, with power law nonlinearity, was obtained. In addition, the constraint conditions were also obtained in order for the solitary wave solutions to exist. In the future, this VP equation will be investigated further. The time-dependent coefficients will be considered. Also, several other solutions, using a variety of other methods, will be obtained. These results will be reported in future.

Acknowledgment

The third, fourth and fifth authors (T. Hayat, O.M. Aldossary and A. Biswas) would like to recognize and thankfully appreciate the support from King Saud University (KSU-VPP-117).

References

1. R. Abazari, Application of (G'/G)-expansion method to travelling wave solutions of three nonlinear evolution equations, *Comput. Fluids*, **39**, pp. 1957–1963, 2010.

2. F. Kangalgil, F. Ayaz, New exact traveling wave solutions for the Ostrovsky equation, *Phys. Lett., A*, **372**, pp. 1831–1835, 2008.

3. C. Koroglu, T. Ozis, A novel traveling wave solution for Ostrovsky equation using Exp-function method, *Comput. Math. Appl.*, **58**, pp. 2142–2146, 2009.

4. B. Li, Y. Ma, J. Sun, The interaction process of the N-soliton solutions for an extended generalization of Vakhnenko equation, *Appl. Math. Comput.*, **216**(12), pp. 3522–3535, 2010.

5. Y. Ma, B. Li, C. Wang, A series of abundant exact traveling wave solutions for a modified generalized Vakhnenko equation using auxiliary equation method, *Appl. Math. Comput.*, **211**(1), pp. 102–107, 2009.

6. A.J. Morrison, E.J. Parkes, The N-soliton solution of the modified generalized Vakhnenko equation (a new nonlinear evolution equation), *Chaos Solitons Fractals*, **16**(1), pp. 13–26, 2003.

7. E.J. Parkes, A note on solitary travelling-wave solutions to the transformed reduced Ostrovsky equation, *Commun. Nonlinear Sci. Numer. Simul.*, **15**, pp. 2769–2771, 2010.

8. E.J. Parkes, A note on "New travelling waves to the Ostrovsky equation", *Appl. Math. Comput.*, **217**, pp. 3575–3577, 2010.

9. E.J. Parkes, Observations on the (G'/G)-expansion method for finding solutions to nonlinear evolution equation, *Appl. Math. Comput.*, **217**, pp. 1759–1763, 2010.

10. E.J. Parkes, Comment on "Application of (G'/G)-expansion method to travelling-wave solutions of three nonlinear evolution equations", *Comput. Fluids*, **42**, pp. 108–109, 2011.

11. V.O. Vakhnenko, E.J. Parkes, The two loop soliton solution of the Vakhnenko-equation, *Nonlinearity*, **11**, pp. 1457–1464, 1998.

12. V.O. Vakhnenko, E.J. Parkes, The calculations of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method, *Chaos Solitons Fractals*, **13**(9), pp. 1819–1826, 2002.

13. V.O. Vakhnenko, E.J. Parkes, A Backlund transformation and the inverse scattering transform method for the generalized Vakhnenko equation, *Chaos Solitons Fractals*, **17**(4), pp. 683–692, 2003.

14. E. Yasar, New traveling wave solutions to the Ostrovsky equation, *Appl. Math. Comput.*, **216**, pp. 3191–3194, 2010.

15. E. Yosufoglu, A. Bekir, A traveling wave solution to the Ostrovsky equation, *Appl. Math. Comput.*, **186**, pp. 256–260, 2007.