Analysis of a frictionless contact problem for elastic-viscoplastic materials

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Abstract. We consider a dynamic frictionless contact problem for elastic-viscoplastic materials with damage. The contact is modelled with normal compliance condition. The adhesion of the contact surfaces is considered and is modelled with a surface variable, the bonding field whose evolution is described by a first order differential equation. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed-point arguments.

Keywords: dynamic process, elastic-viscoplastic materials, evolution equations, parabolic inequalities, differential equations, fixed-point arguments.

1 Introduction

The adhesive contact between bodies, when a glue is added to keep the surfaces from relative motion, is receiving increasing attention in the mathematical literature. Analysis of models for adhesive contact can be found in [1–4], and recently in the monograph [5]. The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; which describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [6, 7], the bonding field satisfies the restrictions $0 \le \beta \le 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [8–10]. In this paper we deal with the study of a dynamic problem of frictionless adhesive contact for general elastic-viscoplastic materials of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}(t)\big) + \mathcal{E}\boldsymbol{\varepsilon}\big(\mathbf{u}(t)\big) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}(s)\big), \boldsymbol{\varepsilon}\big(\mathbf{u}(s)\big), \alpha(s)\big)\,\mathrm{d}s, \tag{1}$$

where $\mathbf{u}$ is the displacement field and $\boldsymbol{\sigma}$ and $\varepsilon(\mathbf{u})$ are the stress and the linearized strain tensor, respectively. Here $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{G}$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material. We also consider that the viscoplastic function $\mathcal{G}$ depends on the internal state variable α describing the damage of the material caused by plastic deformations. In (1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t. It follows from (1) that at each time moment, the stress tensor $\boldsymbol{\sigma}(t)$ is split into two parts: $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^V(t) + \boldsymbol{\sigma}^R(t)$, where $\boldsymbol{\sigma}^V(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))$ represents the purely viscous part of the stress whereas $\boldsymbol{\sigma}^R(t)$ satisfies a rate-type elastic-viscoplastic relation with damage

$$\boldsymbol{\sigma}^R(t) = \mathcal{E}\varepsilon\bigl(\mathbf{u}(t)\bigr) + \int_0^t \mathcal{G}\bigl(\boldsymbol{\sigma}^R(s), \varepsilon\bigl(\mathbf{u}(s)\bigr), \alpha(s)\bigr)\,\mathrm{d}s. \tag{2}$$

When $\mathcal{G} = 0$ (1) reduces to the Kelvin–Voigt viscoelastic constitutive law given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon\bigl(\dot{\mathbf{u}}(t)\bigr) + \mathcal{E}\varepsilon\bigl(\mathbf{u}(t)\bigr). \tag{3}$$

The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [11, 12] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [13]. In all these papers the damage of the material is described with a damage function α, restricted to have values between zero and one. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$ the material is completely damaged, when $0 < \alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [5, 14–19]. In this paper the inclusion used for the evolution of the damage field is

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi\bigl(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(\mathbf{u}), \alpha\bigr),$$

where K denotes the set of admissible damage functions defined by

$$K = \bigl\{\xi \in H^1(\Omega) \bigm| 0 \le \xi \le 1,\ \text{a.e. in } \Omega\bigr\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of the set K and ϕ is a given constitutive function which describes the sources of the damage in the system. Examples and mechanical interpretation of elastic-viscoplastic materials of the form (2) in which the function $\mathcal{G}$ does not depend on the damage parameter α were considered by many authors, see for instance [20, 21] and the references therein. Contact problems for materials of the form (1), (2) without damage parameter and (3) are the topic of numerous papers, e.g. [5, 22–27] and the recent references [25, 28]. Contact problems for elastic-viscoplastic materials of the form (2) are studied in [5, 16]. In this paper we study a dynamic frictionless contact problem. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We model the material's behavior with an elastic-viscoplastic constitutive law with damage. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

The paper is organised as follows. In Section 2 we present the notation and some preliminaries. In Section 3 we present the mechanical problem, list the assumptions on the data, give the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result, Theorem 1. The proof of the theorem is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments.

2 Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [29]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces:

$$
\begin{aligned}
H &= L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\},\\
\mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},\\
H_1 &= \{\mathbf{u} = (u_i) \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H}\},\\
\mathcal{H}_1 &= \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}.
\end{aligned}
$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$
\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).
$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$
\begin{aligned}
(\mathbf{u}, \mathbf{v})_H &= \int_\Omega u_i v_i \,\mathrm{d}x \qquad \forall \mathbf{u}, \mathbf{v} \in H,\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} \,\mathrm{d}x \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},\\
(\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall\, \mathbf{u}, \mathbf{v} \in H_1,\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.
\end{aligned}
$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ for the trace of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{4}$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{5}$$

and the following Green's formula holds:

$$\big(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v})\big)_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v} \, \mathrm{d}a \quad \forall \mathbf{v} \in H_1. \tag{6}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm

$$|\mathbf{f}|_{C(0,T;X)} = \max_{t \in [0,T]} \big|\mathbf{f}(t)\big|_X$$

while $C^1(0,T;X)$ is a real Banach space with the norm

$$|\mathbf{f}|_{C^1(0,T;X)} = \max_{t \in [0,T]} \big|\mathbf{f}(t)\big|_X + \max_{t \in [0,T]} \big|\dot{\mathbf{f}}(t)\big|_X.$$

Finally, for $k \in \mathbb{N}$ and $p \in [1, \infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, for a real number r, we use r_+ to represent its positive part; that is $r_+ = \max\{0, r\}$ and if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.\,,.)_{X_1 \times X_2}$.

3 Problem statement

We consider an elastic-viscoplastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ with the boundary Γ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}(\Gamma_1) > 0$. The time interval of interest is $[0,T]$ where $T > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there. A volume force of density $\mathbf{f}_0$ acts in $\Omega \times (0,T)$ and surface tractions of density $\mathbf{f_2}$ act on $\Gamma_2 \times (0,T)$. We assume that the body is in adhesive frictionless contact with an obstacle, the so-called foundation, over the potential contact surface Γ_3. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. We use an elastic-viscoplastic constitutive law with damage to model the material's behaviour and an ordinary differential equation to describe the evolution of the bonding field. The mechanical formulation of the frictionless problem with normal compliance is as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega\times[0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega\times[0,T] \to S_d$, a damage field $\alpha : \Omega\times[0,T] \to \mathbb{R}$ and a bonding field $\beta : \Gamma_3\times[0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon\big(\dot{\mathbf{u}}(t)\big) + \mathcal{E}\varepsilon\big(\mathbf{u}(t)\big) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}(s)\big), \boldsymbol{\varepsilon}\big(\mathbf{u}(s)\big), \alpha(s)\big)\,\mathrm{d}s \quad \text{in } \Omega\times(0,T), \tag{7}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi\big(\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}), \boldsymbol{\varepsilon}(\mathbf{u}), \alpha\big) \quad \text{in } \Omega\times(0,T), \tag{8}$$

$$\rho\ddot{\mathbf{u}} = \operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{in } \Omega\times(0,T), \tag{9}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1\times(0,T), \tag{10}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2\times(0,T), \tag{11}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3\times(0,T), \tag{12}$$

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta)\mathbf{R}_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_3\times(0,T), \tag{13}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu\big(R_\nu(u_\nu)\big)^2 + \gamma_\tau\big|\mathbf{R}_\tau(\mathbf{u}_\tau)\big|^2\big) - \epsilon_a\Big)_+ \quad \text{on } \Gamma_3\times(0,T), \tag{14}$$

$$\frac{\partial\alpha}{\partial\boldsymbol{\nu}} = 0 \quad \text{on } \Gamma\times(0,T), \tag{15}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0 \quad \text{in } \Omega, \tag{16}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{17}$$

Equation (7) represents the elastic-viscoplastic constitutive law with damage introduced in Section 1, Eq. (8) is the inclusion used for the evolution of the damage field. Equation (9) represents the equation of motion where ρ denotes the material mass density. Equations (10) and (11) are the displacement and traction boundary conditions, respectively. Condition (12) represents the normal compliance condition with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed; that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The contact condition (12) was used in various papers, see e.g. [1, 2, 5, 10]. Condition (13) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } |\mathbf{v}| \le L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if } |\mathbf{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows.

Next, Eq. (14) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [1], see also [9, 10] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ϵ_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from Eq. (14), $\dot{\beta} \le 0$. Boundary condition (15) describes a homogeneous Neumann boundary condition where $\partial\alpha/\partial\boldsymbol{\nu}$ is the normal derivative of α. In Eq. (16) $\mathbf{u}_0$ is the initial displacement, $\mathbf{v}_0$ the initial velocity and α_0 is the initial damage. Finally, in Eq. (17) β_0 denotes the initial bonding. To obtain the variational formulation of the problem (7)–(17) we introduce for the bonding field the set

$$Z = \bigl\{\theta \in L^\infty\bigl(0,T;L^2(\Gamma_3)\bigr) \bigm| 0 \le \theta(t) \le 1 \ \forall t \in [0,T], \text{ a.e. on } \Gamma_3\bigr\}.$$

Let V be the closed subspace of H_1 given by

$$V = \bigl\{\mathbf{v} \in H_1 \bigm| \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\bigr\}.$$

Then, the following Korn's inequality holds:

$$\bigl|\boldsymbol{\varepsilon}(\mathbf{v})\bigr|_{\mathcal{H}} \ge C_k|\mathbf{v}|_{H_1} \quad \forall \mathbf{v} \in V,$$

where $C_k > 0$ is a constant depending only on Ω and Γ_1. On the space V we consider the inner product given by

$$(\mathbf{u},\mathbf{v})_V = \bigl(\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})\bigr)_{\mathcal{H}} \tag{18}$$

and let $|\cdot|_V$ be the associated norm. It follows from Korn's inequality that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V,|\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \le C_0|\mathbf{v}|_V \quad \forall \mathbf{v} \in V. \tag{19}$$

In the study of the mechanical problem (7)–(17), we assume that:

(A1) The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies:

(a) There exists a constant $L_{\mathcal{A}} > 0$ such that

$$\bigl|\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)\bigr| \le L_{\mathcal{A}}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S_d, \ \text{a.e. } \mathbf{x} \in \Omega.$$

(b) There exists a constant $m_{\mathcal{A}} > 0$ such that

$$\big(\mathcal{A}(\mathbf{x},\varepsilon_1) - \mathcal{A}(\mathbf{x},\varepsilon_2)\big) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}}|\varepsilon_1 - \varepsilon_2|^2 \quad \forall \varepsilon_1, \varepsilon_2 \in S_d,$$
$$\text{a.e. } \mathbf{x} \in \Omega.$$

(c) The mapping $\mathbf{x} \to \mathcal{A}(\mathbf{x},\varepsilon)$ is Lebesgue measurable on Ω for any $\varepsilon \in S_d$.

(d) The mapping $\mathbf{x} \to \mathcal{A}(\mathbf{x},\mathbf{0}) \in \mathcal{H}$.

(A2) The elasticity operator $\mathcal{E} : \Omega \times S_d \to S_d$ satisfies:

(a) There exists a constant $L_{\mathcal{E}} > 0$ such that

$$\big|\mathcal{E}(\mathbf{x},\varepsilon_1) - \mathcal{E}(\mathbf{x},\varepsilon_2)\big| \leq L_{\mathcal{E}}|\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1, \varepsilon_2 \in S_d, \ \text{ a.e. } \mathbf{x} \in \Omega.$$

(b) For any $\varepsilon \in S_d$, $\mathbf{x} \to \mathcal{E}(\mathbf{x},\varepsilon)$ is Lebesgue measurable on Ω.

(c) The mapping $\mathbf{x} \to \mathcal{E}(\mathbf{x},\mathbf{0}) \in \mathcal{H}$.

(A3) The viscoplasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R} \to S_d$ satisfies:

(a) There exists a constant $L_{\mathcal{G}} > 0$ such that

$$\big|\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\varepsilon_1,\alpha_1) - \mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\varepsilon_2,\alpha_2)\big| \leq L_{\mathcal{G}}\big(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|\big)$$
$$\forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in S_d \ \text{ and } \ \alpha_1, \alpha_2 \in \mathbb{R}, \ \text{ a.e. } \mathbf{x} \in \Omega.$$

(b) For any $\boldsymbol{\sigma}, \varepsilon \in S_d$ and $\alpha \in \mathbb{R}$, $\mathbf{x} \to \mathcal{G}(\mathbf{x},\sigma,\varepsilon,\alpha)$ is Lebesgue measurable on Ω.

(c) The mapping $\mathbf{x} \to \mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},0) \in \mathcal{H}$.

(A4) The damage source function $\phi : \Omega \times S_d \times S_d \times \mathbb{R} \to \mathbb{R}$ satisfies:

(a) There exists a constant $L_{\phi} > 0$ such that

$$\big|\phi(\mathbf{x},\boldsymbol{\sigma}_1,\varepsilon_1,\alpha_1) - \phi(\mathbf{x},\boldsymbol{\sigma}_2,\varepsilon_2,\alpha_2)\big| \leq L_{\phi}\big(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|\big)$$
$$\forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in S_d \ \text{ and } \ \alpha_1, \alpha_2 \in \mathbb{R}, \ \text{ a.e. } \mathbf{x} \in \Omega.$$

(b) For any $\boldsymbol{\sigma}, \varepsilon \in S_d$ and $\alpha \in \mathbb{R}, \mathbf{x} \to \phi(\mathbf{x}, \boldsymbol{\sigma}, \varepsilon, \alpha)$ is Lebesgue measurable on Ω.

(c) The mapping $\mathbf{x} \to \phi(\mathbf{x},\mathbf{0},\mathbf{0},0) \in \mathcal{H}$.

(A5) The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

(a) There exists a constant $L_\nu > 0$ such that

$$\big|p_\nu(\mathbf{x},r_1) - p_\nu(\mathbf{x},r_2)\big| \leq L_\nu|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \ \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(b) The mapping $\mathbf{x} \to p_\nu(\mathbf{x},r)$ is measurable on Γ_3, for any $r \in \mathbb{R}$.

(c) $p_\nu(\mathbf{x},r) = 0$ for all $r \leq 0$, a.e. $\mathbf{x} \in \Gamma_3$.

(A6) The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

(a) There exists a constant $L_\tau > 0$ such that

$$\big|p_\tau(\mathbf{x},d_1) - p_\tau(\mathbf{x},d_2)\big| \leq L_\tau|d_1 - d_2| \quad \forall d_1, d_2 \in \mathbb{R}, \ \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(b) There exists $M_\tau > 0$ such that $\big|p_\tau(\mathbf{x},d)\big| \leq M_\tau \ \forall d \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_3$.

(c) The mapping $\mathbf{x} \to p_\tau(\mathbf{x}, d)$ is measurable on Γ_3, or any $d \in \mathbb{R}$.
(d) The mapping $\mathbf{x} \to p_\tau(\mathbf{x}, 0) \in L^2(\Gamma_3)$.

(A7) The mass density satisfies:

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(\mathbf{x}) \geq \rho^*, \text{ a.e. } \mathbf{x} \in \Omega.$$

(A8) The adhesion coefficient and the limit bound satisfy:

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_a \geq 0.$$

(A9) The body forces and surface tractions have the regularity

$$\mathbf{f}_0 \in L^2(0, T; H), \quad \mathbf{f}_2 \in L^2\big(0, T; L^2(\Gamma_2)^d\big).$$

(A10) Finally, we assume that the initial data satisfy:
(a) $\mathbf{u}_0 \in V$, $\mathbf{v}_0 \in H$,
(b) $\alpha_0 \in K$,
(c) $\beta_0 \in L^2(\Gamma_3)$, $0 \leq \beta_0 \leq 1$, a.e. on Γ_3.

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi \cdot \nabla\varphi \,\mathrm{d}x. \tag{20}$$

We will use a modified inner product on the Hilbert space $H = L^2(\Omega)^d$ given by

$$\big((\mathbf{u}, \mathbf{v})\big)_H = (\rho\mathbf{u}, \mathbf{v})_H \quad \forall \mathbf{u}, \mathbf{v} \in H,$$

that is, it is weighted with ρ, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$\|\mathbf{v}\|_H = (\rho\mathbf{v}, \mathbf{v})_H^{1/2} \quad \forall \mathbf{v} \in H.$$

It follows from assumptions (A7) that $\|\cdot\|_H$ and $|\cdot|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $(.\,,.)_{V'\times V}$ to represent the duality pairing between V' and V and recall that

$$(\mathbf{u}, \mathbf{v})_{V'\times V} = \big((\mathbf{u}, \mathbf{v})\big)_H \quad \forall \mathbf{u} \in H, \ \forall \mathbf{v} \in V.$$

Assumptions (A9) allow us, for a.e. $t \in (0, T)$, to define $\mathbf{f}(t) \in V'$ by

$$\big(\mathbf{f}(t), \mathbf{v}\big)_{V'\times V} = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \,\mathrm{d}a \quad \forall \mathbf{v} \in V, \tag{21}$$

and note that

$$\mathbf{f} \in L^2(0, T; V'). \tag{22}$$

Finally, we consider the adhesion functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ defined by

$$j(\beta, \mathbf{u}, \mathbf{v}) = \int\limits_{\Gamma_3} p_\nu(u_\nu) v_\nu \,\mathrm{d}a + \int\limits_{\Gamma_3} \bigl(-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + p_\tau(\beta) \mathbf{R}_\tau(\mathbf{u}_\tau) \cdot \mathbf{v}_\tau\bigr) \,\mathrm{d}a. \quad (23)$$

Keeping in mind (A5) and (A6), we observe that integrals in (23) are well defined. Using standard arguments based on Green's formula (6), we can derive the following variational formulation of the frictionless problem with normal compliance (7)–(17) as follows.

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to V$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, a damage field $\alpha : [0,T] \to H^1(\Omega)$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}\bigl(\dot{\mathbf{u}}(t)\bigr) + \mathcal{E}\boldsymbol{\varepsilon}\bigl(\mathbf{u}(t)\bigr) \\ &\quad + \int\limits_0^t \mathcal{G}\bigl(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}\bigl(\dot{\mathbf{u}}(s)\bigr), \boldsymbol{\varepsilon}\bigl(\mathbf{u}(s)\bigr), \alpha(s)\bigr) \,\mathrm{d}s, \quad \text{a.e. } t \in (0,T), \end{aligned} \quad (24)$$

$$\begin{aligned} &\bigl(\ddot{\mathbf{u}}(t), \mathbf{v}\bigr)_{V' \times V} + \bigl(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v})\bigr)_{\mathcal{H}} + j\bigl(\beta(t), \mathbf{u}(t), \mathbf{v}\bigr) = \bigl(\mathbf{f}(t), \mathbf{v}\bigr)_{V' \times V} \quad \forall \mathbf{v} \in V, \\ &\quad \text{a.e. } t \in (0,T), \end{aligned} \quad (25)$$

$$\begin{aligned} &\alpha(t) \in K, \quad \bigl(\dot{\alpha}(t), \xi - \alpha(t)\bigr)_{L^2(\Omega)} + a\bigl(\alpha(t), \xi - \alpha(t)\bigr) \\ &\qquad \geq \bigl(\phi(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}\bigl(\dot{\mathbf{u}}(t)\bigr), \boldsymbol{\varepsilon}\bigl(\mathbf{u}(t)\bigr), \alpha(t)), \xi - \alpha(t)\bigr)_{L^2(\Omega)} \quad \forall \xi \in K, \\ &\quad \text{a.e. } t \in (0,T), \end{aligned} \quad (26)$$

$$\dot{\beta}(t) = -\bigl(\beta(t)\bigl(\gamma_\nu\bigl(R_\nu\bigl(u_\nu(t)\bigr)\bigr)^2 + \gamma_\tau\bigl|\mathbf{R}_\tau\bigl(\mathbf{u}_\tau(t)\bigr)\bigr|^2\bigr) - \epsilon_a\bigr)_+, \quad \text{a.e. } t \in (0,T), \quad (27)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0. \quad (28)$$

The existence of the unique solution to Problem PV is proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 1. We note that, in Problem P and in Problem PV we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, Eq. (27) guarantees that $\beta(\mathbf{x}, t) \leq \beta_0(\mathbf{x})$ and, therefore, assumption (A10c) shows that $\beta(\mathbf{x}, t) \leq 1$ for $t \geq 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\beta(\mathbf{x}, t_0) = 0$ at time t_0, then it follows from (27) that $\dot{\beta}(\mathbf{x}, t) = 0$ for all $t \geq t_0$ and therefore, $\beta(\mathbf{x}, t) = 0$ for all $t \geq t_0$, a.e. $\mathbf{x} \in \Gamma_3$. We conclude that $0 \leq \beta(\mathbf{x}, t) \leq 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.

4 Existence and uniqueness result

The main result in this section is the following existence and uniqueness result.

Theorem 1. *Let the assumptions* (A1)–(A10) *hold. Then Problem* PV *has a unique solution* $(\mathbf{u}, \boldsymbol{\sigma}, \alpha, \beta)$ *which satisfies*

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \quad \ddot{\mathbf{u}} \in L^2\bigl(0,T;V'\bigr), \quad (29)$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \operatorname{Div}\boldsymbol{\sigma} \in L^2\big(0,T;V'\big), \tag{30}$$

$$\alpha \in W^{1,2}\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big), \tag{31}$$

$$\beta \in W^{1,\infty}\big(0,T;L^2(\Gamma_3)\big) \cap Z. \tag{32}$$

A quadruplet $(\mathbf{u},\boldsymbol{\sigma},\alpha,\beta)$ which satisfies (24)–(28) is called a weak solution to the compliance contact Problem P. We conclude that under the stated assumptions, problem (7)–(17) has a unique weak solution satisfying (29)–(32). We turn now to the proof of Theorem 1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in the following that (A1)–(A10) hold. Below, C denotes a generic positive constant which may depend on $\Omega, \Gamma_1, \Gamma_2, \Gamma_3, \mathcal{A}, \mathcal{E}, \mathcal{G}, \phi, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau L$, and T but does not depend on t nor of the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity we suppress in what follows the explicit dependence of various functions on $\mathbf{x} \in \Omega \cup \Gamma$.

Let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given. In the first step we consider the following variational problem.

Problem PV$_{\boldsymbol{\eta}}$. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that

$$\big(\ddot{\mathbf{u}}_\eta(t),\mathbf{v}\big)_{V'\times V} + \big(\mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}_\eta(t)\big),\boldsymbol{\varepsilon}(\mathbf{v})\big)_{\mathcal{H}} + \big(\boldsymbol{\eta}(t),\mathbf{v}\big)_{V'\times V} = \big(\mathbf{f}(t),\mathbf{v}\big)_{V'\times V}$$
$$\forall \mathbf{v} \in V, \ \text{ a.e. } t \in (0,T), \tag{33}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0. \tag{34}$$

In the study of problem PV_η we have the following result.

Lemma 1. *Problem* PV$_{\boldsymbol{\eta}}$ *has a unique solution with the regularity expressed in* (29).

Proof. We use an abstract existence and uniqueness result which may be found in [5, p. 48] or in [30] and proceed like in [5, p. 105]. $\square$

In the second step we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 1 and we consider the following initial-value problem.

Problem PV$_{\boldsymbol{\beta}}$. Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = -\Big(\beta_\eta(t)\Big(\gamma_\nu\big(R_\nu\big(u_{\eta\nu}(t)\big)\big)^2 + \gamma_\tau\big|\mathbf{R}_\tau\big(\mathbf{u}_{\eta\tau}(t)\big)\big|^2\Big) - \epsilon_a\Big)_+, \tag{35}$$

$$\beta_\eta(0) = \beta_0. \tag{36}$$

We have the following result.

Lemma 2. *There exists a unique solution* $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$ *to Problem* PV$_{\boldsymbol{\beta}}$.

Proof. For the sake of simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\big(\beta(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau\big|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t))\big|^2) - \epsilon_a\big)_+ \tag{37}$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\mathbf{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of the classical Cauchy–Lipschitz theorem given in [31, p. 60] we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to Problem PV$_{\boldsymbol{\beta}}$. Also, the arguments used in Remark 1 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\beta_\eta \in Z$, which concludes the proof of Lemma 2. □

In the third step we let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem PV$_{\boldsymbol{\theta}}$. Find a damage field $\alpha_\theta : [0,T] \to H^1(\Omega)$ such that

$$\begin{aligned} &\alpha_\theta(t) \in K, \quad \big(\dot\alpha_\theta(t), \xi - \alpha_\theta(t)\big)_{L^2(\Omega)} + a\big(\alpha_\theta(t), \xi - \alpha_\theta(t)\big) \\ &\qquad\qquad \ge \big(\theta(t), \xi - \alpha_\theta(t)\big)_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T), \end{aligned} \tag{38}$$

$$\alpha_\theta(0) = \alpha_0. \tag{39}$$

In the study of Problem PV$_{\boldsymbol{\theta}}$ we have the following result.

Lemma 3. *Problem* PV$_{\boldsymbol{\theta}}$ *has a unique solution* α_θ *satisfying*

$$\alpha_\theta \in H^1\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big). \tag{40}$$

Proof. We use a standard result for parabolic variational inequalities (see, e.g., [30, p. 124] or [5, p. 47]. □

Now we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 1 and α_θ obtained in Lemma 3 to construct the following Cauchy problem for the stress field.

Problem PV$_{\boldsymbol{\eta\theta}}$. Find a stress field $\boldsymbol{\sigma}_{\eta\theta} : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\theta}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\mathbf{u}_\eta(t)\big) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}_{\eta\theta}(s), \boldsymbol{\varepsilon}\big(\mathbf{u}_\eta(s)\big), \alpha_\theta(s)\big)\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{41}$$

In the study of Problem PV$_{\boldsymbol{\eta\theta}}$ we have the following result.

Lemma 4. *There exists a unique solution to Problem* $\mathrm{PV}_{\boldsymbol{\eta}\theta}$ *and it satisfies* $\boldsymbol{\sigma}_{\eta\theta} \in W^{1,2}(0,T,\mathcal{H})$. *Moreover, if* $\boldsymbol{\sigma}_i$, $\mathbf{u}_i$ *and* α_i *represent the solutions of Problem* $\mathrm{PV}_{\eta_i\theta_i}$, PV_{η_i} *and* PV_{θ_i}, *respectively, for* $(\boldsymbol{\eta}_i,\theta_i) \in L^2(0,T;V' \times L^2(\Omega))$, $i=1,2$, *then there exists* $C>0$ *such that*

$$\bigl|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\bigr|_{\mathcal{H}}^2 \le C\left(\bigl|\mathbf{u}_1(t)-\mathbf{u}_2(t)\bigr|_V^2 + \int_0^t \bigl|\mathbf{u}_1(s)-\mathbf{u}_2(s)\bigr|_V^2\,\mathrm{d}s + \int_0^t \bigl|\alpha_1(s)-\alpha_2(s)\bigr|_{L^2(\Omega)}^2\,\mathrm{d}s\right) \quad \forall t\in[0,T]. \tag{42}$$

Proof. Let $\Lambda_{\eta\theta} : L^2(0,T,\mathcal{H}) \to L^2(0,T,\mathcal{H})$ be the operator given by

$$\Lambda_{\eta\theta}\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\bigl(\mathbf{u}_\eta(t)\bigr) + \int_0^t \mathcal{G}\bigl(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}\bigl(\mathbf{u}_\eta(s)\bigr),\alpha_\theta(s)\bigr)\,\mathrm{d}s \tag{43}$$

for all $\boldsymbol{\sigma} \in L^2(0,T,\mathcal{H})$ and $t\in[0,T]$. For $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2 \in L^2(0,T,\mathcal{H})$ we use (43) and (A3) to obtain for all $t\in[0,T]$

$$\bigl|\Lambda_{\eta\theta}\boldsymbol{\sigma}_1(t) - \Lambda_\eta\boldsymbol{\sigma}_2(t)\bigr|_{\mathcal{H}} \le L_{\mathcal{G}}\int_0^t \bigl|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\bigr|_{\mathcal{H}}\,\mathrm{d}.$$

It follows from this inequality that for p large enough, a power $\Lambda_{\eta\theta}^p$ of the operator $\Lambda_{\eta\theta}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\eta\theta} \in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\theta}\boldsymbol{\sigma}_{\eta\theta} = \boldsymbol{\sigma}_{\eta\theta}$. Moreover, $\boldsymbol{\sigma}_{\eta\theta}$ is the unique solution to Problem $\mathrm{PV}_{\boldsymbol{\eta}\theta}$ and, using (41), the regularity of $\mathbf{u}_\eta$, the regularity of α_θ and the properties of the operators $\mathcal{E}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_{\eta\theta} \in W^{1,2}(0,T,\mathcal{H})$. Consider now $(\boldsymbol{\eta}_1,\theta_1),(\boldsymbol{\eta}_2,\theta_2) \in L^2(0,T;V'\times L^2(\Omega))$ and for $i=1,2$, denote $\mathbf{u}_{\eta_i}=\mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta_i\theta_i}=\boldsymbol{\sigma}_i$ and $\alpha_{\theta_i}=\alpha_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{E}\boldsymbol{\varepsilon}\bigl(\mathbf{u}_i(t)\bigr) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_i(s),\boldsymbol{\varepsilon}\bigl(\mathbf{u}_i(s)\bigr),\alpha_i(s))\,\mathrm{d}s \quad \forall t\in[0,T],$$

and, using the properties (A2) and (A3) of $\mathcal{E}$ and $\mathcal{G}$, we find

$$\begin{aligned}\bigl|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\bigr|_{\mathcal{H}}^2 &\le C\left(\bigl|\mathbf{u}_1(t)-\mathbf{u}_2(t)\bigr|_V^2 + \int_0^t \bigl|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\bigr|_{\mathcal{H}}^2\,\mathrm{d}s \right.\\ &\quad\left. + \int_0^t \bigl|\mathbf{u}_1(s)-\mathbf{u}_2(s)\bigr|_V^2\,\mathrm{d}s + \int_0^t \bigl|\alpha_1(s)-\alpha_2(s)\bigr|_{L^2(\Omega)}^2\,\mathrm{d}s\right) \quad \forall t\in[0,T]. \end{aligned} \tag{44}$$

We use Gronwall argument in the obtained inequality to deduce (42), which concludes the proof of Lemma 4. $\square$

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$, the operator $\mathcal{E}$, the functional j and the function ϕ, for $t \in [0, T]$, we consider the element

$$\Lambda(\boldsymbol{\eta}, \theta)(t) = \big(\Lambda^1(\boldsymbol{\eta}, \theta)(t), \Lambda^2(\boldsymbol{\eta}, \theta)(t)\big) \in V' \times L^2(\Omega), \tag{45}$$

defined by the equations

$$\begin{aligned}
&\big(\Lambda^1(\boldsymbol{\eta}, \theta)(t), \mathbf{v}\big)_{V'\times V} \\
&\quad = \big(\mathcal{E}\boldsymbol{\varepsilon}\big(\mathbf{u}_\eta(t)\big), \boldsymbol{\varepsilon}(\mathbf{v})\big)_{\mathcal{H}} + \left(\int_0^t \mathcal{G}\big(\boldsymbol{\sigma}_{\eta\theta}(s), \boldsymbol{\varepsilon}\big(\mathbf{u}_\eta(s)\big), \alpha_\theta(s)\big)\,\mathrm{d}s,\ \boldsymbol{\varepsilon}(\mathbf{v})\right)_{\mathcal{H}} \\
&\qquad + j\big(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}\big) \quad \forall \mathbf{v} \in V, \qquad (46)\\
&\Lambda^2(\boldsymbol{\eta}, \theta)(t) = \phi\big(\boldsymbol{\sigma}_{\eta\theta}(t), \boldsymbol{\varepsilon}\big(\mathbf{u}_\eta(t)\big), \alpha_\theta(t)\big). \qquad (47)
\end{aligned}$$

Here, for every $(\boldsymbol{\eta}, \theta) \in L^2(0, T; V' \times L^2(\Omega))$ $\mathbf{u}_\eta, \beta_\eta, \alpha_\theta$ and $\boldsymbol{\sigma}_{\eta\theta}$ represent the displacement field, the bonding field, the damage field and the stress field obtained in Lemmas 1, 2, 3 and 4 respectively. We have the following result.

Lemma 5. *The operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0, T; V' \times L^2(\Omega))$ such that $\Lambda(\boldsymbol{\eta}^*, \theta^*) = (\boldsymbol{\eta}^*, \theta^*)$.*

Proof. Let now $(\boldsymbol{\eta}_1, \theta_1), (\boldsymbol{\eta}_2, \theta_2) \in L^2(0, T; V' \times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i$, $\boldsymbol{\sigma}_{\eta_i\theta_i} = \boldsymbol{\sigma}_i$, $\alpha_{\theta_i} = \alpha_i$ and $\beta_{\eta_i} = \beta_i$ for $i = 1, 2$. Using (19), (A2), (A3), (A5), (A6), the definition of R_ν, $\mathbf{R}_\tau$ and the Remark 1, we have

$$\begin{aligned}
&\big|\Lambda^1(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2, \theta_2)(t)\big|_{V'}^2 \\
&\quad \le \big|\mathcal{E}\boldsymbol{\varepsilon}\big(\mathbf{u}_1(t)\big) - \mathcal{E}\boldsymbol{\varepsilon}\big(\mathbf{u}_2(t)\big)\big|_{\mathcal{H}}^2 \\
&\qquad + \int_0^t \big|\mathcal{G}\big(\boldsymbol{\sigma}_1(s), \boldsymbol{\varepsilon}\big(\mathbf{u}_1(s)\big), \alpha_1(s)\big) - \mathcal{G}\big(\boldsymbol{\sigma}_2(s), \boldsymbol{\varepsilon}\big(\mathbf{u}_2(s)\big), \alpha_2(s)\big)\big|_{\mathcal{H}}^2\,\mathrm{d}s \\
&\qquad + C\big|p_\nu\big(u_{1\eta\nu}(t)\big) - p_\nu\big(u_{2\eta\nu}(t)\big)\big|_{L^2(\Gamma_3)}^2 \\
&\qquad + C\big|\beta_1^2(t)R_\nu\big(u_{1\eta\nu}(t)\big) - \beta_2^2(t)R_\nu\big(u_{1\eta\nu}(t)\big)\big|_{L^2(\Gamma_3)}^2 \\
&\qquad + C\big|p_\tau\big(\beta_1(t)\big)\mathbf{R}_\tau\big(\mathbf{u}_{1\eta\tau}(t)\big) - p_\tau\big(\beta_2(t)\big)\mathbf{R}_\tau\big(\mathbf{u}_{1\eta\tau}(t)\big)\big|_{L^2(\Gamma_3)}^2 \\
&\quad \le C\Bigg(\big|\mathbf{u}_1(t) - \mathbf{u}_2(t)\big|_V^2 + \big|\beta_1(t) - \beta_2(t)\big|_{L^2(\Gamma_3)}^2 + \int_0^t \big|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\big|_{\mathcal{H}}^2\,\mathrm{d}s \\
&\qquad + \int_0^t \big|\alpha_1(s) - \alpha_2(s)\big|_{L^2(\Omega)}^2\,\mathrm{d}s + \int_0^t \big|\mathbf{u}_1(s) - \mathbf{u}_2(s)\big|_V^2\,\mathrm{d}s\Bigg). \qquad (48)
\end{aligned}$$

We use estimate (42) to obtain

$$\begin{aligned} &\left|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\right|_{V'}^2 \\ &\quad\le C\Biggl(\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2+\int_0^t\left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|_V^2\,\mathrm{d}s \\ &\qquad+\int_0^t\left|\alpha_1(s)-\alpha_2(s)\right|_{L^2(\Omega)}^2\,\mathrm{d}s+\left|\beta_1(t)-\beta_2(t)\right|_{L^2(\Gamma_3)}^2\Biggr). \end{aligned} \tag{49}$$

Recall that above $u_{\eta\nu}$ and $\mathbf{u}_{\eta\tau}$ denote the normal and the tangential component of the function $\mathbf{u}_\eta$ respectively. By similar arguments, from (47), (42) and (A4) it follows that

$$\begin{aligned} &\left|\Lambda^2(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^2(\boldsymbol{\eta}_2,\theta_2)(t)\right|_{L^2(\Omega)}^2 \\ &\quad\le C\Biggl(\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2+\int_0^t\left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|_V^2\,\mathrm{d}s \\ &\qquad+\left|\alpha_1(t)-\alpha_2(t)\right|_{L^2(\Omega)}^2+\int_0^t\left|\alpha_1(s)-\alpha_2(s)\right|_{L^2(\Omega)}^2\,\mathrm{d}s\Biggr). \end{aligned} \tag{50}$$

Therefore,

$$\begin{aligned} &\left|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\right|_{V'\times L^2(\Omega)}^2 \\ &\quad\le C\Biggl(\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2+\int_0^t\left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|_V^2\,\mathrm{d}s+\int_0^t\left|\alpha_1(s)-\alpha_2(s)\right|_{L^2(\Omega)}^2\,\mathrm{d}s \\ &\qquad+\left|\alpha_1(t)-\alpha_2(t)\right|_{L^2(\Omega)}^2+\left|\beta_1(t)-\beta_2(t)\right|_{L^2(\Gamma_3)}^2\Biggr). \end{aligned} \tag{51}$$

Moreover, from (33) we obtain

$$\begin{aligned} &(\dot{\mathbf{v}}_1-\dot{\mathbf{v}}_2,\mathbf{v}_1-\mathbf{v}_2)_{V'\times V}+\bigl(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_1)-\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_2),\boldsymbol{\varepsilon}(\mathbf{v}_1-\mathbf{v}_2)\bigr)_{\mathcal{H}} \\ &\quad+(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\mathbf{v}_1-\mathbf{v}_2)_{V'\times V}=0. \end{aligned}$$

We integrate this equality with respect to time, we use the initial conditions $\mathbf{v}_1(0)=\mathbf{v}_2(0)=\mathbf{v}_0$ and condition (A1) to find

$$m_{\mathcal{A}}\int_0^t\left|\mathbf{v}_1(s)-\mathbf{v}_2(s)\right|_V^2\,\mathrm{d}s\le-\int_0^t\bigl(\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s),\mathbf{v}_1(s)-\mathbf{v}_2(s)\bigr)_{V'\times V}\,\mathrm{d}s$$

for all $t\in[0,T]$. Then, using the inequality $2ab\le a^2/m_{\mathcal{A}}+m_{\mathcal{A}}b^2$ we obtain

$$\int_0^t\left|\mathbf{v}_1(s)-\mathbf{v}_2(s)\right|_V^2\,\mathrm{d}s\le C\int_0^t\left|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\right|_{V'}^2\,\mathrm{d}s\quad\forall t\in[0,T]. \tag{52}$$

On the other hand, from the Cauchy problem (35)–(36) we can write

$$\beta_i(t) = \beta_0 - \int_0^t \left(\beta_i(s)\left(\gamma_\nu\left(R_\nu\left(u_{i\nu}(s)\right)\right)^2 + \gamma_\tau\left|\mathbf{R}_\tau\left(\mathbf{u}_{i\tau}(s)\right)\right|^2\right) - \epsilon_a\right)_+ \mathrm{d}s,$$

and then

$$\begin{aligned}
&\left|\beta_1(t) - \beta_2(t)\right|_{L^2(\Gamma_3)} \\
&\quad \le C \int_0^t \left|\beta_1(s)\left(R_\nu\left(u_{1\nu}(s)\right)\right)^2 - \beta_2(s)\left(R_\nu\left(u_{2\nu}(s)\right)\right)^2\right|_{L^2(\Gamma_3)} \mathrm{d}s \\
&\qquad + C \int_0^t \left|\beta_1(s)\left|\mathbf{R}_\tau\left(\mathbf{u}_{1\tau}(s)\right)\right|^2 - \beta_2(s)\left|\mathbf{R}_\tau\left(\mathbf{u}_{2\tau}(s)\right)\right|^2\right|_{L^2(\Gamma_3)} \mathrm{d}s.
\end{aligned}$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\begin{aligned}
&\left|\beta_1(t) - \beta_2(t)\right|_{L^2(\Gamma_3)} \\
&\quad \le C\left(\int_0^t \left|\beta_1(s) - \beta_2(s)\right|_{L^2(\Gamma_3)} \mathrm{d}s + \int_0^t \left|\mathbf{u}_1(s) - \mathbf{u}_2(s)\right|_{L^2(\Gamma_3)^d} \mathrm{d}s\right).
\end{aligned} \tag{53}$$

Next, we apply Gronwall's inequality to deduce

$$\left|\beta_1(t) - \beta_2(t)\right|_{L^2(\Gamma_3)} \le C \int_0^t \left|\mathbf{u}_1(s) - \mathbf{u}_2(s)\right|_{L^2(\Gamma_3)^d} \mathrm{d}s,$$

and from the relation (19) we obtain

$$\left|\beta_1(t) - \beta_2(t)\right|_{L^2(\Gamma_3)}^2 \le C \int_0^t \left|\mathbf{u}_1(s) - \mathbf{u}_2(s)\right|_V^2 \mathrm{d}s. \tag{54}$$

From (38) we deduce that

$$\begin{aligned}
&(\dot{\alpha_1} - \dot{\alpha_2}, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \\
&\quad \le (\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \quad \text{a.e. } t \in (0, T).
\end{aligned}$$

Integrating the previous inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$ and inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \ge 0$ to find

$$\frac{1}{2}\left|\alpha_1(t) - \alpha_2(t)\right|_{L^2(\Omega)}^2 \le \int_0^t \left(\theta_1(s) - \theta_2(s), \alpha_1(s) - \alpha_2(s)\right)_{L^2(\Omega)} \mathrm{d}s,$$

which implies that

$$\left|\alpha_1(t)-\alpha_2(t)\right|^2_{L^2(\Omega)} \le \int_0^t \left|\theta_1(s)-\theta_2(s)\right|^2_{L^2(\Omega)} \,\mathrm{d}s + \int_0^t \left|\alpha_1(s)-\alpha_2(s)\right|^2_{L^2(\Omega)} \,\mathrm{d}s.$$

This inequality, combined with Gronwall's inequality, leads to

$$\left|\alpha_1(t)-\alpha_2(t)\right|^2_{L^2(\Omega)} \le C\int_0^t \left|\theta_1(s)-\theta_2(s)\right|^2_{L^2(\Omega)} \,\mathrm{d}s \quad \forall t\in[0,T]. \tag{55}$$

We substitute (54) in (51) to obtain

$$\begin{aligned}
&\left|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\right|^2_{V'\times L^2(\Omega)} \\
&\quad\le C\Biggl(\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|^2_V + \int_0^t \left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|^2_V \,\mathrm{d}s \\
&\qquad + \left|\alpha_1(t)-\alpha_2(t)\right|^2_{L^2(\Omega)} + \int_0^t \left|\alpha_1(s)-\alpha_2(s)\right|^2_{L^2(\Omega)} \,\mathrm{d}s\Biggr) \\
&\quad\le C\Biggl(\int_0^t \left|\mathbf{v}_1(s)-\mathbf{v}_2(s)\right|^2_V \,\mathrm{d}s + \left|\alpha_1(t)-\alpha_2(t)\right|^2_{L^2(\Omega)} \\
&\qquad + \int_0^t \left|\alpha_1(s)-\alpha_2(s)\right|^2_{L^2(\Omega)} \,\mathrm{d}s\Biggr).
\end{aligned}$$

From the previous inequality and estimates (52) and (55), it follows now that

$$\begin{aligned}
&\left|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\right|^2_{V'\times L^2(\Omega)} \\
&\quad\le C\int_0^t \left|\bigl(\boldsymbol{\eta}_1,\theta_1\bigr)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\right|^2_{V'\times L^2(\Omega)} \,\mathrm{d}s.
\end{aligned}$$

Reiterating this inequality m times leads to

$$\begin{aligned}
&\left|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)\right|^2_{L^2(0,T;V'\times L^2(\Omega))} \\
&\quad\le \frac{C^m T^m}{m!}\left|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\right|^2_{L^2(0,T;V'\times L^2(\Omega))}.
\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0,T;V'\times L^2(\Omega))$, and so Λ has a unique fixed point. $\square$

Now, we have all the ingredients to prove Theorem 1.

Proof. Let $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0,T;V' \times L^2(\Omega))$ be the fixed point of Λ defined by (45)–(47) and denote

$$\mathbf{u} = \mathbf{u}_{\eta^*}, \quad \boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \boldsymbol{\sigma}_{\eta^*\theta^*}, \tag{56}$$

$$\beta = \beta_{\eta^*}, \quad \alpha = \alpha_{\theta^*}. \tag{57}$$

We prove that the quadruplet $(\mathbf{u}, \boldsymbol{\sigma}, \alpha, \beta)$ satisfies (24)–(28) and (29)–(32). Indeed, we write (41) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$, $\theta = \theta^*$ and use (56)–(57) to obtain that (24) is satisfied. Now we consider (33) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use the first equality in (56) to find

$$\begin{aligned}&\big(\ddot{\mathbf{u}}(t), \mathbf{v}\big)_{V' \times V} + \big(\mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}(t)\big), \boldsymbol{\varepsilon}(\mathbf{v})\big)_{\mathcal{H}} + \big(\boldsymbol{\eta}^*(t), \mathbf{v}\big)_{V' \times V} = \big(\mathbf{f}(t), \mathbf{v}\big)_{V' \times V} \\ &\quad \forall \mathbf{v} \in V, \ \text{ a.e. } t \in (0,T).\end{aligned} \tag{58}$$

Equalities $\Lambda^1(\boldsymbol{\eta}^*, \theta^*) = \boldsymbol{\eta}^*$ and $\Lambda^2(\boldsymbol{\eta}^*, \theta^*) = \theta^*$ combined with (46), (47), (56) and (57) show that

$$\begin{aligned}&\big(\boldsymbol{\eta}^*(t), \mathbf{v}\big)_{V' \times V} \\ &\quad = \big(\mathcal{E}\boldsymbol{\varepsilon}\big(\mathbf{u}(t)\big), \boldsymbol{\varepsilon}(\mathbf{v})\big)_{\mathcal{H}} + \left(\int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}(s)\big), \boldsymbol{\varepsilon}\big(\mathbf{u}(s)\big), \alpha(s)\big)\,\mathrm{d}s, \boldsymbol{\varepsilon}(\mathbf{v})\right)_{\mathcal{H}} \\ &\qquad + j\big(\beta(t), \mathbf{u}(t), \mathbf{v}\big) \quad \forall \mathbf{v} \in V,\end{aligned} \tag{59}$$

$$\theta^*(t) = \phi\big(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}\big(\dot{\mathbf{u}}(t)\big), \boldsymbol{\varepsilon}\big(\mathbf{u}(t)\big), \alpha(t)\big). \tag{60}$$

We now substitute (59) in (58) and use (24) to see that (25) is satisfied. We write (38) for $\theta = \theta^*$ and use (57) and (60) to find that (26) is satisfied. We consider now (35) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (56)–(57) to obtain that (27) is satisfied. Next, (28) and the regularities (29), (31) and (32) follow Lemmas 1, 2 and 3. The regularity $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$ follows from Lemmas 1, 4, the second equality in (56) and (A1). Finally (25) implies that

$$\rho\ddot{\mathbf{u}}(t) = \operatorname{Div}\boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } V', \ \text{ a.e. } t \in (0,T),$$

and therefore by (A7) and (A9) we obtain that $\operatorname{Div}\boldsymbol{\sigma} \in L^2(0,T;V')$. We deduce that the regularity (30) holds which concludes the existence part of the theorem. The uniqueness part of Theorem 1 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (45)–(47) and the unique solvability of the Problems $\mathrm{PV}_{\boldsymbol{\eta}}$, $\mathrm{PV}_{\boldsymbol{\beta}}$, $\mathrm{PV}_{\boldsymbol{\theta}}$ and $\mathrm{PV}_{\boldsymbol{\eta}\theta}$. □

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