

The Kaldor–Kalecki stochastic model of business cycle

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Received: 28 January 2011 / **Revised:** 12 May 2011 / **Published online:** 30 May 2011

Abstract. This paper is concerned with the deterministic and the stochastic delayed Kaldor–Kalecki nonlinear business cycle models of the income. They will take into consideration the investment demand in the form suggested by Rodano. The existence of the Hopf bifurcation is studied and the direction and the local stability of the Hopf bifurcation is also taken into consideration. For the stochastic model, the dynamics of the mean values and the square mean values of the model's variables are set. Numerical examples are given to illustrate our theoretical results.

Keywords: nonlinear business cycle model, Kaldor–Kalecki model, normal form, stochastic system.

1 Introduction

The model proposed by Kaldor [1] is one of earliest and simplest nonlinear models of business cycles. This model cannot be considered as a satisfying description of actual economies. Nevertheless, it continues to generate a considerable amount of economic, pedagogical and methodological interest, for the researchers in applied dynamics and economics.

Kalecki introduced the idea that there is a time delay for investment before a business decision. Krawiek and Sydlowski [2, 3] used the Kalecki's idea into Kaldor's model and considered the Kaldor–Kalecki model of business cycles.

The parameters of the real models are subject to perturbations that can be considered as stochastic or uncertain. Starting with these considerations, the associated stochastic model can be taken into consideration.

In the present paper, we investigate the effects of the random perturbation for the Kaldor–Kalecki model analyzing the steady state of the model with stochastic perturbation.

The analysis of this model is related to the equilibrium point. The obtained results are connected to the stability or the existence of the limit cycle and the existence of the limit cycle for the expected values and variances in the stochastic case.

The reminder of the paper develops as follows. In Section 2, we describe a deterministic Kaldor–Kalecki model using the investment demand proposed in [4]. We set the conditions for the existence of the delay parameter value for which the model displays a Hopf bifurcation. Also, the normal form is given. In Section 3, the stochastic system is presented and the locally asymptotic stability is analyzed according to the mean of variables and the square mean. Numerical simulations are carried out in Section 4. Finally, concluding remarks are given in Section 5.

2 The deterministic model of a Kaldor–Kalecki business cycle with delay

In the last decade, the study of delayed differential equations in business cycles has received much attention. The first model of business cycles can be traced back to Kaldor [1], who used a system of ordinary differential equations to study business cycles in 1940 by proposing nonlinear investment and saving functions so that the system may have a cyclic behavior or limit cycles, which are important from the point of view of economics. Kalecki introduced the idea that there is a time delay for investment before a business decision. Krawiec and Szydłowski incorporated Kalecki’s idea into Kaldor’s model by proposing the following Kaldor–Kalecki model of business cycles [2, 3, 5]:

$$\begin{aligned}\dot{Y}(t) &= \alpha(I(Y(t), K(t)) - S(Y(t), K(t))), \\ \dot{K}(t) &= I(Y(t - \tau), K(t)) - qK(t),\end{aligned}\tag{1}$$

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, $q \in (0, 1)$ is the depreciation rate of capital stock, $I(Y, K)$ and $S(Y, K)$ are investment and saving functions, and $\tau \geq 0$ is a time lag representing delay for investment due to the past investment decision.

The discrete systems associated to the Kaldor and Kaldor–Kalecki models were analyzed in [6, 7].

Consider that past investment decisions also influence the change in the capital stock and the model (1) is extended by imposing delays in both the gross product and capital stock [8, 9]. Thus, by adding the same delay to the capital stock K in the investment function $I(Y, K)$ in the second equation of the system (1), the following Kaldor–Kalecki model business cycles is obtained [8–10]:

$$\begin{aligned}\dot{Y}(t) &= \alpha(I(Y(t), K(t)) - S(Y(t), K(t))), \\ \dot{K}(t) &= I(Y(t - \tau), K(t - \tau)) - qK(t),\end{aligned}\tag{2}$$

As usual in a Keynesian framework, savings are assumed to be proportional to the current level of income:

$$S(Y, K) = pY,\tag{3}$$

where the coefficient p , $p \in (0, 1)$ represents the propensity to save.

As usual, the investment demand is assumed to be an increasing and sigmoid-shaped function of the income. Without loss of generality, in what follows we shall consider the form proposed in [4]:

$$I(Y, K) = pu + r \left(\frac{pu}{q} - K \right) + f(Y - u), \quad (4)$$

where $\frac{pu}{q}$ is the “normal” level of the capital stock u .

In (4), two short-run investment components are considered: the first one is proportional to the difference between normal capital stock and current stock, according to a coefficient r ($r > 0$), usually explained by the presence of adjustment costs; the second one is an increasing, but non-linear function of the difference between current income and its normal level. Function f has the properties $f(0) = 0$ and $f'(0) \neq 0$.

This second component of the short-run investment function is a convenient specification of the sigmoid-shaped relationship between investment and income proposed by Kaldor. We note that this analytic specification does not compromise the generality of the results.

From (2) with (3) and (4) we obtain the following system:

$$\begin{aligned} \dot{Y}(t) &= \alpha \left(pu + r \left(\frac{pu}{q} - K(t) \right) + f(Y(t) - u) - pY(t) \right), \\ \dot{K}(t) &= pu + r \left(\frac{pu}{q} - K(t - \tau) \right) + f(Y(t - \tau) - u) - qK(t). \end{aligned} \quad (5)$$

The system (5) with the initial conditions:

$$Y(\theta) = h_1(\theta), \quad K(\theta) = h_2(\theta), \quad \theta \in [-\tau, 0]$$

and $h_1, h_2 : [-\tau, 0] \rightarrow \mathbb{R}$ of C^1 -class functions, represent a system of differential equations with delay [11].

By carrying out the translation $u_1(t) = Y(t) - u$, $u_2(t) = K(t) - \frac{pu}{q}$ from (5) we get the system:

$$\begin{aligned} \dot{u}_1(t) &= -\alpha pu_1(t) - \alpha ru_2(t) + \alpha f(u_1(t)), \\ \dot{u}_2(t) &= -qu_2(t) - ru_2(t - \tau) + f(u_1(t - \tau)), \\ u_1(\theta) &= h_1(\theta) - u, \quad u_2(\theta) = h_2(\theta) - \frac{pu}{q}, \quad \theta \in [-\tau, 0], \quad \alpha \in (0, 1). \end{aligned} \quad (6)$$

The linearized system of (6) in $(0, 0)^T$ is given by:

$$\begin{aligned} \dot{y}_1(t) &= a_{11}y_1(t) + a_{12}y_2(t), \\ \dot{y}_2(t) &= b_{21}y_1(t - \tau) + b_{22}y_2(t - \tau) + a_{22}y_2(t), \end{aligned} \quad (7)$$

where

$$\begin{aligned} a_{11} &= -\alpha(p - \rho_1), & a_{12} &= \alpha r, & a_{22} &= -q, \\ b_{21} &= \rho_1, & b_{22} &= -r, & \rho_1 &= f'(0). \end{aligned}$$

The characteristic function for (7) is given by:

$$f(\lambda, \tau) = \lambda^2 + b\lambda + c + (d\lambda + g)e^{-\lambda\tau},$$

where $b = \alpha(p - \rho_1) + q$, $c = \alpha q(p - \rho_1)$, $d = r$, $g = r\alpha(p - 2\rho_1)$.

The analysis of the characteristic equation $f(\lambda, \tau) = 0$ with respect to parameter τ is done using methods from [8, 12]. This analysis leads to the following proposition:

Proposition 1. (i) If $\tau = 0$ and

$$\rho_1 < \min \left\{ p + \frac{q+r}{\alpha}, \frac{p(q+r)}{q+2r} \right\}$$

then equation $f(\lambda, 0) = 0$ has roots with a negative real part.

(ii) If $\tau > 0$, then there exists $\tau_0 > 0$ so that equation $f(\lambda, \tau) = 0$ admits the roots $\lambda(\tau_0) = \pm i\omega_0$, where ω_0 is given by:

$$\omega_0 = \sqrt{\frac{d^2 + 2c - b^2 + \sqrt{\Delta}}{2}},$$

where $\Delta = (\alpha^2(p - \rho_1)^2 - q^2)^2 + r^4 + 6r^2\alpha^2(p - \rho_1)^2$ and

$$\tau_0 = \arctan \frac{\omega_0(d\omega_0^2 - cd + bg)}{\omega_0^2(g - bd) + cg}.$$

(iii) The solution of equation $f(\lambda, \tau) = 0$, denoted by $\lambda = \lambda(\tau)$, depends on τ . For $\tau = \tau_0$, $\lambda = i\omega_0$, we have:

$$\begin{aligned} M &:= \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)_{\tau=\tau_0, \lambda=i\omega_0} \\ &= \operatorname{Re} \left(\frac{i\omega_0(d i\omega_0 + g)}{2i\omega_0 + b + (d - d i\omega_0\tau_0 - g\tau_0)e^{-\omega_0\tau_0}} \right), \\ N &:= \operatorname{Im} \left(\frac{d\lambda}{d\tau} \right)_{\tau=\tau_0, \lambda=i\omega_0}. \end{aligned} \quad (8)$$

From Proposition 1, if $M \neq 0$ then $\tau = \tau_0$ is a Hopf bifurcation.

In what follows we analyze the direction and the local stability of the Hopf bifurcation as in [13, 14]. For notational convenience, let $\tau = \tau_0 + \mu$, $\mu \in (-\varepsilon, \varepsilon)$. Then $\varepsilon = 0$ is the Hopf bifurcation value of the system (6). In the study of the Hopf bifurcation problem, first we transform the system (6) into an operator equation of the form:

$$\dot{u}_t = \mathcal{A}(\mu)u_t + \mathcal{R}(u_t), \quad (9)$$

where $u = (u_1, u_2)^T$, $u_t = u(t + \theta)$, $\theta \in [-\tau, 0]$. The operators \mathcal{A} and \mathcal{R} are defined as:

$$\mathcal{A}(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau, 0), \\ A\phi(0) + B\phi(-\tau), & \theta = 0, \end{cases} \quad (10)$$

where $\phi \in C^1([-\tau, 0], \mathbb{C}^2)$, A, B are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_{21} & b_{22} \end{pmatrix} \tag{11}$$

and

$$\mathcal{R}(\phi(\theta)) = \begin{cases} (0, 0)^T, & \theta \in [-\theta, 0), \\ (F_1(\mu, \phi), F_2(\mu, \phi))^T, & \theta = 0, \end{cases} \tag{12}$$

$$\begin{aligned} F_1(\mu, \phi) &= -\frac{1}{2}\alpha\rho_2\phi_1^2(0) - \frac{1}{6}\alpha\rho_3\phi_1^3(0), \\ F_2(\mu, \phi) &= -\frac{1}{2}\rho_2\phi_1^2(-\tau) - \frac{1}{6}\rho_3\phi_1^3(-\tau). \end{aligned} \tag{13}$$

For $\psi \in C^1([0, \tau], \mathbb{C}^2)$, the adjoint operator \mathcal{A}^* of \mathcal{A} is defined as:

$$\mathcal{A}^*(\mu)(\psi(s)) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [0, \tau), \\ A\psi^T(0) + B\psi^T(\tau), & s = \tau, \end{cases}$$

For $\phi \in C^1([-\tau, 0], \mathbb{C}^2)$ and $\psi \in C^1([0, \tau], \mathbb{C}^2)$ define the bilinear form:

$$\langle \psi, \phi \rangle = \bar{\psi}^T(0)\phi(0) - \int_{-\tau_1}^0 \left(\int_0^\theta \bar{\psi}^T(\xi - \theta)B\phi(\xi) d\xi \right) d\theta. \tag{14}$$

To determine the Poincaré normal form of the operator $\mathcal{A}(\mu)$, we need to calculate the eigenvector ϕ of \mathcal{A} associated with the eigenvalue $\lambda_1 = i\omega_0$ and the eigenvector ψ of \mathcal{A}^* associated with the eigenvalue $\lambda_2 = \bar{\lambda}_1$.

Using (10), (11) we obtain:

Proposition 2. (i) *The eigenvector ϕ of \mathcal{A} associated with the eigenvalue λ_1 is given by:*

$$\phi(\theta) = ve^{\lambda_1\theta}, \quad \theta \in [-\tau, 0],$$

where $v = (v_1, v_2)^T$, $v_1 = a_{12}$, $v_2 = \lambda_1 - a_{11}$.

(ii) *The eigenvector ψ of \mathcal{A}^* associated with the eigenvalue $\lambda_2 = \bar{\lambda}_1$ is given by:*

$$\psi(s) = we^{\lambda_2s}, \quad s \in [0, \tau],$$

where

$$\begin{aligned} w &= (w_1, w_2)^T, \quad w_1 = h\eta, \quad w_2 = \eta, \\ h &= \frac{b_{21}e^{\lambda_1\tau}}{\lambda_2 - a_{11}}, \quad \eta = \frac{1}{(\bar{h} - \tau b_{21})v_1 + (1 - \tau h b_{22})v_2}. \end{aligned}$$

(iii) *With respect to (14) we have:*

$$\langle \psi(s), \phi(\theta) \rangle = 1, \quad \langle \psi(s), \bar{\phi}(\theta) \rangle = \langle \bar{\psi}(s), \phi(\theta) \rangle = 0, \quad \langle \bar{\psi}(s), \bar{\phi}(\theta) \rangle = 1.$$

Next, we construct the coordinates of the center of the manifold Ω_0 at $\varepsilon = 0$, [13].
Let

$$z(t) = \langle \psi, u_t \rangle, \quad w(t, \theta) = u_t - 2 \operatorname{Re} \{ z(t) \phi(\theta) \}.$$

On the center manifold Ω_0 , we consider $w(t, \theta) = w(z(t), \bar{z}(t), \theta)$, where

$$w(z, \bar{z}, \theta) = w_{20}(\theta) \frac{z^2}{2} + w_{11}(\theta) z \bar{z} + w_{02}(\theta) \frac{\bar{z}^2}{2} + \dots$$

and z, \bar{z} are the local coordinates of the center manifold Ω_0 in the direction of ϕ and ψ , respectively. Notice that for $\mu = 0$, for the solution $u_t \in \Omega_0$ of (9), we have:

$$\dot{z}(t) = \lambda_1 z(t) + \langle \psi, \mathcal{R}(w(t, \theta) + 2 \operatorname{Re} \{ z(t) \phi(\theta) \}) \rangle.$$

We rewrite this as $\dot{z}(t) = \lambda_1 z(t) + g(z, \bar{z})$ on the center manifold Ω_0 in the powers of z and \bar{z} :

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2}. \quad (15)$$

From (12), (13) and (15) we obtain:

Proposition 3. *For the system (6) we have:*

$$\begin{aligned} g_{20} &= \bar{w}_1 F_{120} + \bar{w}_2 F_{220}, & g_{11} &= \bar{w}_1 F_{111} + \bar{w}_2 F_{211}, \\ g_{02} &= \bar{w}_1 F_{102} + \bar{w}_2 F_{202}, & g_{21} &= \bar{w}_1 F_{121} + \bar{w}_2 F_{221}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} F_{120} &= -\alpha \rho_2 v_1^2, & F_{102} &= \bar{F}_{120}, & F_{111} &= -\alpha \rho_2 v_1^2, \\ F_{220} &= \rho_2 v_1^2, & F_{202} &= \bar{F}_{220}, & F_{211} &= \rho_2 v_1^2, \\ F_{121} &= -\alpha \rho_2 (w_{120}(0) \bar{v}_1 + 2w_{111}(0) v_1) - \alpha \rho_3 v_1^2 \bar{v}_1, \\ F_{221} &= -\rho_2 (w_{220}(-\tau_0) \bar{v}_1 + 2w_{211}(-\tau_0) v_1) - \rho_3 v_1^2 \bar{v}_1, \end{aligned}$$

and

$$w_{20}(\theta) = (w_{120}(\theta), w_{220}(\theta))^T, \quad w_{11}(\theta) = (w_{111}(\theta), w_{211}(\theta))^T$$

are given by:

$$\begin{aligned} w_{20}(\theta) &= \frac{g_{20}}{\lambda_1} v e^{\lambda_1 \theta} - \frac{\bar{g}_{20}}{3\lambda_1} \bar{v} e^{\lambda_2 \theta} + E_1 e^{2\lambda_1 \theta}, \\ w_{11}(\theta) &= \frac{g_{11}}{\lambda_1} v e^{\lambda_1 \theta} - \frac{\bar{g}_{11}}{\lambda_1} \bar{v} e^{\lambda_2 \theta} + E_2, \quad \theta \in [-\tau, 0), \\ E_1 &= -(A + e^{2\lambda_2 \tau_0} B - 2\lambda_1 I)^{-1} F_{20}, \quad E_2 = -(A + B) F_{11}, \\ F_{20} &= (F_{120}, F_{220})^T, \quad F_{11} = (F_{111}, F_{211})^T. \end{aligned} \quad (17)$$

Let

$$D_1 = \det(A + e^{2\lambda_2\tau_0}B - 2\lambda_1I), \quad D_2 = \det(A + B)$$

and

$$d_{11}^1 = \frac{a_{22} - 2\lambda_1 + b_{22}e^{2\lambda_2\tau_0}}{D_1}, \quad d_{12}^1 = -\frac{a_{12}}{D_1}, \quad d_{21}^1 = -\frac{b_{21}e^{2\lambda_2\tau_0}}{D_1}, \quad d_{22}^1 = \frac{a_{11} - 2\lambda_1}{D_1},$$

$$d_{11}^2 = \frac{a_{22} + b_{22}}{D_2}, \quad d_{12}^2 = -\frac{a_{12}}{D_2}, \quad d_{21}^2 = -\frac{b_{21}}{D_2}, \quad d_{22}^2 = \frac{a_{11}}{D_2}.$$

From (17) we have:

$$E_{11} = (\alpha d_{11}^1 - d_{12}^1)\rho_2 a_{12}^2, \quad E_{12} = (\alpha d_{21}^1 - d_{22}^1)\rho_2 a_{12}^2,$$

$$E_{21} = (\alpha d_{11}^2 - d_{12}^2)\rho_2 a_{12}^2, \quad E_{22} = (\alpha d_{21}^2 - d_{22}^2)\rho_2 a_{12}^2, \tag{18}$$

$$g_{20} = \bar{\eta}(1 - \alpha\bar{h})a_{12}^2, \quad g_{11} = \bar{\eta}(1 - \alpha\bar{h})a_{12}^2, \quad g_{02} = \eta(1 - \alpha h)a_{12}^2 = \bar{g}_{20},$$

$$w_{120}(0) = \frac{g_{20}}{\lambda_1}a_{12} - \frac{g_{02}}{3\lambda_1}a_{12} + E_{11}, \quad w_{111}(0) = \frac{g_{11}}{\lambda_1}a_{12} - \frac{\bar{g}_{11}}{\lambda_1}a_{12} + E_{21},$$

$$w_{220}(-\tau_0) = \frac{g_{20}}{\lambda_1}(\lambda_1 - a_{11})e^{\lambda_2\tau_0} - \frac{\bar{g}_{20}}{3\lambda_1}(\lambda_2 - a_{11})e^{\lambda_1\tau_0} + E_{12}e^{2\lambda_2\tau_0}, \tag{19}$$

$$w_{211}(-\tau_0) = \frac{g_{11}}{\lambda_1}(\lambda_1 - a_{11})e^{\lambda_2\tau_0} - \frac{\bar{g}_{11}}{\lambda_1}(\lambda_2 - a_{11})e^{\lambda_1\tau_0} + E_{22}.$$

From (16) and (19) we obtain:

$$g_{21} = \bar{h}\bar{\eta}F_{121} + \bar{\eta}F_{221}, \tag{20}$$

where

$$F_{121} = -\alpha\rho_2 a_{12} \left(\frac{g_{20}}{\lambda_1}a_{12} - \frac{g_{02}}{3\lambda_1}a_{12} + 2\frac{g_{11}}{\lambda_1}a_{12} - 2\frac{\bar{g}_{11}}{\lambda_1}a_{12} + E_{11} + 2E_{21} \right) - \alpha\rho_3 a_{12}^3,$$

$$F_{221} = -\rho_2 a_{12} \left(\frac{g_{20}}{\lambda_1}(\lambda_1 - a_{11})e^{\lambda_2\tau_0} - \frac{\bar{g}_{20}}{3\lambda_1}(\lambda_2 - a_{11})e^{\lambda_1\tau_0} + 2\frac{g_{11}}{\lambda_1}(\lambda_1 - a_{11})e^{\lambda_2\tau_0} - 2\frac{\bar{g}_{11}}{\lambda_1}(\lambda_2 - a_{11})e^{\lambda_1\tau_0} + 2E_{22} + E_{12}e^{2\lambda_2\tau_0} \right) - \rho_3 a_{12}^3. \tag{21}$$

Therefore, we can compute the following parameters:

$$C(0) = \frac{i}{2\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \tag{22}$$

$$\mu_2 = -\frac{\text{Re}(C(0))}{M}, \quad \beta_2 = 2\text{Re}(C(0)), \quad T_0 = -\frac{\text{Im}(C(0)) + \mu_2 N}{\omega_0},$$

where M and N are given by (8).

In the formulas (22), μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcation periodic solutions; T_0 determines the period of the bifurcating periodic solution.

Proposition 4. (See [12, 13].) (i) If $\mu_2 > 0$ (< 0) the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0$ ($< \tau_0$).

(ii) If $\beta_0 < 0$ (> 0) the solutions are orbitally stable (unstable).

(iii) If $T_0 > 0$ (< 0) the period increases (decreases).

The solution of the system (5) is:

$$Y(t) = u + v_1 z(t) + \overline{v_1 z(t)} + w_{120}(0) \frac{z(t)^2}{2} + w_{111}(0) z(t) \overline{z(t)} + w_{120}(0) \frac{\overline{z(t)}^2}{2},$$

$$K(t) = \frac{pu}{q} + v_2 z(t) + \overline{v_2 z(t)} + w_{220}(0) \frac{z(t)^2}{2} + w_{211}(0) z(t) \overline{z(t)} + w_{220}(0) \frac{\overline{z(t)}^2}{2},$$

where $z(t)$ is the solution of the equation:

$$\dot{z}(t) = i\omega_0 z(t) + g_{20} \frac{z(t)^2}{2} + g_{11} z(t) \overline{z(t)} + g_{02} \frac{\overline{z(t)}^2}{2}.$$

3 The analysis of the stochastic Kaldor–Kalecki model associated to (5)

For the dynamical system (5), we are interested in finding the effect of the noise perturbation on the equilibrium point ($Y_0 = u$, $K_0 = \frac{pu}{q}$). Let the perturbed stochastic model of (5) given by a system of stochastic differential equations with delay:

$$\begin{aligned} dY(t) &= \alpha \left(pu + r \left(\frac{pu}{q} - K(t) \right) + f(Y(t) - u) - pY(t) \right) dt \\ &\quad - \sigma_1 (Y(t) - u) dw(t), \\ dK(t) &= \left(pu + r \left(\frac{pu}{q} - K(t - \tau) \right) + f(Y(t - \tau) - u) - qK(t - \tau) \right) dt \\ &\quad - \sigma_2 \left(K(t) - \frac{pu}{q} \right) dw(t), \end{aligned} \quad (23)$$

where $\sigma_1 > 0$, $\sigma_2 > 0$.

The solution of (23) is a stochastic process denoted by $Y(t) = Y(t, \omega)$, $K(t) = K(t, \omega)$, $\omega \in \Omega$. From the Chebyshev inequality the possible rang of Y , K at a time t is “almost” determined by its mean and variance at time t . So, the first and the second moments are important for investigating the solution’s behavior.

Consider the stochastic system given by (23). Linearizing (23) around the equilibrium ($u, pu/q$) yields the linear stochastic differential delay equation:

$$dy(t) = (A_2 y(t) + B_2 y(t - \tau)) dt - C y(t) dw(t), \quad (24)$$

where $y(t) = (y_1(t), y_2(t))^T$ and

$$A_2 = \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ k_{21} & k_{22} \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$

$$h_{11} = \alpha(\rho_1 - p), \quad h_{12} = -\alpha r, \quad h_{22} = -q, \quad k_{21} = \rho_1, \quad k_{22} = -r.$$

Let $y(t)$ be the fundamental solution of the system:

$$\dot{y}(t) = A_2 y(t) + B_2 y(t - \tau).$$

The solution of (23) is a stochastic process given by:

$$y(t, \Phi) = y_{\Phi}(t) - \int_0^t Y(t-s) C y(t-s, \Phi) dw(s),$$

where $y_{\Phi}(t)$ is the solution given by:

$$y_{\Phi}(t) = Y(t)\Phi(0) + \int_{-\tau}^0 Y(t-\tau-s)\Phi(s) ds$$

and $\Phi : [-\tau, 0] \rightarrow \mathbb{R}^2$ is the family of continuous functions.

The existence and uniqueness theorem for the stochastic differential delay equation has been established in [15].

The solution $y(t, \Phi)$ is a stochastic process with the distribution at any time t , determined by the initial function $\Phi(\theta)$. From the Chebyshev inequality, the possible range of y , at the time t is “almost” determined by its mean and variance at the time t . Thus, the first and second moments of the solutions are important for the investigation of the solutions’ behavior.

We have used E to denote the mathematical expectation and we denote $y(t, \Phi)$ by $y(t)$. From (24), we obtain:

Proposition 5. (i) *The moments of the solution for the system (24) are given by:*

$$\dot{E}(y(t)) = A_2 E(y(t)) + B_2 E(y(t - \tau)), \quad (25)$$

where $E(y(t)) = (E(y_1(t)), E(y_2(t)))^T$.

(ii) *The characteristic equation of the system (25) is given by:*

$$\lambda^2 + (\alpha(p - \rho_1) + q)\lambda + \alpha q(p - \rho_1) + (r\lambda - \alpha(p - \rho_1) + \alpha r \rho_1)e^{-\lambda\tau} = 0. \quad (26)$$

(iii) *If $\tau = 0$, the roots of the equation (26) have a negative real part if and only if ρ_1 satisfies the relation:*

$$\frac{p(1-q)}{2-q} < \rho_1 < \frac{\alpha p + q + r}{\alpha}.$$

(iv) If $\tau \neq 0$ and $\Delta_2 = (a_2^2 - 2b_2 - c_2)^2 - 4(b_2^2 - d_2^2) > 0$, $b_2^2 - d_2^2 > 0$, where

$$a_2 = \alpha(p - \rho_1) + q, \quad b_2 = \alpha q(p - \rho_1), \quad c_2 = r, \quad d_2 = \alpha r \rho_1 - \alpha(p - \rho_1),$$

then there exists ω_0 and τ_0 so that $\lambda_0 = \pm i\omega_0$, $\tau = \tau_0$ is a solution of the equation (26); ω_0 is a positive solution of the following equation:

$$\omega^4 + (a_2^2 - 2b_2 - c_2)\omega^2 + b_2^2 - d_2^2 = 0$$

and

$$\tau_0 = \frac{1}{\omega_0} \arctan \frac{\omega_0 a_2 d_2 + \omega_0 c_2 (\omega_0^2 - b_2)}{d_2 (\omega_0^2 - b_2) - \omega_0^2 c_2 a_2}. \quad (27)$$

From (27) we have $\lambda = \lambda(\tau)$ and $\operatorname{Re} \lambda(\tau_0) = 0$ and $\frac{d\lambda(\tau)}{d\tau} \Big|_{\tau=\tau_0, \lambda=i\omega_0} \neq 0$.

Thus, τ_0 is a Hopf bifurcation. The solutions of the system (25) on the center manifold are given by:

$$E(t) = z(t)\Phi(0) + \bar{z}(t)\bar{\Phi}(0),$$

where

$$\Phi(0) = (h_{12}, \lambda_0 - h_{11})^T, \quad \lambda_0 = i\omega_0,$$

and

$$\dot{z}(t) = \lambda_0 z(t), \quad z(t) = x(t) + iy(t).$$

For $\tau \in [0, \tau_0)$ the mean values of the variables for the system (24) are asymptotically stable; for $\tau > \tau_0$, they are unstable and while for $\tau = \tau_0$ they are periodical.

To examine the stability of the second moments of $y(t)$ for the linear stochastic differential delay equation (24) we use the Itô rule. We have:

$$\begin{aligned} \frac{d}{dt} E(y(t)y^T(t)) &= E(dy(t)y^T(t) + y(t)dy^T(t) + Cy(t)y^T(t)C) \\ &= E(A_2 y(t)y^T(t) + y(t)y^T(t)A_2 + B_2 y(t-\tau)y^T(t) \\ &\quad + y(t)y^T(t-\tau)B_2^T + Cy(t)y^T(t)C). \end{aligned} \quad (28)$$

Let $R(t, s) = E(y(t)y^T(s))$ be the covariance matrix of the process $y(t)$ so that $R(t, t)$ satisfies:

$$\begin{aligned} \dot{R}(t, t) &= A_2 R(t, t) + R(t, t)A_2^T + B_2 R(t-\tau, t) \\ &\quad + R(t, t-\tau)B_2^T + CR(t, t)C. \end{aligned} \quad (29)$$

From (29), $R(t, s) = (R_{ij}(t, s))_{i,j=\overline{1,2}}$ and $R_{ij}(t, s) = E(y_i(t)y_j(s))$ we obtain:

Proposition 6. (i) The differential system (29) is given by:

$$\begin{aligned} \dot{R}_{11}(t, t) &= (2h_{11} + \sigma_1^2)R_{11}(t, t) + 2h_{12}R_{12}(t, t-\tau), \\ \dot{R}_{22}(t, t) &= \sigma_2^2 R_{22}(t, t) + k_{22}R_{22}(t, t-\tau) + k_{21}R_{12}(t, t-\tau), \\ \dot{R}_{12}(t, t) &= (h_{12} + h_{22} + \sigma_1\sigma_2)R_{12}(t, t) + k_{21}R_{11}(t, t-\tau) + k_{22}R_{12}(t, t-\tau). \end{aligned} \quad (30)$$

(ii) The characteristic function of (30) is given by:

$$f_2(\lambda, \tau) = (2\lambda - \sigma_2^2 + re^{-\lambda\tau})(4\lambda^2 - 2(\sigma_1^2 + \sigma_1\sigma_2 + 2\alpha(\rho_1 - p) - \alpha r)\lambda + 2\alpha(\rho_1 - p)(\sigma_1\sigma_2 - \alpha r - r) + 2\alpha r\rho_1 e^{-\lambda\tau}). \quad (31)$$

For the proof of Proposition 6(ii), consider $R_{ij}(t, s) = e^{\lambda(t+s)}K_{ij}$, $i = 1, 2$, where K_{ij} are constants. Replacing $R_{ij}(t, s)$ in (30) and setting the condition that the resulting system we should accept nontrivial solution, we obtain $f_2(\lambda, \tau) = 0$.

The analysis of the second moments is done studying the roots of the characteristic equation $f_2(\lambda, \tau) = 0$.

From (31) we have:

Proposition 7. (i) If $\tau = 0$, the roots of the characteristic equation $f_2(\lambda, \tau) = 0$ have negative real parts if and only if

$$\sigma_2^2 < r, \quad H_1 < \sigma_1 < H_2, \quad (32)$$

where

$$H_1 = \frac{-\sigma_2 - \sqrt{\sigma_2^2 - 4\alpha r - 8\alpha(\rho_1 - p)}}{2}, \quad H_2 = \frac{-\sigma_2 + \sqrt{\sigma_2^2 - 4\alpha r - 8\alpha(\rho_1 - p)}}{2}.$$

(ii) If $\tau \neq 0$, $\sigma_2^2 < r$ then for $\tau \in (0, \tau_1)$ the roots of the equation $2\lambda - \sigma_2^2 + re^{-\lambda\tau} = 0$ have negative real parts, where

$$\tau_1 = \frac{1}{\omega_1} \arctan \frac{2\omega_1}{\sigma_2^2}, \quad \omega_1 = \frac{\sqrt{r^2 - \sigma_2^4}}{2}.$$

(iii) If $\tau \neq 0$ and the relations (32) hold, then for $\tau \in (0, \tau_2)$ the roots of the equation:

$$2\lambda^2 + a_1\lambda + b_1 + c_1e^{-\lambda\tau} = 0,$$

where

$$a_1 = -(\sigma_1^2 + \sigma_1\sigma_2 + 2\alpha(\rho_1 - p) - \alpha r), \\ b_1 = \alpha(\rho_1 - p)(\sigma_1\sigma_2 - \alpha r - r), \quad c_1 = \alpha r\rho_1$$

have negative real parts, where

$$\tau_2 = \frac{1}{\omega_2} \arctan \frac{a_1\omega_2}{2\omega_2^2 - b_1}, \quad \omega_2 = \frac{\sqrt{4b_1^2 - a_1^2 + \sqrt{(a_1^2 - 4b_1^2)^2 + 16c_1^2}}}{4}.$$

Because the solution of the equation $f_2(\lambda, \tau) = 0$ is $\lambda = \lambda(\tau)$ from (31) we have:

$$\operatorname{Re} \lambda(\tau_i) = \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \right) \Big|_{\tau=\tau_i, \lambda=i\omega_i} \neq 0, \quad i = 1, 2.$$

Thus, $\tau = \tau_i$, $i = 1, 2$ is a Hopf bifurcation.

Let $\tau_3 = \min\{\tau_1, \tau_2\}$. The square mean values are asymptotically stable if $\tau \in (0, \tau_3)$. For $\tau = \tau_3$ the system (28) has a limit cycle. The solutions of the system (30) on the center manifold are given by:

$$R(t) = z(t)\phi(0) + \bar{z}(t)\bar{\phi}(0),$$

where

$$\phi(0) = \begin{pmatrix} 2h_{12}(2\lambda_3 - k_{22}e^{-\lambda_3\tau_3}) \\ k_{21}(2\lambda_3 - 2h_{11} - \sigma_1^2)e^{-\lambda_2\tau_3} \\ (2\lambda_3 - 2h_{11} - \sigma_1^2)(2\lambda_3 - k_{32}e^{-\lambda_3\tau_3}) \end{pmatrix}, \quad \lambda_3 = i\omega_3,$$

and

$$\dot{z}(t) = \lambda_3 z(t), \quad z(t) = x(t) + iy(t).$$

Let $\tau_4 = \min\{\tau_0, \tau_3\}$, where τ_0 is given by (27). From Proposition 4 and Proposition 7 we find that for $\tau \in [0, \tau_4)$ the mean values and the square mean values of the variables are locally asymptotically stable.

4 Numerical simulations

The numerical simulation was made using a program in Maple 12.

In our numerical simulations we examine the Kaldor–Kalecki uncertainty and stochastic models.

For the uncertainty model we consider: $f(x) = 0.06x - 0.2x^2 + 0.05x^3$, $p = 0.3$, $r = 1$, $q = 0.2$, $u = 3$, $\alpha = 0.8$, $c_1 = 0.2$, $c_2 = 0.8$, $a = 0.2$. The equilibrium point is $(Y_0 = 3, K_0 = 4.5)$ and it is asymptotically stable for $\tau = 0$ and $\omega_0 = 0.933$, $\tau_0 = 1.19$, $\mu_2 = -0.037$, $\beta_2 = 0.051$, $T_0 = -0.010$. The orbits $(t, Y(t))$, $(t, K(t))$, $(Y(t), K(t))$ are given by:

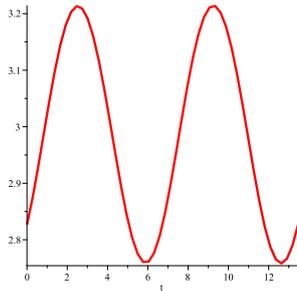


Fig. 1. $(t, Y(t))$.

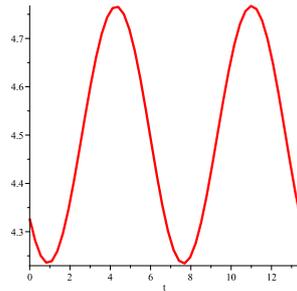


Fig. 2. $(t, K(t))$.

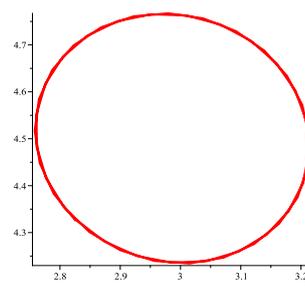


Fig. 3. $(Y(t), K(t))$.

For the stochastic case we consider $f(x) = 0.4/(1+\exp(-4x)) - 0.5$, $p = 0.3$, $r = 1$, $q = 0.2$, $u = 3$, $\alpha = 0.8$, $\sigma_2 = 0.4$, $\sigma_1 = 0.7$. The roots of the characteristic equation of the square mean value have negative real parts if $\tau = 0$. In this case $\tau_1 = 3.18$, $\tau_2 = 5.09$.

If $\tau \in (0, \tau_1)$ the square mean values are asymptotically stable. For $\tau = \tau_1$ we obtain the orbits $(t, R_{11}(t, t))$, $(t, R_{22}(t, t))$, $(t, R_{12}(t, t))$ given in Fig. 4, Fig. 5 and Fig. 6:

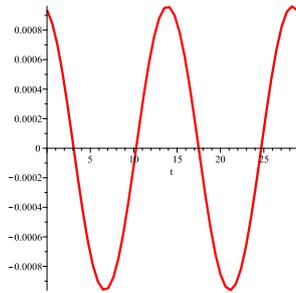


Fig. 4. $(t, R_{11}(t, t))$.

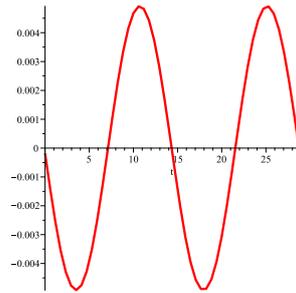


Fig. 5. $(t, R_{22}(t, t))$.

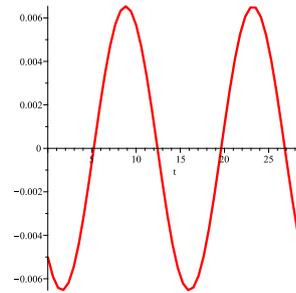


Fig. 6. $(t, R_{12}(t, t))$.

For $\tau_0 = 4.58$, in Fig. 7 and Fig. 8 are displayed the orbits of the mean values.

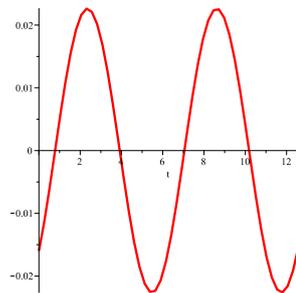


Fig. 7. $(t, E_{11}(t, t))$.

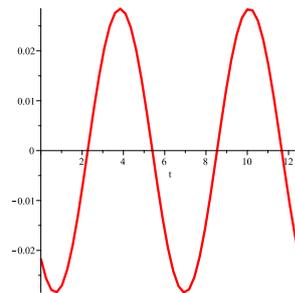


Fig. 8. $(t, E_{22}(t, t))$.

For $\tau \in (0, 3.18)$ the square mean values and the mean values are asymptotically stable.

The numerical simulations verify the theoretical results. Also, we can consider the functions $f(x) = 0.02 \sin x$, $f(x) = 0.2 \arctan(x)$.

5 Conclusions

The analysis of a Kaldor–Kalecki business cycle model in this paper allowed us to obtain some new dynamic scenarios which may be interesting for researchers in applied dynamics and economics.

The paper has analyzed the Kaldor–Kalecki model and the equilibrium point of the model with stochastic perturbation.

We have determined the values of the delay for which the Kaldor–Kalecki system is asymptotically stable and for which the system displays a limit cycle. The direction and stability of the bifurcating periodic solutions are determined.

For the stochastic model, we have analyzed the square mean and the variance of the model's variables.

We have determined the values of τ for which the square mean values and the variances are stable.

As in [16] the hybrid Kaldor–Kalecki model will be taken into consideration in our next paper.

Acknowledgements

The research was done under the Grant with the title “The qualitative analysis and numerical simulation for some economic models which contain evasion and corruption”, CNCSIS-UEFISCU Romania (Grant No. 1085/2008).

The authors would like to express their gratitude for valuable comments on this manuscript from the anonymous referee.

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