On the third order boundary value problems with asymmetric nonlinearity*

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Abstract. The author considers two point third order boundary value problem with asymmetric nonlinearity. The structure and oscillatory properties of solutions of the third order nonlinear autonomous ordinary differential equation are discussed. Results on the estimation of the number of solutions to boundary value problem are provided. An illustrative example is given.

Keywords: boundary value problem with asymmetric nonlinearity, oscillatory properties of solutions, estimation of the number of solutions.

1 Introduction

Boundary value problem for the autonomous equation

$$x''' + f(x) = 0 \tag{1}$$

together with the boundary conditions

$$x(a) = x'(a) = x(b) = 0 \tag{2}$$

is considered. We assume that the function $f(x) : \mathbb{R} \to \mathbb{R}$ is continuous. By a solution of (1) we mean $C^3[a, b]$ function $x(t)$, which satisfies the equation. We will use also the following assumptions:

(A1) $xf(x) > 0$ if $x \neq 0$;

(A1′) $\exists\, m, M > 0$ such that $|f(x)| > M$ when $|x| > m$;

(A2) for $x > 0$ $f(Ax) = A^k f(x)$, for some $k > 1$, and for $x < 0$ $f(Ax) = A^l f(x)$, for some $0 < l < 1$, $A > 0$.

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The first two assumptions (A1) and (A1′) provide the oscillatory behavior of solutions of the equation (1) and the last assumption (A2) ensure the asymmetry of nonlinearity in the equation (1). Conditions (A1) and (A1′) are independent. It is easy to verify, that if the conditions (A1) and (A2) are satisfied, then (A1′) holds also. First, let chose $0 < x_1 < x_2$. Obviously, there exists $A > 1$ such, that $x_2 = Ax_1$. Now consider $f(x_2) = f(Ax_1) = A^k f(x_1) > f(x_1)$. Thus $f(x)$ is strictly increasing function for $x > 0$. Analogously we can show that $f(x)$ is strictly increasing for $x < 0$. Let chose $x_2 < x_1 < 0$. Then there exists $A > 1$ such, that $x_2 = Ax_1$. Now consider $f(x_2) = f(Ax_1) = A^l f(x_1) < f(x_1)$. Therefore, from conditions (A1) and (A2) (A1′) follows. We state (A1′) because some statements below hold if (A1) and (A1′) only are satisfied. The typical example of function $f(x)$ which satisfies the conditions (A1), (A1′) and (A2), for instance, is:

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ x^{\frac{1}{3}}, & x < 0, \end{cases} \quad \text{with } k = 2 \text{ and } l = \frac{1}{3}.$$

Nonlinear boundary value problems appear in almost all branches of science, engineering and technology, for instance, boundary layer theory in fluid mechanics, heat power transmission theory, control and optimization theory etc, and as a consequence, have generated a lot of interest over the years. The theory of nonlinear boundary value problems is an important and actual area of research since it is aimed to applications. Classical results in the theory concern the existence and uniqueness of solutions. A variety of techniques are employed in the theory, for example, methods that involve differential inequalities, shooting and angular function techniques, lower and upper solutions method, functional analytic approaches, topological methods, etc. Let us mention books by P. Bailey, L. Shampine, P. Waltman [1], S. Bernfeld and V. Lakshmikantham [2], N.I. Vasilyev and Yu.A. Klokov [3], C. de Coster, P. Habets [5], W. Kelley, A. Peterson [4].

The more complicated, more actual questions are about the number of solutions to boundary value problem, of their properties, nodal structure etc. These types of problems are insufficiently investigated in the literature even for second order problems. There are few results on multiple solutions of third order nonlinear problems. Results concerning two point third order nonlinear boundary value problems were obtained by E. Rovderova [7], F. Sadyrbaev [9]. In [7], the author states some results on the number of solutions of two point boundary value problems. In [9], the author established multiplicity results for certain classes of third order nonlinear boundary value problems. His approach was based on the Hanan's theory [6] of conjugate points for third order linear differential equations. Results which ensure the existence of infinitely many solutions of two point higher order nonlinear boundary value problem under superlinear growth condition are given by C. de Coster and M. Gaudenzi [8]. J. Graef and B. Yang give sufficient conditions for the existence of multiple positive solutions to the third order three point boundary value problem in [11]. Three point third order boundary value problem with symmetric nonlinearity is considered in [14].

On the other hand, there is voluminous literature concerning asymmetric nonlinearities. For example, investigation of the Fučik type equation of the second order ([12] and references therein), study of the third order asymmetric equations, where the right side is a piece-wise linear function [13]. These types of investigations are motivated also by technical applications, for instance the theory of suspension bridges.

We are looking for multiple solutions of the problem (1), (2). The aim of the present paper is to provide results which ensure the existence of a given number of solutions to (1), (2) under additional conditions. The shooting method is used for treating the number of solutions to boundary value problems. The shooting method reduces solving a boundary value problem to solving of an initial value problem. So we consider the auxiliary initial value problem for the equation (1) with initial data $x(a) = 0$, $x'(a) = 0$, $x''(a) = \gamma$ and we are looking for such γ that the solution of initial value problem vanishes at $t = b$.

The paper is organized as follows. In Section 2 we consider some basic results and notions which are used in later sections. The 3rd section is devoted to the oscillatory properties of solutions of the equation (1). In the 4th section we investigate the structure of solutions and in the 5th section we deal with the estimation of the number of solutions to two point boundary value problem. Examples and figures are given in the 6th section to illustrate the results.

2 Preliminary results

Proposition 1. *Suppose* $x(t) \in C^3[a,b]$. *If* $x(a) \geq 0$, $x'(a) \leq 0$, $x''(a) \geq 0$ (*but not all zero*) *and* $x'''(t)x(t) \leq 0$. *Then* $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$ *for* $t < a$.

Proof. Let $x(a) \geq 0$, $x'(a) \leq 0$, $x''(a) \geq 0$ and $(x(a))^2 + (x'(a))^2 + (x''(a))^2 > 0$. In all cases $x(t)$ will be positive in some open interval whose right boundary point is $t = a$.

Suppose that there exists a point $t = t_0$ such that $x(t_0) = 0$ and $x(t) > 0$ for $t_0 < t < a$.

Since $x(t_0) = 0$, there will exist a point $t = t_1$, $t_0 \leq t_1 < a$ such that $x'(t_1) = 0$ and there will exist a point $t = t_2$, $t_0 \leq t_2 < a$ such that $x''(t_2) = 0$.

Since $x'''(t)x(t) \leq 0$, it follows that $x'''(t) < 0$ for $t_0 < t < a$.

Consider

$$x''(t) = x''(a) - \int_t^a x'''(s)\,\mathrm{d}s, \quad t_0 \leq t < a.$$

The right-hand side is positive, and increases as $t \to -\infty$, as long as $x'''(t)$ remains negative. We thus conclude that $x''(t)$ is positive for $t_0 \leq t < a$.

Consider

$$x'(t) = x'(a) - \int_t^a x''(s)\,\mathrm{d}s, \quad t_0 \leq t < a.$$

The right-hand side is negative, and decreases as $t \to -\infty$, as long as $x''(t)$ remains positive. We thus conclude that $x'(t)$ is negative for $t_0 \leq t < a$.

Consider

$$x(t) = x(a) - \int_t^a x'(s)\,\mathrm{d}s, \quad t_0 \le t < a.$$

The right-hand side is positive, and increases as $t \to -\infty$, as long as $x'(t)$ remains negative. We thus conclude that $x(t)$ is positive for $t_0 \le t < a$. These contradictions prove the proposition. □

Proposition 2. *Suppose* $x(t) \in C^3[a,b]$. *If* $x(a) \le 0$, $x'(a) \ge 0$, $x''(a) \le 0$ (*but not all zero*) *and* $x'''(t)x(t) \le 0$. *Then* $x(t) < 0$, $x'(t) > 0$, $x''(t) < 0$ *for* $t < a$.

Proof. The proof is analogous to the proof of Proposition 1 above. □

Remark. The function $x(t)$ from Propositions 1 and 2 may be thought as a solution of differential equation (1).

3 Oscillatory properties of solutions

The next two corollaries follows from Propositions 1 and 2.

Corollary 1. *Suppose that condition* (A1) *holds. If* $x(t)$ *is a nontrivial solution of* (1) *and* $x(a) = x(b) = 0$ ($a < b$), *then* $x'(b)x''(b) > 0$.

Proof. Assume $x'(b)x''(b) \le 0$.

(i) $x'(b) \le 0$, $x''(b) \ge 0$. Then, by the Proposition 1 $x(t) > 0$ for $t < b$. This is a contradiction, since $x(a) = 0$.

(ii) $x'(b) \ge 0$, $x''(b) \le 0$. Then, by the Proposition 2 $x(t) < 0$ for $t < b$. This is a contradiction, since $x(a) = 0$. □

Corollary 2. *Assume that condition* (A1) *is satisfied. If* $x(t)$ *is a nontrivial solution of* (1) *and* $x(a) = x(b) = 0$, $a < b$, *then* $x'(b) \ne 0$.

Proof. Let $x'(b) = 0$, and, without loss of generality, let $x''(b) > 0$. Then, by the Proposition 1 $x(t) > 0$ for $t < a$. But $x(a) = 0$, $a < b$. The contradiction proves the corollary. □

Proposition 3. *Let* $x(t)$ *be a solution of the equation* (1) *such that* $x(a) = x'(b) = 0$ ($a < b$), $x(t) \ne 0$ *for* $t \in (a,b)$. *If the condition* (A1) *is fulfilled, then* $x(t)$ *vanishes in* $(b, +\infty)$.

Proof. Assume that $x(t)$ does not change sign for $t > b$. Without loss of generality, let $x(t) > 0$, $t > b$. Multiplying the equation (1) by $x(t)$ and integrating from a to t, we obtain

$$\int_a^t x(s)x'''(s)\,\mathrm{d}s + \int_a^t x(s)f\big(x(s)\big)\,\mathrm{d}s = 0.$$

Integrating the first term by parts, we get

$$x(t)x''(t) - x(a)x''(a) - \int_a^t x''(s)x'(s)\,\mathrm{d}s + \int_a^t x(s)f\bigl(x(s)\bigr)\,\mathrm{d}s = 0,$$

or

$$x(t)x''(t) = \frac{1}{2}x'^2(t) - \frac{1}{2}x'^2(a) - \int_a^t x(s)f\bigl(x(s)\bigr)\,\mathrm{d}s.$$

If $t = b$ we obtain

$$x(b)x''(b) = \frac{1}{2}x'^2(b) - \frac{1}{2}x'^2(a) - \int_a^b x(s)f\bigl(x(s)\bigr)\,\mathrm{d}s < 0.$$

Since $x(b) > 0$, then $x''(b) < 0$. Since $x(t) > 0$, then (in view of (A1) and (1)) $x'''(t) < 0$ and $x''(t)$ is strictly decreasing. Thus, $x''(t) < 0$ for $t > b$ and $x'(t)$ is strictly decreasing for $t > b$. Since $x'(b) = 0$ and $x'(t)$ is strictly decreasing for $t > b$, then $x'(t) < 0$ for $t > b$. Thus, $x(t)$ is strictly decreasing for $t > b$. If two consecutive derivatives of $x(t)$ are negative then $x(t)$ must ultimately be negative. This completes the proof of the proposition. □

Proposition 4. *Let $x(t)$ be a solution of the equation* (1) *such that $x(a) = 0$. If the conditions* (A1) *and* (A1′) *hold, then $x(t)$ vanishes in $(a, +\infty)$.*

Proof. Suppose that $x(t)$ does not vanish for $t > a$. Without loss of generality, let $x(t) > 0$ for $t > a$. If there exists $b > a$ such that $x'(b) = 0$, then the proof follows from the Proposition 3 above. Therefore, assume that $x'(t)$ does not vanish for $t > a$. Since $x'(t) > 0$ for t immediately to the right of a, it follows that $x'(t) > 0$ for $t > a$. As $x(t) > 0$, then in view of (A1) and (1), $x'''(t) < 0$ and $x''(t)$ is strictly decreasing.

First suppose there exists $t_1 \geq a$ such that $x''(t_1) = 0$. Then $x''(t) < 0$ for $t > t_1$. If two consecutive derivatives of $x'(t)$ are negative then $x'(t)$ must ultimately be negative.

Now assume that $x''(t) > 0$ for $t > a$. So $x'(t)$ is strictly increasing for $t > a$. Integrating equation (1) between $t_0 > a$ and t we obtain

$$\int_{t_0}^t x'''(s)\,\mathrm{d}s + \int_{t_0}^t f\bigl(x(s)\bigr)\,\mathrm{d}s = 0,$$

or

$$x''(t_0) = x''(t) + \int_{t_0}^t f\bigl(x(s)\bigr)\,\mathrm{d}s \geq \int_{t_0}^t f\bigl(x(s)\bigr)\,\mathrm{d}s \geq \int_{t_0}^t M\,\mathrm{d}s.$$

The left side is independent of t and thus the integral on the right must converge as $t \to +\infty$. This contradiction proves the proposition. □

Next we assume that conditions (A1) and (A1$'$) are satisfied.

Corollary 3. *If $x(t)$ is a nontrivial solution of* (1) *and $t = a$ is a zero of $x(t)$, then $x(t)$ has an infinity of simple zeros in $(a, +\infty)$. If $t = a$ is a double zero of $x(t)$, then $x(t)$ does not vanish in $(-\infty, a)$.*

4 Structure of solutions

Positive part. Consider the nontrivial solution of the equation (1) with initial conditions $x(0) = 0$, $x'(0) = \alpha_0$, $x''(0) = \beta_0$. Let us denote this solution by $x_0(t)$. Assume that $t = t_1$ is the first zero of the solution $x_0(t)$ to the right of $t = 0$ and $x_0(t) > 0$ for $t \in (0, t_1)$. Let $x_0'(t_1) = \alpha_1$ and $x_0''(t_1) = \beta_1$.

Consider the function $y(t) = B^{\frac{3}{k-1}} x_0(Bt)$ with $B > 0$ (parameter). Obviously $y(0) = 0$, $y'(0) = B^{\frac{3}{k-1}+1}\alpha_0$, $y''(0) = B^{\frac{3}{k-1}+2}\beta_0$. Moreover $y(\tau_1) = 0$, $y'(\tau_1) = B^{\frac{3}{k-1}+1}\alpha_1$, $y''(\tau_1) = B^{\frac{3}{k-1}+2}\beta_1$, where

$$\tau_1 = t_1/B \tag{3}$$

and $y(t) > 0$ for $t \in (0, \tau_1)$.

Proposition 5. *If condition* (A2) *fulfilled, then the function $y(t)$ for $t \in [0, \tau_1]$ is a solution of the equation* (1)*.*

Remark. A similar statement for higher order Emden–Fowler type equation can be found in [10].

Proof. The proposition can be proved by direct substitution. Consider $y'''(t) = B^{\frac{3}{k-1}+3} \times x_0'''(Bt)$. For $y(t) > 0$ $f(y(t)) = B^{\frac{3k}{k-1}} f(x_0(Bt))$. Thus $y(t)$ satisfies the equation (1) for $t \in [0, \tau_1]$. Moreover $y(t) \in C^3_{[0,\tau_1]}$. □

Let $y'(0) = \alpha$, $y''(0) = \beta$, $y'(\tau_1) = \alpha_\tau$, $y''(\tau_1) = \beta_\tau$, and consider α, β, α_τ and β_τ as functions from B.

Proposition 6. *Four statements are equivalent:*

(i) *Parameter B continuously and monotonically tends to $+\infty$ (zero);*

(ii) *Point $t = \tau_1$ continuously and monotonically tends to $t = 0$ ($t = +\infty$);*

(iii) *$|\alpha(B)| + |\beta(B)|$ continuously and monotonically tends to $+\infty$ (zero);*

(iv) *$|\alpha_\tau(B)| + |\beta_\tau(B)|$ continuously and monotonically tends to $+\infty$ (zero).*

Proof. The equivalence of the first and the second statements follows from (3). Now we prove the equivalence of the first and the third statements.

Consider the auxiliary function

$$\begin{aligned} G_1(B) &= \left|\alpha(B)\right| + \left|\beta(B)\right| = \left|B^{\frac{3}{k-1}+1}\alpha_0\right| + \left|B^{\frac{3}{k-1}+2}\beta_0\right| \\ &= B^{\frac{3}{k-1}+1}|\alpha_0| + B^{\frac{3}{k-1}+2}|\beta_0|. \end{aligned}$$

The function $G_1(B)$ is continuous and for every $B > 0$ is strictly increasing, because

$$G_1'(B) = \left(\frac{3}{k-1} + 1\right) B^{\frac{3}{k-1}} |\alpha_0| + \left(\frac{3}{k-1} + 2\right) B^{\frac{3}{k-1}+1} |\beta_0| > 0, \quad k > 1.$$

Moreover $\lim\limits_{B\to+\infty} G_1(B) = +\infty$ and $\lim\limits_{B\to 0+} G_1(B) = 0$.

The proof for the equivalence of the first and the fourth statements is analogous. The equivalence of the second and the third statements, second and fourth, third and fourth follows from the transitivity. □

Negative part. Now consider a solution $u_0(t)$ of the equation (1) with initial conditions $u_0(0) = 0$, $u_0'(0) = \mu_0$, $u_0''(0) = \nu_0$. Assume that $t = \xi_1$ is the first zero of the solution $u_0(t)$ to the right from point $t = 0$ and $u_0(t) < 0$ for $t \in (0, \xi_1)$. Let denote $u_0'(\xi_1) = \mu_1$, $u_0''(\xi_1) = \nu_1$.

Consider the function $v(t) = A^{\frac{3}{l-1}} u_0(At)$ with $A > 0$ (parameter). Obviously $v(0) = 0$, $v'(0) = A^{\frac{3}{l-1}+1}\mu_0$, $v''(0) = A^{\frac{3}{l-1}+2}\nu_0$. Moreover $v(\zeta_1) = 0$, $v'(\zeta_1) = A^{\frac{3}{l-1}+1}\mu_1$, $v''(\zeta_1) = A^{\frac{3}{l-1}+2}\nu_1$, where

$$\zeta_1 = \xi_1/A \tag{4}$$

and $v(t) < 0$ for $t \in (0, \zeta_1)$.

Proposition 7. *If condition* (A2) *holds, then the function* $v(t)$ *for* $t \in [0, \zeta_1]$ *is a solution of the equation* (1).

Proof. The proposition can be proved by direct substitution. Consider $v'''(t) = A^{\frac{3}{l-1}+3} \times u_0'''(At)$. For $v(t) < 0$ $f(v(t)) = A^{\frac{3l}{l-1}} f(u_0(At))$. Thus $v(t)$ satisfies the equation (1) for $t \in [0, \zeta_1]$. Moreover $v(t) \in C^3_{[0,\zeta_1]}$. □

Let denote $v'(0) = \mu$, $v''(0) = \nu$, $v'(\zeta_1) = \mu_\zeta$, $v''(\zeta_1) = \nu_\zeta$ and consider μ, ν, μ_ζ and ν_ζ as the functions from A.

Proposition 8. *Four statements are equivalent:*

(i) *Parameter* A *continuously and monotonically tends to* $+\infty$ (*zero*);

(ii) *Point* $t = \zeta_1$ *continuously and monotonically tends to* $t = 0$ ($t = +\infty$);

(iii) $|\mu(A)| + |\nu(A)|$ *continuously and monotonically tends to zero* ($+\infty$);

(iv) $|\mu_\zeta(A)| + |\nu_\zeta(A)|$ *continuously and monotonically tends to zero* ($+\infty$).

Proof. The equivalence of the first and the second statements follows from (4). Now we prove the equivalence of the first and the third statements.

Consider the auxiliary function

$$\begin{aligned} G_2(A) &= \bigl|\mu(A)\bigr| + \bigl|\nu(A)\bigr| = \bigl|A^{\frac{3}{l-1}+1}\mu_0\bigr| + \bigl|A^{\frac{3}{l-1}+2}\nu_0\bigr| \\ &= A^{\frac{3}{l-1}+1}|\mu_0| + A^{\frac{3}{l-1}+2}|\nu_0|. \end{aligned}$$

The function $G_2(A)$ is continuous and for every $A > 0$ is strictly decreasing, because

$$G_2'(A) = \left(\frac{3}{l-1} + 1\right) A^{\frac{3}{l-1}} |\mu_0| + \left(\frac{3}{l-1} + 2\right) A^{\frac{3}{l-1}+1} |\nu_0| < 0, \quad 0 < l < 1.$$

Moreover $\lim\limits_{A\to+\infty} G_2(A) = 0$ and $\lim\limits_{A\to 0+} G_2(A) = +\infty$. The proof for the equivalence of the first and the fourth statements is analogous. The equivalence of the second and the third statements, second and fourth, third and fourth follows from the transitivity. □

5 Two point boundary value problem

Let $z(t)$ be a nontrivial solution of the equation (1) with the initial conditions $z(0) = 0$, $z'(0) = 0$, $z''(0) = \gamma$. Let denote simple zeros of $z(t)$ to the right from $t_0 = 0$ by t_1, $t_2, \dots, t_i, \dots$.

First suppose that $\gamma > 0$; then $z(t) > 0$ for $t \in (t_{2i-2}, t_{2i-1})$ and $z(t) < 0$ for $t \in (t_{2i-1}, t_{2i})$ and let $\delta^+_{2i-1} = t_{2i-1} - t_{2i-2}$, $\delta^-_{2i} = t_{2i} - t_{2i-1}$, $i = 1, 2, \dots$.

The next two statements follow from Propositions 6 and 8.

Corollary 4. *Assume that condition* (A2) *is satisfied and consider the solution* $z(t)$*. If* γ *continuously and monotonically tends to* $+\infty$ (*resp.*: *zero*)*, then* δ^+_{2i-1} *continuously and monotonically tend to zero* (*resp.*: $+\infty$) *and* δ^-_{2i} *continuously and monotonically tend to* $+\infty$ (*resp.*: *zero*)*.*

Proof. Let denote $z'(t_i) = a_i$ and $z''(t_i) = b_i$. Now suppose that γ continuously and monotonically tends to $+\infty$. Then in view of Proposition 6, the point $t = t_1$ continuously and monotonically tends to $t = 0$ and $|a_1| + |b_1|$ continuously and monotonically tends to $+\infty$. So δ^+_1 continuously and monotonically tends to zero.

Since $|a_1| + |b_1|$ continuously and monotonically tends to $+\infty$, then in view of Proposition 8, the point $t = t_2$ continuously and monotonically tends to $t = +\infty$ and $|a_2| + |b_2|$ continuously and monotonically tends to $+\infty$. So δ^-_2 continuously and monotonically tends to $+\infty$.

Since $|a_2| + |b_2|$ continuously and monotonically tends to $+\infty$, then in view of Proposition 6, the point $t = t_3$ continuously and monotonically tends to $t = t_2$ and $|a_3| + |b_3|$ continuously and monotonically tends to $+\infty$. So δ^+_3 continuously and monotonically tends to zero.

We can continue this process. If γ continuously and monotonically tends to zero, the process is analogous. □

Now suppose that $\gamma < 0$; then $z(t) > 0$ for $t \in (t_{2i-1}, t_{2i})$ and $z(t) < 0$ for $t \in (t_{2i-2}, t_{2i-1})$ and let $\delta^+_{2i} = t_{2i} - t_{2i-1}$, $\delta^-_{2i-1} = t_{2i-1} - t_{2i-2}$, $i = 1, 2, \dots$.

Corollary 5. *Assume that the condition* (A2) *is satisfied and consider the solution* $z(t)$*. If* γ *continuously and monotonically tends to* $-\infty$ (*resp.*: *zero*)*, then* δ^+_{2i} *continuously and monotonically tend to zero* (*resp.*: $+\infty$) *and* δ^-_{2i-1} *continuously and monotonically tend to* $+\infty$ (*resp.*: *zero*)*.*

Proof. The proof is analogous to the proof of Corollary 4. □

Therefore, we can describe t_i as the functions of γ. Any function $t_i(\gamma)$ consists from two branches. The first one defined for $\gamma > 0$ and we denote it by $t_i(\gamma+)$ and the second one defined for $\gamma < 0$ and we denote it by $t_i(\gamma-)$. Thus, $t_1(\gamma+) = \delta_1^+$, $t_1(\gamma-) = \delta_1^-$, $t_2(\gamma+) = \delta_1^+ + \delta_2^-$, $t_2(\gamma-) = \delta_1^- + \delta_2^+$, $t_3(\gamma+) = \delta_1^+ + \delta_2^- + \delta_3^+$, $t_3(\gamma-) = \delta_1^- + \delta_2^+ + \delta_3^-$ and so on.

Since for every $i = 2, 3, \dots$ $\lim_{\gamma\to 0} t_i(\gamma+) = +\infty$, $\lim_{\gamma\to +\infty} t_i(\gamma+) = +\infty$, $\lim_{\gamma\to 0} t_i(\gamma-) = +\infty$, $\lim_{\gamma\to -\infty} t_i(\gamma-) = +\infty$, branches $t_i(\gamma+)$ and $t_i(\gamma-)$ for $i = 2, 3, \dots$ have minimums. Let denote $t_i^+ = \min_{(0,+\infty)} t_i(\gamma+)$, $t_i^- = \min_{(-\infty,0)} t_i(\gamma-)$ and $t_i^* = \min\{t_i^+, t_i^-\}$ $i = 2, 3, \dots$.

Remark. Any point of intersection of $t_i(\gamma)$ with the line $t = b$ yields a solution to the problem (1), (2). The nodal structure of the certain solution of the main problem depends on the branch $t_i(\gamma)$ which intersects the line $t = b$. Suppose this point belongs to $t_m(\gamma)$ for certain m. Then the corresponding solution has exactly $(m-1)$ simple zeros in (a, b). The number of these intersections is the number of solutions of the main problem. Since branches $t_i(\gamma+)$ and $t_i(\gamma-)$ for $i = 2, 3, \dots$ have minimums, the number of intersection depends on the value of b.

Theorem 1. *Assume that the condition* (A2) *is satisfied. If* $b \in (a, t_2^*)$*, then the problem* (1)*,* (2) *has two nontrivial solutions. The first one exists for* $\gamma > 0$ *and is positive for* $t \in (a, b)$; *the second one exists for* $\gamma < 0$ *and is negative for* $t \in (a, b)$.

Proof. Since $b \in (a, t_2^*)$ then the line $t = b$ intersects the branch $t_1(\gamma+)$ and the branch $t_1(\gamma-)$. Thus, the problem (1), (2) has two nontrivial solutions. The first one exists for $\gamma > 0$ and is positive for $t \in (a, b)$, the second one exists for $\gamma < 0$ and is negative for $t \in (a, b)$. □

Theorem 2. *Assume that the condition* (A2) *is satisfied. If* $b \in (t_i^*, t_{i+1}^*)$ *then the problem* (1)*,* (2) *has at least* $2i$ *nontrivial solutions,* $i = 2, 3, \dots$.

Proof. Since $b \in (t_i^*, t_{i+1}^*)$ then the line $t = b$ intersects the branch $t_1(\gamma+)$, the branch $t_1(\gamma-)$, the branch $t_2(\gamma)$ at least twice, the branch $t_3(\gamma)$ at least twice, ..., the branch $t_i(\gamma)$ at least twice. Thus, the problem (1), (2) has at least $2i$ nontrivial solutions, $i = 2, 3, \dots$. □

Theorem 3. *Assume that the condition* (A2) *is satisfied. If* $b = t_i^*$ *then the problem* (1)*,* (2) *has at least* $2i - 1$ *nontrivial solutions,* $i = 2, 3, 4, \dots$.

Proof. Since $b = t_i^*$, the line $t = b$ intersects the branch $t_1(\gamma+)$, the branch $t_1(\gamma-)$, the branch $t_2(\gamma)$ at least twice, the branch $t_3(\gamma)$ at least twice, ... and the branch $t_i(\gamma)$. Thus the problem (1), (2) has at least $2i - 1$ nontrivial solutions, $i = 3, 4, \dots$. □

6 Example

Consider the equation

$$x''' = -\begin{cases} x^3, & x \geq 0, \\ x^{\frac{3}{5}}, & x < 0, \end{cases} \tag{5}$$

together with initial conditions

$$x(0) = 0, \quad x'(0) = 0, \quad x''(0) = \gamma. \tag{6}$$

Zero functions $t_i(\gamma)$, $i = 1, 2, 3$ for initial value problem (5), (6) are depicted in Fig. 1. This figure was obtained by using programm *Mathematica 7.0.*

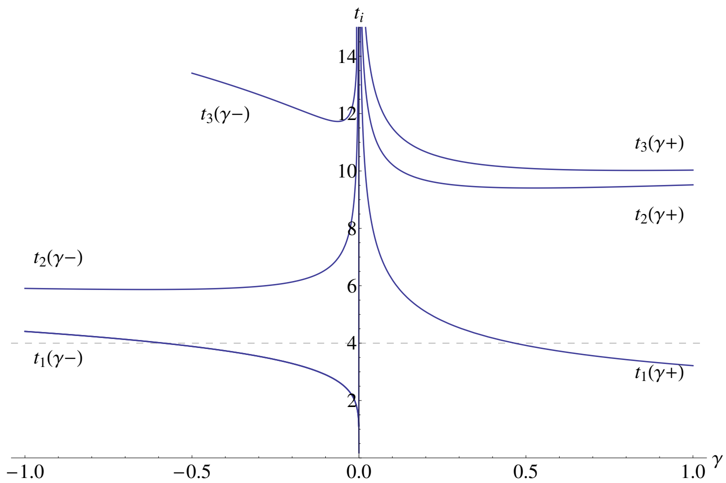

Fig. 1. Zero functions $t_i(\gamma)$, $i = 1, 2, 3$, for initial value problem (5), (6). All the branches except $t_1(\gamma+)$ tend to $+\infty$ as $|\gamma| \to +\infty$.

If we consider equation (5) together with boundary conditions (2), then the number of solutions to problem (5), (2) depends on the length of the interval (a, b). For instance, if $b - a = 4$, then the number of solutions to problem (5), (2) is exactly two. We can estimate the number of solutions to problem (5), (2) for the greater interval (a, b) by using Theorems 2 and 3.

Conclusions

- We have shown that the case of asymmetric nonlinearity substantially differs from symmetric one (for example $f(x) = x^3$).
- The number of solutions to problem (1), (2) in contrast to the case $x''' + x^3 = 0$, (2) is finite.
- The number of solutions depends on the length $b - a$ of the interval.

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