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# Symmetry analysis for steady boundary-layer stagnation-point flow of Rivlin–Ericksen fluid of second grade subject to suction

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**Abstract.** An analysis for the steady two-dimensional boundary-layer stagnation-point flow of Rivlin–Ericksen fluid of second grade with a uniform suction is carried out via symmetry analysis. By employing Lie-group method to the given system of nonlinear partial differential equations, the symmetries of the equations are determined. Using these symmetries, the solution of the given equations is found. The effect of the visco-elastic parameter $k$ and the suction parameter $R$ on the tangential and normal velocities, temperature profiles, heat transfer coefficient and the wall shear stress, have been studied. Also, the effect of the Prandtl number *Pr* on the temperature and the heat transfer coefficient has been studied.

**Keywords:** stagnation point flow, Rivlin–Ericksen fluid, Lie-group, symmetries, mechanics of fluids.

## 1 Introduction

The two-dimensional stagnation point flow of an incompressible viscous fluid appears in several manufacturing processes of industry such as the extrusion of polymers, the cooling of metallic plates, the aerodynamic extrusion of plastic sheets, etc. In the glass industry, blowing, floating or spinning of fibers are processes, which involve the flow due to a stretching surface, [1].

Non-Newtonian fluids are more appropriate in industrial applications than Newtonian fluids. Engineers and rheologists proposed simple generalizations such as the power-law model for the non-Newtonian fluids. In addition to viscosity, another parameter, namely the power-law index is needed to characterize the fluid. Walters [2] presented the relation between the different constitutive relations used for elasto-viscous liquids such as the models due to Coleman–Noll, Rivlin–Ericksen, Green–Rivlin and Oldroyd. There are several constitutive relations which are used to model the non-Newtonian (non-linear) characteristics exhibited by some fluids. Take the past history of deformation into account; integral models such as the K-BKZ model are used when memory effects are important. However, for many fluids, only a very short part of the history of the deformation influences the stress. The constitutive relations for these fluids may be expressed as a function of the deformation gradient and its derivatives evaluated at the current time. A material in which the stress depends only on a finite number of these time derivatives is called a material of differential type. The Newtonian fluid and the second grade fluid are the simplest fluids of the differential type when the dependence upon the velocity gradient is given a more specific form, [3].

The constitutive equation for an incompressible homogeneous Rivlin–Ericksen fluid of second grade is given by the following relation

$$S = -p\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2{\mathbf{A_1}}^2, \tag{1}$$

where $S$ is the stress tensor, $p$ is the pressure, $\mathbf{I}$ is the identity tensor, $\mu$ is the coefficient of viscosity, $\alpha_1$ and $\alpha_2$ are the material moduli and denote the first and second normal stress coefficients which are not always constants, [4].

The kinematical tensors $A_1$ and $A_2$ are the first two Rivlin–Ericksen tensors [5] which are discussed in detail by Rajagopal et al. [6] and given by

$$\mathbf{A_1} = \nabla\mathbf{v} + (\nabla\mathbf{v})^T, \tag{2}$$

$$\mathbf{A_2} = \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{A_1} + \mathbf{A_1}\cdot\nabla\mathbf{v} + (\nabla\mathbf{v})^T\cdot\mathbf{A_1}, \tag{3}$$

where $\nabla$ is the gradient operator, $\mathbf{v}$ is the velocity vector, a superscript $T$ denotes transpose, and $\mathrm{D}/\mathrm{D}t$ is the material time derivative which is define as follows:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \big[\nabla(\cdot)\big]\mathbf{v}, \tag{4}$$

where $\partial/\partial t$ is the partial derivative.

Dunn and Fosdick [7] studied the thermodynamics stability in the model (1). They concluded that, if the fluid is modeled by (1), thermodynamics compatibility in the sense that all motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum in equilibrium requires that

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{5}$$

The fluids characterized by above restrictions are called the second grade fluids.

Massoudi and Ramezan [8], following Rajagopal et al. [9], showed that, by substituting (1) and using (5) into the balance of linear momentum

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \operatorname{div}\mathbf{S} + \rho\mathbf{b}, \tag{6}$$

and using the incompressibility constrain, i.e., $\operatorname{div} \mathbf{v} = 0$, the following boundary layer equation is obtained

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial p^*}{\partial \bar{x}} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\alpha_1}{\rho}\left[\frac{\partial}{\partial \bar{x}}\left(\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right], \quad (7)$$

where

$$p^* = p - (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2 + \rho\varphi, \quad (8)$$

and, $\rho$, $\mathbf{b}$ and $\varphi$ are the fluid density, the body force and the scalar function, respectively.

We can eliminate the pressure term from the boundary layer equation (7) by using Bernoulli's equation

$$U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x}. \quad (9)$$

Also, we have assumed that the body force $\mathbf{b}$ is conservative, i.e., $\mathbf{b} = \operatorname{grad} \varphi$.

Massoudi [10] provided solutions for the flow and heat transfer in the boundary layer stagnation flow of second grade fluids with arbitrary power law heat flux variation. The governing equations were solved using second order central difference approximation along with the appropriate boundary conditions. In his paper, the effect of mass injection/suction was neglected. Also, the effects of viscous dissipation are neglected. The effect of the non-Newtonian nature of the fluid on the velocity profile and heat transfer coefficient at the wall for different Prandtl numbers and wall surface flux variation was investigated. He concluded that, the smaller the Prendtl number, the thicker is the thermal boundary layer.

Mahapatra and Gupta [11] analyzed the steady two-dimensional stagnation-point flow of an incompressible viscoelastic fluid over a flat deformable surface when the surface is stretched in its own plane with a velocity proportional to the distance from the stagnation-point. They concluded that, for a fluid of small kinematic viscosity, a boundary layer is formed when the stretching velocity of the surface is less than the inviscid free stream velocity and an inverted boundary layer is formed when the stretching velocity exceeds the free stream velocity. Temperature distribution in the boundary layer is determined when the surface is held at constant temperature giving the so called surface heat flux. In their analysis, they used the finite-differences scheme along with the Thomas algorithm to solve the resulting system of ordinary differential equations.

Liu [12] examined the flow and heat transfer of a steady, laminar flow of an electrically conducting second grade fluid in a porous medium subject to a transverse uniform magnetic field over a stretching sheet with power law surface temperature or power law surface temperature gradient. The energy equation includes the viscous dissipation, work done due to deformation and internal heat generation or absorption. The resulting closed form solutions of the velocity components show that the magnitude of dimensionless surface velocity gradient depends on the viscoelastic parameter of the second grade fluid and the combined parameter, associated with the permeability parameter of porous medium and the contribution due to the magnetic field. He concluded that, the magnitude of dimensionless surface velocity gradient, increases with the combined parameter but decreases with the effect of viscoelastic parameter, which implies that, the viscoelasticity of the second grade fluid reduces the skin friction at the sheet, while, the combined effects of the permeability and magnetic field increase the power needed to stretch the sheet.

Cortell [13] studied the flow and heat transfer of a steady, laminar flow of an incompressible and electrically conducting second grade fluid subject to suction and to a transverse uniform magnetic field past a stretching sheet. In his study, the viscoelastic parameter is the parameter of interest. The equations for the heat transfer analysis were solved by the Runge–Kutta method of fourth order. Two different cases have been analyzed for the equations of heat transfer. At the first case (constant surface temperature) he concluded that, the temperature decreases as the viscoelastic parameter increases and the dimensionless heat transfer coefficient increases with increase in the viscoelastic parameter. At the second case (prescribed surface temperature), his study included both viscous dissipation and work due to deformation. Also, he analyzed the effect on both temperature and temperature-gradient profiles when the contribution of heat due to elastic deformation is taken into account in the energy equation.

Ariel [14] developed a numerical algorithm to compute the stagnation point flow of second grade fluids with/without suction. The algorithm caters for the situation when the plate is impermeable or porous, it is applicable for all values of the physical parameters such as the viscoelasticity of the fluid and the suction velocity. The viability of the algorithm is checked by comparing the numerical results obtained by it with the asymptotic solutions for large suction for the problem of the stagnation point flow and with the exact analytical solution for the problem of the flow over a stretching sheet with suction. It is proposed to apply the algorithm developed in his paper to compute the flow of second grade fluids in other situations, e.g., the flow near a rotating disk, and particularly in finite domains such as the flow through the channels, between disks, etc.

This paper is concerned with the solution of steady laminar flow of an incompressible Rivlin–Ericksen fluid of second grade with a uniform suction. Lie-group method is applied to the equations of motion for determining symmetry reductions of partial differential equations, [15–21]. The resulting system of nonlinear differential equations is then solved numerically using shooting method coupled with Runge–Kutta scheme.

## 2 Mathematical formulation of the problem

Consider the steady, laminar two-dimensional flow of an incompressible Rivlin–Ericksen fluid of second grade impinging perpendicular on a permeable wall and flows away along the $\bar{x}$-axis. The wall is placed in the plane $\bar{y} = 0$ of a Cartesian system of coordinates $O\bar{x}\bar{y}$ ($\bar{y} = 0$). The wall has a uniform temperature $\bar{T}_w$ and the free stream temperature is $\bar{T}_\infty$. A uniform suction is applied at the wall along $\bar{y}$-direction with a velocity given by $\bar{v} = -v_0$, where $v_0$ is a constant in which $v_0 > 0$, indicates the suction, Fig. 1. The velocity components for the potential flow are $(\bar{U}, \bar{V})$, where their distribution in the frictionless flow in the neighborhood of the stagnation point is given by $\bar{U} = a\bar{x}$ and $\bar{V} = -a\bar{y}$, where $a$ is a constant.

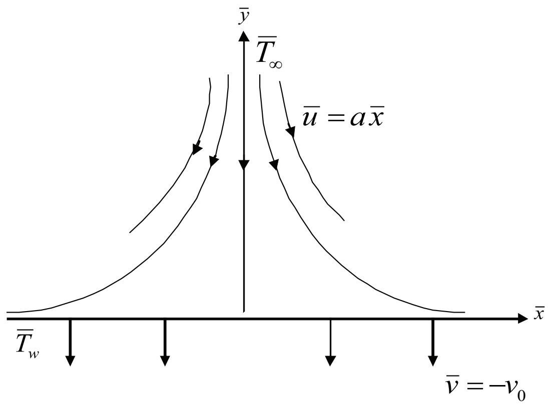


Fig. 1. Physical model and coordinate system.

The steady two-dimensional boundary layer equations, as discussed before, and the energy equation corresponding to the boundary layer analysis (without dissipation), for this fluid, are given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{10}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{U}\frac{\mathrm{d}\bar{U}}{\mathrm{d}\bar{x}} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\alpha_1}{\rho}\left[\frac{\partial}{\partial \bar{x}}\left(\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right], \tag{11}$$

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{12}$$

together with the boundary conditions

$$\bar{u} = 0, \qquad \bar{v} = -v_0, \quad \bar{T} = \bar{T}_w \quad \text{at} \;\; \bar{y} = 0, \tag{13a}$$

$$\bar{u} \to a\bar{x}, \quad \frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{T} \to \bar{T}_\infty \quad \text{as} \;\; \bar{y} \to \infty, \tag{13b}$$

where $\bar{u}$ and $\bar{v}$ are the fluid tangential and normal velocities, respectively, $\bar{U}$ is the free stream velocity over the body surface, $\nu$ is the kinematic viscosity, $c_p$ is the specific heat at constant pressure, $\bar{T}$ is the fluid temperature, $\alpha$ is the thermal conductivity, $\bar{T}_w$ is the wall temperature, and $\bar{T}_\infty$ is the ambient temperature.

We assumed an additional condition that is $\partial\bar{u}/\partial\bar{y} = 0$ in (13b), since the boundary conditions (13) are not sufficient to completely determine the solution. This extra condition is fully valid from the physical considerations, since there is no shear in the free stream. Indeed, it was used by Ariel [14], explicitly, in computing the stagnation point flow of second grade fluids when there is a suction at the wall, and also implicitly, in computing the flows of viscoelastic fluids due to rotating disk by Ariel [22], and radial stretching of a sheet by Ariel [23], and by Ayub et al. [24]. For further details on this issue, we refer the reader to Rajagopal [25], Rajagopal and Kaloni [26], and Chan Man Fong et al. [27].

Let us introduce the non-dimensional variables

$$x = \frac{a\bar{x}}{U_1}, \quad y = \sqrt{\frac{a}{\nu}}\bar{y}, \quad u = \frac{\bar{u}}{U_1}, \quad v = \frac{\bar{v}}{\sqrt{a\nu}}, \quad U = \frac{\bar{U}}{U_1}, \quad T = \frac{\bar{T} - \bar{T}_\infty}{\bar{T}_w - \bar{T}_\infty}, \tag{14}$$

where $U_1$ is the characteristic velocity.

Invoking (14), equations (10)–(12) are reduced to the dimensionless form as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{15}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{\partial^2 u}{\partial y^2} + k\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right], \tag{16}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial y^2}, \tag{17}$$

where $k = a\alpha_1/\mu$ is the viscoelastic parameter, $Pr = \mu c_p/\alpha$ is the Prandtl number and $\mu = \nu\rho$ is the molecular viscosity coefficient.

The boundary conditions (13) will be

$$u = 0, \quad v = -\frac{v_0}{\sqrt{a\nu}}, \quad T = 1 \quad \text{at } y = 0, \tag{18a}$$

$$u \to x, \quad \frac{\partial u}{\partial y} = 0, \qquad T \to 0 \quad \text{as } y \to \infty. \tag{18b}$$

From the continuity equation (15), there exists a stream function $\Psi(x, y)$ such that

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}, \tag{19}$$

which satisfies equation (15) identically.

Substituting from (19) into (16) and (17), gives

$$\begin{aligned}&\Psi_y\Psi_{xy} - \Psi_x\Psi_{yy} - UU_x - \Psi_{yyy}\\ &\quad - k(\Psi_{xy}\Psi_{yyy} + \Psi_y\Psi_{xyyy} - \Psi_x\Psi_{yyyy} - \Psi_{yy}\Psi_{xyy}) = 0,\end{aligned} \tag{20}$$

and

$$\Psi_y T_x - \Psi_x T_y - \frac{1}{Pr}T_{yy} = 0, \tag{21}$$

where subscripts denote partial derivatives.

The boundary conditions (18) will be

$$\Psi_y = 0, \quad \Psi_x = R, \quad T = 1 \quad \text{at } y = 0, \tag{22a}$$

$$\Psi_y \to x, \quad \Psi_{yy} = 0, \quad T \to 0 \quad \text{as } y \to \infty, \tag{22b}$$

where $R = v_0/\sqrt{a\nu}$ is the suction parameter.

## 3 Solution of the problem

At first, we derive the similarity solutions using Lie-group method under which (20)–(21) and the boundary conditions (22) are invariant, and then we use these symmetries to determine similarity variables.

### 3.1 Lie point symmetries

Consider the one-parameter $(\varepsilon)$ Lie group of infinitesimal transformations in $(x, y; \Psi, U, T)$ given by

$$\begin{aligned}
x^* &= x + \varepsilon\phi(x, y; \Psi, U, T) + O(\varepsilon^2),\\
y^* &= y + \varepsilon\zeta(x, y; \Psi, U, T) + O(\varepsilon^2),\\
\Psi^* &= \Psi + \varepsilon\eta(x, y; \Psi, U, T) + O(\varepsilon^2),\\
U^* &= U + \varepsilon F(x, y; \Psi, U, T) + O(\varepsilon^2),\\
T^* &= T + \varepsilon g(x, y; \Psi, U, T) + O(\varepsilon^2),
\end{aligned} \tag{23}$$

where $\varepsilon$ is the group parameter.

A system of partial differential equations (20) and (21) is said to admit a symmetry generated by the vector field

$$\mathrm{X} \equiv \phi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \Psi} + F\frac{\partial}{\partial U} + g\frac{\partial}{\partial T}, \tag{24}$$

if it is left invariant by the transformation $(x, y; \Psi, U, T) \to (x^*, y^*; \Psi^*, U^*, T^*)$.

The solutions $\Psi = \Psi(x, y)$, $T = T(x, y)$ and $U = U(x)$, are invariant under the symmetry (24) if

$$\Phi_\Psi = X\big(\Psi - \Psi(x, y)\big) = 0 \quad \text{when } \Psi = \Psi(x, y), \tag{25}$$

$$\Phi_T = X\big(T - T(x, y)\big) = 0 \quad \text{when } T = T(x, y), \tag{26}$$

and

$$\Phi_U = X\big(U - U(x)\big) = 0 \qquad \text{when } U = U(x). \tag{27}$$

Assume,

$$\begin{aligned}
\Delta_1 &= \Psi_y\Psi_{xy} - \Psi_x\Psi_{yy} - UU_x - \Psi_{yyy}\\
&\quad - k(\Psi_{xy}\Psi_{yyy} + \Psi_y\Psi_{xyyy} - \Psi_x\Psi_{yyyy} - \Psi_{yy}\Psi_{xyy}),\\
\Delta_2 &= \Psi_y T_x - \Psi_x T_y - \frac{1}{Pr}T_{yy}.
\end{aligned} \tag{28}$$

A vector $X$ given by (24), is said to be a Lie point symmetry vector field for (20) and (21) if

$$X^{[4]}(\Delta_j)|_{\Delta_j=0} = 0, \quad j = 1, 2, \tag{29}$$

where

$$X^{[4]} \equiv \phi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \Psi} + F\frac{\partial}{\partial U} + g\frac{\partial}{\partial T} + \eta^{x}\frac{\partial}{\partial \Psi_x} + \eta^{y}\frac{\partial}{\partial \Psi_y} + g^{x}\frac{\partial}{\partial T_x}$$
$$+ g^{y}\frac{\partial}{\partial T_y} + F^{x}\frac{\partial}{\partial U_x} + \eta^{xy}\frac{\partial}{\partial \Psi_{xy}} + \eta^{yy}\frac{\partial}{\partial \Psi_{yy}} + g^{yy}\frac{\partial}{\partial T_{yy}}$$
$$+ \eta^{xyy}\frac{\partial}{\partial \Psi_{xyy}} + \eta^{yyy}\frac{\partial}{\partial \Psi_{yyy}} + \eta^{xyyy}\frac{\partial}{\partial \Psi_{xyyy}} + \eta^{yyyy}\frac{\partial}{\partial \Psi_{yyyy}}, \tag{30}$$

is the fourth prolongation of $X$.

To calculate the prolongation of the given transformation, we need to differentiate (23) with respect to each of the variables, $x$ and $y$. To do this, we introduce the following total derivatives

$$\begin{aligned} D_x &\equiv \partial_x + \Psi_x\partial_\Psi + U_x\partial_U + T_x\partial_T + \Psi_{xx}\partial_{\Psi_x} + U_{xx}\partial_{U_x} + T_{xx}\partial_{T_x} \\ &\quad + \Psi_{xy}\partial_{\Psi_y} + \dots, \\ D_y &\equiv \partial_y + \Psi_y\partial_\Psi + T_y\partial_T + \Psi_{yy}\partial_{\Psi_y} + T_{yy}\partial_{T_y} + \Psi_{xy}\partial_{\Psi_x} + \dots. \end{aligned} \tag{31}$$

Equation (29) gives the following system of linear partial differential equations

$$\begin{aligned} &-U_xF + (k\Psi_{yyyy} - \Psi_{yy})\eta^x + (\Psi_{xy} - k\Psi_{xyyy})\eta^y - UF^x \\ &\quad + (\Psi_y - k\Psi_{yyy})\eta^{xy} + (k\Psi_{xyy} - \Psi_x)\eta^{yy} + k\Psi_{yy}\eta^{xyy} \\ &\quad - (1 + k\Psi_{xy})\eta^{yyy} - k\Psi_y\eta^{xyyy} + k\Psi_x\eta^{yyyy} = 0, \\ &-T_y\eta^x + T_x\eta^y + \Psi_y g^x - \Psi_x g^y - \frac{1}{Pr}g^{yy} = 0. \end{aligned} \tag{32}$$

The components $\eta^x$, $\eta^y$, $g^x$, $g^y$, $F^x$, $\eta^{xy}$, $\eta^{yy}$, $g^{yy}$, $\eta^{xyy}$, $\eta^{yyy}$, $\eta^{xyyy}$, $\eta^{yyyy}$ can be determined from the following expressions:

$$\begin{aligned} \eta^S &= D_S\eta - \Psi_x D_S\phi - \Psi_y D_S\zeta, \\ g^N &= D_N g - T_x D_N\phi - T_y D_N\zeta, \\ F^x &= D_x F - U_x D_x\phi, \\ \eta^{JS} &= D_S\eta^J - \Psi_{Jx}D_S\phi - \Psi_{Jy}D_S\zeta, \\ g^{JN} &= D_N g^J - T_{Jx}D_N\phi - T_{Jy}D_N\zeta, \end{aligned} \tag{33}$$

where $S$, $J$ and $N$ stand for $x, y$.

Substitution from (33) into (32) and solving the resulting linear system, which is called determining equations, in view of the invariance of the boundary conditions (22), yields [28–31]

$$\phi = C_1x, \quad \zeta = C_2, \quad \eta = C_1\Psi + C_3, \quad F = C_1U, \quad g = 0. \tag{34}$$

From which, the system of nonlinear equations (20)–(21) has the three-parameter Lie group of point symmetries generated by

$$X_1 \equiv x\frac{\partial}{\partial x} + \Psi\frac{\partial}{\partial \Psi} + U\frac{\partial}{\partial U}, \quad X_2 \equiv \frac{\partial}{\partial \Psi} \quad \text{and} \quad X_3 \equiv \frac{\partial}{\partial y}. \tag{35}$$

The one-parameter group generated by $X_1$ consists of scaling, whereas $X_2$ and $X_3$ consists of translation. The commutator table of the symmetries is given in Table 1, where the entry in the $i$th row and $j$th column is defined as $[X_i, X_j] = X_iX_j - X_jX_i$.

Table 1. Table of commutators of the basis operators.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | 0 | $-X_2$ | 0 |
| $X_2$ | $X_2$ | 0 | 0 |
| $X_3$ | 0 | 0 | 0 |

The finite transformations corresponding to the symmetries $X_1$, $X_2$ and $X_3$ are respectively

$$\begin{array}{llllll} X_1\colon & x^* = e^{\varepsilon_1}x, & y^* = y, & \Psi^* = e^{\varepsilon_1}\Psi, & U^* = e^{\varepsilon_1}U, & T^* = T, \\ X_2\colon & x^* = x, & y^* = y, & \Psi^* = \Psi + \varepsilon_2, & U^* = U, & T^* = T, \\ X_3\colon & x^* = x, & y^* = y + \varepsilon_3, & \Psi^* = \Psi, & U^* = U, & T^* = T, \end{array} \tag{36}$$

where $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are the group parameters.

For $X_2$, the characteristic

$$\Phi = (\Phi_\Psi, \Phi_U, \Phi_T) \tag{37}$$

has the components

$$\Phi_\Psi = 1, \quad \Phi_U = 0, \quad \Phi_T = 0, \tag{38}$$

which means that no solutions are invariant under the group generated by $X_2$.

For $X_3$, the characteristic (37) has the components

$$\Phi_\Psi = -\Psi_y, \quad \Phi_U = 0, \quad \Phi_T = -T_y. \tag{39}$$

Therefore, the general solutions of the invariant surface conditions (25) and (26) are

$$\Psi \equiv \Psi(x), \quad T \equiv T(x). \tag{40}$$

From (40) into (19), we get

$$u = 0, \quad v = v(x), \quad T \equiv T(x). \tag{41}$$

Equation (41) is a solution of the continuity equation (15) and the energy equation (17), even though it is not a particularly interesting one since it contradicts the boundary conditions.

For $X_1$, the characteristic (37) has the components

$$\Phi_\Psi = \Psi - x\Psi_x, \quad \Phi_U = U - xU_x, \quad \Phi_T = -xT_x. \tag{42}$$

Hence, the general solutions of the invariant surface conditions (25)–(27) are

$$\Psi = xG(y), \quad U(x) = x, \quad T = h(y). \tag{43}$$

Substitution from (43) into (20)–(21) yields

$$\frac{\mathrm{d}^3G}{\mathrm{d}y^3} + G\frac{\mathrm{d}^2G}{\mathrm{d}y^2} - \left(\frac{\mathrm{d}G}{\mathrm{d}y}\right)^2 + 1 - k\left[G\frac{\mathrm{d}^4G}{\mathrm{d}y^4} - 2\frac{\mathrm{d}G}{\mathrm{d}y}\frac{\mathrm{d}^3G}{\mathrm{d}y^3} + \left(\frac{\mathrm{d}^2G}{\mathrm{d}y^2}\right)^2\right] = 0, \tag{44}$$

and

$$\frac{\mathrm{d}^2h}{\mathrm{d}y^2} + Pr\, G\frac{\mathrm{d}h}{\mathrm{d}y} = 0. \tag{45}$$

The boundary conditions (22) will be

$$\frac{\mathrm{d}G}{\mathrm{d}y} = 0, \quad G = R, \qquad h = 1 \quad \text{at } y = 0, \tag{46a}$$

$$\frac{\mathrm{d}G}{\mathrm{d}y} \to 1, \quad \frac{\mathrm{d}^2G}{\mathrm{d}y^2} = 0, \quad h \to 0 \quad \text{as } y \to \infty. \tag{46b}$$

For $X_1 + \beta X_2$, the characteristic (37) has the components

$$\Phi_\Psi = \Psi + \beta - x\Psi_x, \quad \Phi_U = U - xU_x, \quad \Phi_T = -xT_x. \tag{47}$$

Hence, the general solutions of the invariant surface conditions (25)–(27) are

$$\Psi = xG(y) - \beta, \quad U(x) = x, T = h(y). \tag{48}$$

Substitution from (48) into (20)–(21) yields the same ordinary differential equations (44)–(45) with the same boundary conditions (46). So, the solutions invariant under both $X_1$ and $X_1 + \beta X_2$ are the same.

For $X_1 + \beta X_3$, the characteristic (37) has the components

$$\Phi_\Psi = \Psi - x\Psi_x - \beta\Psi_y, \quad \Phi_U = U - xU_x, \quad \Phi_T = -xT_x - \beta T_y. \tag{49}$$

Hence, the general solution of the invariant surface condition (25) is

$$\Psi = xL(\lambda), \tag{50}$$

where $\lambda = \beta \ln x - y$ is the similarity variable which contradicts the boundary conditions.

For $X_2 + \beta X_3$, the characteristic (37) has the components

$$\Phi_\Psi = 1 - \beta\Psi_y, \quad \Phi_U = 0, \quad \Phi_T = -\beta T_y. \tag{51}$$

Hence, the general solutions of the invariant surface condition (25)–(26) are

$$\Psi = \frac{1}{\beta}y + H(x), \quad T \equiv T(x). \tag{52}$$

Substitution from (52) into (21) gives

$$T_x = 0. \tag{53}$$

Equation (53) yields $T = const$, which not a particularly interesting one since it contradicts the boundary conditions.

For $X_1 + \beta X_2 + \gamma X_3$, the characteristic (37) has the components

$$\Phi_\Psi = \Psi + \beta - x\Psi_x - \gamma\Psi_y, \quad \Phi_U = U - xU_x, \quad \Phi_T = -xT_x - \gamma T_y. \tag{54}$$

Hence, the general solution of the invariant surface condition (25) is

$$\Psi = xM(\lambda) - \beta, \tag{55}$$

where $\lambda = \gamma \ln x - y$ is the similarity variable which contradicts the boundary conditions.

### 3.2 Numerical solution

The system of nonlinear differential equations (44)–(45) with the boundary conditions (46) is solved numerically using the shooting method, coupled with Runge–Kutta scheme.

From (19) and (43), we get

$$\frac{u}{x} = \frac{\mathrm{d}G}{\mathrm{d}y}, \quad v = -G \quad \text{and} \quad T = h(y). \tag{56}$$

## 4 Results and discussion

### 4.1 Tangential velocity

#### 4.1.1 *The effect of the viscoelastic parameter $k$*

Fig. 2 illustrates the behaviour of the tangential velocity $u/x$ for $R = 1.0$, over a range of the viscoelastic parameter $k$. As seen, the tangential velocity increases as $k$ decreases. That is because a decrease in the viscoelastic parameter $k$ causes a decrease in the velocity boundary layer thickness and as a result an increase in the tangential velocity.

#### 4.1.2 *The effect of the suction parameter $R$*

Fig. 3 illustrates the behaviour of the tangential velocity $u/x$ for $k = 5.0$, over a range of the suction parameter $R$. The tangential velocity increases as $R$ increases. That is because an increase in the suction parameter $R$ causes a decrease in the velocity boundary layer thickness and as a result an increase in the tangential velocity.

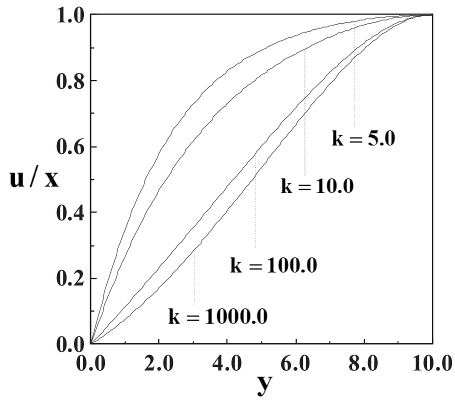


Fig. 2. Tangential velocity profiles over a range of $k$ with $R = 1.0$.

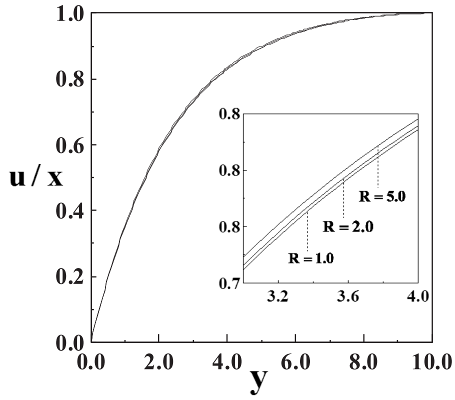


Fig. 3. Tangential velocity profiles over a range of $R$ with $k = 5.0$.

## 4.2 Normal velocity

### 4.2.1 *The effect of the viscoelastic parameter $k$*

Fig. 4 illustrates the behaviour of the normal velocity $G(y) = -v$ for $R = 1.0$, over a range of the viscoelastic parameter $k$. It is clear that, the normal velocity increases as $k$ decreases, as mentioned before.

### 4.2.2 *The effect of the suction parameter $R$*

Fig. 5 illustrates the behaviour of the normal velocity $G(y) = -v$ for $k = 5.0$, over a range of the suction parameter $R$. We noticed that, the normal velocity increases as $R$ increases.

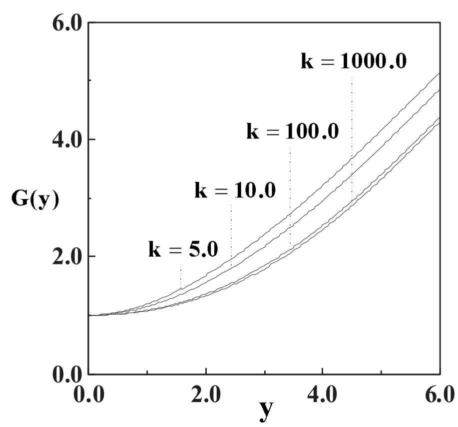


Fig. 4. Normal velocity profiles over a range of $k$ with $R = 1.0$.

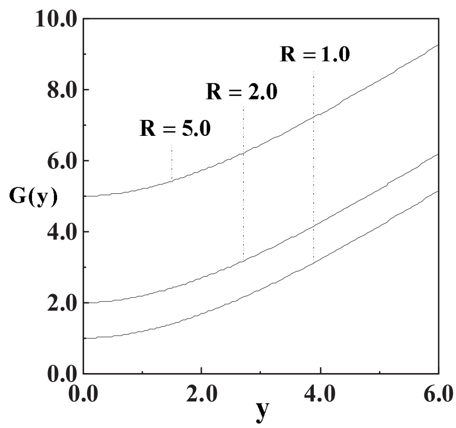


Fig. 5. Normal velocity profiles over a range of $R$ with $k = 5.0$.

### 4.3 The temperature profiles

#### 4.3.1 *The effect of the viscoelastic parameter $k$*

Fig. 6 illustrates the variation of the temperature profiles $T$ for $Pr = 0.5$ and $R = 1.0$, over a range of the viscoelastic parameter $k$. We noticed that, the temperature profiles decrease as $k$ decreases and therefore the thinning of the thermal boundary layer. That is because, for a fixed value for $Pr$, a decrease in the viscoelastic parameter $k$ causes a decrease in the velocity boundary layer thickness and as a result a decrease in the thermal boundary layer thickness. Our result is in complete agreement with that reported by Massoudi [10]. A small variation in the rate of increase of $T$ appears with higher values $k$ and this variation becomes more evident with lower values of $k$.

#### 4.3.2 *The effect of the suction parameter $R$*

Fig. 7 illustrates the variation of the temperature profiles $T$ for $Pr = 0.5$ and $k = 5.0$, over a range of the suction parameter $R$. As seen, the temperature increases as $R$ decreases.

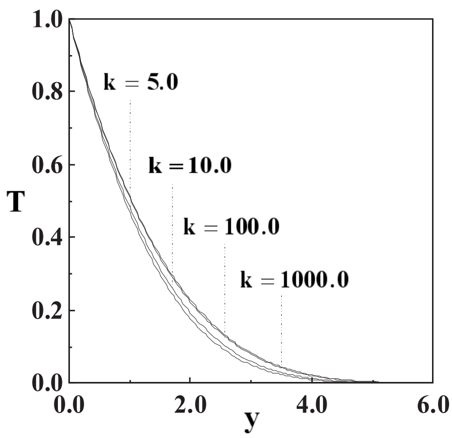


Fig. 6. Temperature profiles over a range of $k$ with $Pr = 0.5$ and $R = 1.0$.

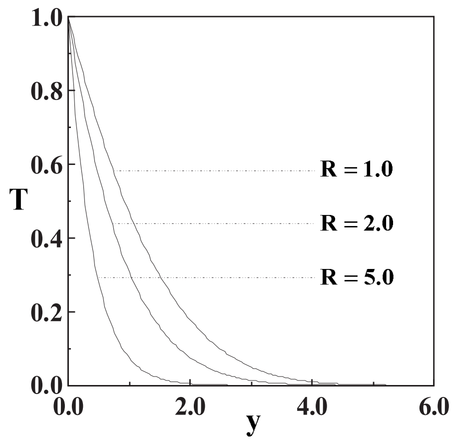


Fig. 7. Temperature profiles over a range of $R$ with $Pr = 0.5$ and $k = 5.0$.

#### 4.3.3 *The effect of the Prandtl mumber $Pr$*

Fig. 8 illustrates the variation of the temperature profiles $T$ for $k = 10.0$ and $R = 1.0$, over a range of Prandtl number $Pr$. It is noticed that, as $Pr$ decreases, the thickness of the thermal boundary layer becomes greater than the thickness of the velocity boundary layer according to the well known relation $\delta_T/\delta \approx Pr^{-1/2}$, where $\delta_T$ is the thickness of the thermal boundary layer and $\delta$ is the thickness of the velocity boundary layer. So, the thickness of the thermal boundary layer increases as $Pr$ decreases and hence, the temperature $T$ increases with the decrease of $Pr$.

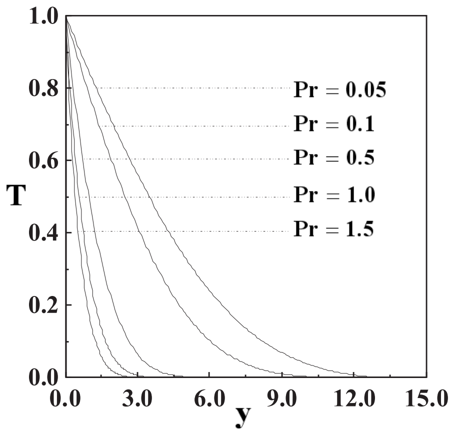


Fig. 8. Temperature profiles over a range of $Pr$ with $k = 10.0$ and $R = 1.0$.

### 4.4 Heat transfer

As mentioned before, when the Prandtl number increases, the thickness of the thermal boundary layer becomes thinner and this causes an increase in the gradient of the temperature. Therefore, the heat transfer coefficient $(-h'(0))$ increases as $Pr$ increases. Our result is in complete agreement with that reported by Liu [12]. For different values of the viscoelastic parameter $k$ and Prandtl number $Pr$ at $R = 1.0$, computed values of the heat transfer coefficient were calculated, Table 2.

Table 2. Values of $-h'(0)$ at $R = 1.0$ for different values of $k$ and $Pr$.

| | $-h'(0)$ | | |
|---|---|---|---|
| $k$ | $Pr = 0.5$ | $Pr = 1.0$ | $Pr = 1.5$ |
| 5.0 | 0.6796938 | 1.1648321 | 1.6460344 |
| 10.0 | 0.6603802 | 1.1416527 | 1.6223326 |
| 100.0 | 0.6271725 | 1.1019078 | 1.5823146 |
| 1000.0 | 0.6214285 | 1.0949981 | 1.5754198 |

From Table 2, it is noticed that, for a fixed value of $Pr$, the heat transfer coefficient $(-h'(0))$ decreases as the viscoelastic parameter $k$ increases. Also, the value of $(-h'(0))$ is positive which is consistent with the fact that the heat flows from the surface to the fluid in the absence of viscous dissipation.

### 4.5 Wall shear stress

The dimensionless wall shear stress $G''(0)$is computed for different values of the viscoelastic parameter $k$ and suction parameter $R$. As seen from Table 3, the value of the dimensionless wall shear stress $G''(0)$ decreases as $k$ increases which is consistent with the fact that, there is progressive thinning of the boundary layer with increasing $k$. Also, the value of the dimensionless wall shear stress $G''(0)$ increases as $R$ increases for fixed value of viscoelastic parameter $k$. The computed values of $G''(0)$ are compared with those obtained by Ariel [14], for different values of the viscoelastic parameter $k$ with suction parameter $R = 10$. The results agree very well, Table 4.

Table 3. Values of dimensionless wall shear stress $G''(0)$ for different $k$ and $R$.

| | $G''(0)$ | | |
|---|---|---|---|
| $k$ | $R = 2.0$ | $R = 5.0$ | $R = 10.0$ |
| 5.0 | 0.4310454 | 0.4374629 | 0.4418974 |
| 10.0 | 0.3103676 | 0.3117219 | 0.3132360 |
| 20.0 | 0.2216492 | 0.2221664 | 0.2224629 |
| 50.0 | 0.1410701 | 0.1430986 | 0.1441271 |
| 100.0 | 0.0999251 | 0.1001764 | 0.1028501 |

Table 4. Comparison between the values of $G''(0)$ for different $k$ with $R = 10$.

| | $G''(0)$ | | |
|---|---|---|---|
| | Ariel [14] | | Present work |
| $k$ | Exact numerical solution | Asymptotic solution | Numerical solution |
| 5.0 | 0.440519 | 0.442889 | 0.4418974 |
| 10.0 | 0.313218 | 0.315207 | 0.3132360 |
| 20.0 | 0.222283 | 0.223909 | 0.2224629 |
| 50.0 | 0.140989 | 0.142190 | 0.1441271 |
| 100.0 | 0.099819 | 0.100750 | 0.1028501 |

## 5 Conclusion

Lie-group method is applicable to both linear and non-linear partial differential equations, which leads to similarity variables that used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which the given partial differential equations are invariant, we can obtain information about the invariants and symmetries of these equations. This information can be used to determine the similarity variables that will reduce the number of independent variables in the system. In this work, we have used Lie-group method to obtain similarity reductions of nonlinear boundary layer equations (10)–(12), for the two-dimensional steady, laminar flow of an incompressible Rivlin–Ericksen fluid of second grade with a uniform suction. By determining the transformation group under which the given partial differential equations are invariant, we obtained the invariants and the symmetries of these equations. In turn, we used these invariants and symmetries to determine the similarity variables that reduced the number of independent variables.

The resulting system of non-linear ordinary differential equations (44)–(45) subjected to the boundary conditions (46), is solved numerically using the shooting method coupled with Runge–Kutta scheme. We have studied the effects of the viscoelastic parameter $k$ and the suction parameter $R$ on the tangential velocity $u/x$, normal velocity $G(y) = -v$, temperature profiles $T$, heat transfer coefficient $(-h'(0))$ and the wall shear stress. Also, the effect of the Prandtl number $Pr$ on the temperature and the heat transfer coefficient has been studied. Our results are in complete agreement with those reported by Massoudi [10] and Liu [12]. Particular computed values of the dimensionless wall shear stress $G''(0)$ are compared with those obtained by Ariel [14] and were found to agree very well with his results.

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