Exact solutions to the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity by using the first integral method

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Abstract. The first integral method is an efficient method for obtaining exact solutions of some nonlinear partial differential equations. In this paper, the first integral method is used to construct exact solutions of the perturbed nonlinear Schrödinger's equation (NLSE) with Kerr law nonlinearity. It is shown that the proposed method is effective and general.

Keywords: first integral method, nonlinear PDE, NLSE with Kerr law nonlinearity.

1 Introduction

The nonlinear equations of mathematical physics are major subjects in physical science [1]. Recently many new approaches for finding the exact solutions to nonlinear wave equations have been proposed, for example, tanh-sech method [2–4], extended tanh method [5–7], sine-cosine method [8–10], homogeneous balance method [11, 12], Jacobi elliptic function method [13–16], F-expansion method [17–19], exp-function method [20, 21], trigonometric function series method [22], (G'/G)-expansion method [23, 24] and so on. All methods mentioned above have limitation in their applications. The first integral method was first proposed by Feng [25] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method is widely used by many such as in [26, 27] and by the reference therein.

In this paper, we will consider the perturbed NLSE with Kerr law nonlinearity [28] with following form:

$$iu_t + u_{xx} + \alpha|u|^2u + i\left[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3\left(|u|^2\right)_x u\right] = 0, \tag{1}$$

where γ_1 is third order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is a also a version of nonlinear dispersion [29, 30]. Eq. (1) describes the propagation of optical solitons in nonlinear optical fibers that exhibits a Kerr law nonlinearity. Eq. (1) has important application in various fields, such as semiconductor materials, optical fiber communications, plasma physics, fluid and solid mechanics. More details are presented [29, 31]. In this paper, we would like to obtain the exact solution of Eq. (1) by using the first integral method.

The rest of this paper is organized as follows. In Section 2, we give the description of the first integral method in Section 3, we apply this method to Eq. (1). Concluding remarks are given in Section 4.

2 The first integral method

Let us consider the nonlinear partial differential equation:

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0. \tag{2}$$

We use the transformations

$$u(x,t) = f(\xi), \tag{3}$$

where $\xi = x - ct$. Based on this we obtain

$$\frac{\partial}{\partial t}(.) = -c\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \ldots. \tag{4}$$

We use (4) to change the nonlinear partial differential equation (2) to nonlinear ordinary differential equation

$$G\biggl(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, \ldots\biggr) = 0. \tag{5}$$

Next, we introduce a new independent variable

$$X(\xi) = f(\xi), \quad Y(\xi) = \frac{\partial f(\xi)}{\partial \xi}, \tag{6}$$

which leads a system of nonlinear ordinary differential equations

$$\begin{aligned} \frac{\partial X(\xi)}{\partial \xi} &= Y(\xi), \\ \frac{\partial Y(\xi)}{\partial \xi} &= F_1\bigl(X(\xi), Y(\xi)\bigr). \end{aligned} \tag{7}$$

By the qualitative theory of ordinary differential equations [23], if we can find the integrals to Eq. (7) under the same conditions, then the general solutions to Eq. (7) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division theorem to obtain one first integral to Eq. (7) which reduces Eq. (7) to a first order integrable ordinary differential equation. An exact solution to Eq. (2) is then obtained by solving this equation. Now, let us recall the Division theorem:

Division theorem. *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$; and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that*

$$Q(w,z) = P(w,z)G(w,z).$$

3 The NLSE with Kerr law nonlinearity equation

In this section we consider the NLSE with Kerr law nonlinearity equation (1).

We seek its traveling wave solution of the form [28]:

$$u(x,t) = \phi(\xi)\exp\bigl(i(Kx - \varOmega t)\bigr), \quad \xi = x - ct. \tag{8}$$

Substituting equation (8) into equation (1), we have

$$\begin{aligned} &i\bigl(\gamma_1\phi''' - 3\gamma_1K^2\phi' + \gamma_2\phi^2\phi' + 2\gamma_3\phi^2\phi' - c\phi' + 2K\phi'\bigr) \\ &\quad + \bigl(\varOmega\phi + \phi'' - K^2\phi + \alpha\phi^3 + 3\gamma_1K\phi'' + \gamma_1K^3\phi - \gamma_2K\phi^3\bigr) = 0, \end{aligned} \tag{9}$$

where γ_i ($i = 1,2,3$), α are positive constants and prime meaning differentiation with respect to ξ. Then we have [28]:

$$A\phi''(\xi) + B\phi(\xi) + C\phi^3(\xi) = 0. \tag{10}$$

Where $A = \gamma_1, B = 2K - c - 3\gamma_1K^2, C = \frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3$.

Using (6) and (7), we get

$$\dot{X}(\xi) = Y(\xi), \tag{11}$$

$$\dot{Y}(\xi) = -\frac{B}{A}X(\xi) - \frac{C}{A}X^3(\xi). \tag{12}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (11) and (12), also

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i\bigl(X(\xi)\bigr)Y^i(\xi) = 0, \tag{13}$$

where $a_i(X)$ $(i = 0, 1, \dots, m)$, are polynomials of X and $a_m(X) \neq 0$. Eq. (13) is called the first integral to (11), (12). Due to the Division theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C[X, Y]$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}\xi} = \frac{\mathrm{d}Q}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}\xi} + \frac{\mathrm{d}Q}{\mathrm{d}Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi} = \big(g(X) + h(X)Y\big)\sum_{i=0}^{m} a_i(X)Y^i. \tag{14}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in (13).

Case A. Suppose that $m = 1$, by comparing with the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (14), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{15}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{16}$$

$$a_1(X)\left[-\frac{B}{A}X - \frac{C}{A}X^3\right] = g(X)a_0(X). \tag{17}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (15) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_0(X)$.

$$a_0(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{18}$$

where A_0 is arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (17) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\begin{aligned} &B_0 = 0, \quad A_1 = \frac{1}{3\gamma_1}\sqrt{-6\gamma_1(\gamma_2 + 2\gamma_3)}, \\ &c = 2K - 3\gamma_1K^2 + \frac{A_0}{3}\sqrt{-6\gamma_1(\gamma_2 + 2\gamma_3)}, \end{aligned} \tag{19}$$

$$\begin{aligned} &B_0 = 0, \quad A_1 = -\frac{1}{3\gamma_1}\sqrt{-6\gamma_1(\gamma_2 + 2\gamma_3)}, \\ &c = 2K - 3\gamma_1K^2 - \frac{A_0}{3}\sqrt{-6\gamma_1(\gamma_2 + 2\gamma_3)}, \end{aligned} \tag{20}$$

where A_0 is arbitrary constant.

Using the conditions (19) in (13), we obtain

$$Y(\xi) = -\frac{1}{6\gamma_1}\sqrt{-6\gamma_1(\gamma_2 + 2\gamma_3)}X^2(\xi) - A_0. \tag{21}$$

Combining (21) with (11), we obtain the exact solution to equation (10) and then the exact solution to Eq. (1) can be written as

$$u(x,t) = -\left\{\sqrt{\frac{6A_0\gamma_1}{\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}}}\tan\left[\sqrt{\frac{A_0\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}}{6\gamma_1}}\right.\right.$$
$$\left.\left.\times\left(x-\left(2K-3\gamma_1K^2+\frac{A_0}{3}\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}\right)t+\xi_0\right)\right]\right\}e^{i(Kx-\Omega t)}. \tag{22}$$

Similarly, in the case of (20), from (13), we obtain

$$Y(\xi) = \frac{1}{6\gamma_1}\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}X^2(\xi) - A_0, \tag{23}$$

and then the exact solution of the Eq. (1) can be written as

$$u(x,t) = -\left\{\sqrt{\frac{6A_0\gamma_1}{\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}}}\tanh\left[\sqrt{\frac{A_0\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}}{6\gamma_1}}\right.\right.$$
$$\left.\left.\times\left(x-\left(2K-3\gamma_1K^2-\frac{A_0}{3}\sqrt{-6\gamma_1(\gamma_2+2\gamma_3)}\right)t+\xi_0\right)\right]\right\}e^{i(Kx-\Omega t)}, \tag{24}$$

where ξ_0 is arbitrary constant.

Case B. Suppose that $m = 2$, by equating the coefficients of Y^i $(i = 3,2,1,0)$ on both sides of (14), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{25}$$
$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{26}$$
$$\dot{a}_0(X) = 2a_2(X)\left[\frac{B}{A}X + \frac{C}{A}X^3\right] + g(X)a_1(X) + h(X)a_0(X), \tag{27}$$
$$a_1(X)\left[-\frac{B}{A}X - \frac{C}{A}X^3\right] = g(X)a_0(X). \tag{28}$$

Since $a_i(X)$ $(i = 0,1,2)$ are polynomials, then from (25) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$a_1(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{29}$$
$$a_0(X) = d + B_0A_0X + \left(\frac{2K-3\gamma_1K^2-c}{\gamma_1} + \frac{B_0^2}{2} + \frac{A_0A_1}{2}\right)X^2$$
$$+\frac{1}{2}B_0A_1X^3 + \left(\frac{A_1^2}{8} + \frac{\gamma_2+2\gamma_3}{6\gamma_1}\right)X^4. \tag{30}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (28) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\begin{aligned}
d &= -\frac{3}{2}\frac{9\gamma_1^2K^4 + 6\gamma_1K^2c - 12\gamma_1K^3 + 4K^2 + c^2 - 4Kc}{\gamma_1(\gamma_2 + 2\gamma_3)},\\
A_1 &= \pm\frac{2}{3}\frac{\sqrt{-\gamma_1(6\gamma_2 + 12\gamma_3)}}{\gamma_1},\\
A_0 &= \pm\frac{(-2K + 3\gamma_1K^2 + c)\sqrt{-\gamma_1(6\gamma_2 + 12\gamma_3)}}{\gamma_1(\gamma_2 + 2\gamma_3)}.
\end{aligned} \tag{31}$$

Using the conditions (31) into (13), we get

$$Y(\xi) = \pm\frac{\sqrt{-6\gamma_1(\gamma_2 + 2\gamma_3)}}{6\gamma_1(\gamma_2 + 2\gamma_3)}\big[-6K + 9\gamma_1K^2 + 3c - (\gamma_2 + 2\gamma_3)X^2(\xi)\big]. \tag{32}$$

Combining (32) with (11), we obtain the exact solution to equation (10) and then the exact solutions to the Eq. (1) can be written as

$$\begin{aligned}
u(x,t) = \pm\Bigg\{&\sqrt{\frac{3(-2K + 3\gamma_1K^2 + c)}{(\gamma_2 + 2\gamma_3)}}\\
&\times \tanh\Bigg[\sqrt{\frac{(2K - 3\gamma_1K^2 - c)}{2\gamma_1}}(x - ct + \xi_0)\Bigg]\Bigg\}e^{i(Kx - \Omega t)},
\end{aligned} \tag{33}$$

where ξ_0 is an arbitrary constant.

4 CONCLUSION

In this work, we obtained exact solutions of the perturbed nonlinear Schrrödinger' equation (NLSE) with Kerr law nonlinearity by using the first integral method. The results shown that this method is efficient.

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