The effect of environmental tax on the survival of biological species in a polluted environment: a mathematical model

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Received: 2009-11-03 **Revised:** 2010-04-01 **Published online:** 2010-09-17

Abstract. In this paper, a nonlinear mathematical model is proposed and analyzed for the survival of biological species affected by a pollutant present in the environment. It is considered that the emission of the pollutant into the environment is dynamic in nature and depends on the environmental tax imposed on the emitters. It is also assumed that the environmental tax is imposed to control the emission of pollutants only when the concentration level of pollutants in the environment crosses a limit over which the pollutants starts causing harm to the population under consideration. Criteria for local stability, global stability and permanence are obtained using theory of ordinary differential equations. Numerical simulations are carried out to investigate the dynamics of the system using fourth order Runge–Kutta Method. It is found that, as the emission rate of pollutants in the environment increases, the density of biological species decreases. It may also be pointed out that the biological species may even become extinct if the rate of emission of pollutants increases continuously. However, if some environmental taxes are imposed to control the rate of emission of these pollutants into the environment, the density of biological species can be maintained at a desired level.

Keywords: biological species, toxicant, environmental tax, numerical simulation, Runge–Kutta.

1 Introduction

Due to the rapid pace of industrialization, various kind of pollutants like oxides of sulphur or oxides of carbon enter into both aquatic and terrestrial environment. These pollutants may be emitted into the environment from different sources (e.g. industries, vehicles, thermal power plant, refineries, etc.) as well as by incessant use of natural resources without recharging and cleaning them. All these pollutants adversely affect the ecosystem – water, air, vegetation, forestry resources and the land which in turn affect the survival of large number of biological species directly as well as indirectly by deteriorating the resource biomass on which some biological species are dependent [1–5]. The examples of this kind of problems may be found in the ecosystems in which the air pollution affects the forests and then the survival of forest dependent biological species. In order to use and irregulate these toxic substances wisely, we must assess the risk of the populations exposed to toxicants. Therefore, it is very important to study the effects of toxicants on biological populations and to obtain conditions for sustainability of population.

In recent decades, several investigators have proposed and analyzed mathematical models to study the effects of toxicants on biological species [6–9]. In particular, Hallam et al. [8, 9] have proposed and analyzed mathematical models to study the effects of toxicants on biological species when these are emitted into the environment from external sources. Hauping and Zhien [10] have proposed a mathematical model to study the effect of a toxicant on naturally stable two species communities. In these investigations the effects of a toxicant simultaneously on growth rate and carrying capacity of the species have not been considered. However, Freedman and Shukla [7] proposed models to study the effects of a toxicant on single-species and predator-prey systems by assuming that the intrinsic growth rate of species decreases as the uptake concentration of the toxicant increases while its carrying capacity decreases with the environmental concentration of the toxicant. Shukla et al. [11] have studied the effects of primary and secondary pollutants on a renewable resource using the same consideration. Shukla et al. [12] have also studied the survival of two competing species in a polluted environment using similar assumptions and showed that the usual competitive outcomes may be altered in the presence of a toxicant, see also [13]. Buonomo et al. [14] have proposed a mathematical model to study the effects of toxicants on a biological population and obtained a threshold value which determines permanence or extinction of biological population.

The environmental problem in India is growing rapidly. Industrial pollution, soil erosion, deforestation, rapid industrialization, urbanization, and land degradation are all worsening problems that need to be addressed. So there must be some strategies to improve the quality of environment declines and whosoever is responsible for causing damage must pay the costs of measures taking the remedy to ill effects. In the recent years, environmental policy makers have suggested that environmental taxes including pollution charges (emission/effluent tax/pollution tax) are one answer to improve the environment at right time. They also suggested for India to tackle the issue of increasing pollution that “Reduce tax on employees and employers and put a tax on pollution”. Such taxes make the polluters pay and thus internalize environmental externalities, revealing the true social cost of production in prices. Examples of environmental taxes include (i) carbon taxes (where industry is taxed for every unit of carbon dioxide emitted, which incentivizes firms to find energy-efficient and low-carbon alternatives), the more carbon dioxide one emits the more he pays in taxes (ii) congestion charges (where motorists pay a fee for entering the “congestion zone” which encourages them to make fewer journeys in these areas), and (iii) fuel duties (where petrol is taxed to encourage to motorists to buy more-efficient cars or use their cars loss frequently). This type of strategies have already been adopted in some countries and the concept of polluters must pay has been popularized. The government of India is considering to impose a special fee on vehicles entering the central business districts of metropolitan cities as part of steps to cut carbon emission and reduce traffic congestion (25/12/2008, Economy Times (New Delhi)). Thomas et al. [15] proposed and analyzed a mathematical model to study the effect of pollutant on a single species population and obtained some criteria to restrict the amount of toxicant in the environment to ensure the survival of the species. Schultink [1] showed that environmental pollution limits or impact standard may be used to define the public risk tolerance limits and carrying capacity constraints. He also proved that thematic indicators and derived indices may be effective in resource assessment and economic evaluation. But in the above mentioned studies, impacts of environmental tax to limit the pollution and its effect on biological species have not been considered.

Keeping in mind the above, in this paper, we propose and analyze a mathematical model to introduce the concept of environmental tax (pollution tax) and its effects on the survival of biological population in a polluted environment. Stability theory of ordinary differential equations is used to analyze the model. To compliment these analytical findings, we present a numerical simulation using fourth order Runge–Kutta method.

2 The mathematical model

Consider a biological species such as plant/tree population in a forest stand affected by the pollutant emitted into the environment by different type of industrial process. It is assumed that the growth rate of the species decreases with the uptake of pollutant by the species. It is also assumed that the introduction of the pollutant from the different sources is dynamic in nature and depends on the environmental tax imposed on the emitters. Tax is imposed only if concentration of pollutant in the environment crosses a permissible limit (limit up to which there is no harm to the population) and method of imposing tax is devised on the basis of emission of pollutants by a particular industry e.g. if an industry emits one unit of carbon mono oxide in a day, tax of one unit rupee would be imposed on the same. A four-dimensional mathematical model governing the situation is given as follows

$$\begin{aligned}
\dot{x}(t) &= x(t)\big(r_0 - r_1 U(t) - f x(t)\big),\\
\dot{U}(t) &= kT(t) - lU(t) - mU(t),\\
\dot{T}(t) &= Q - k_1 T(t)x(t) - hT(t) + l_1 U(t)x(t) - \rho g\big(I(t)\big),\\
\dot{I}(t) &= u_{T_0}\big(T(t)\big)\big[\theta\big(T(t) - T_0\big) - \theta_0 I(t)\big],
\end{aligned} \tag{1}$$

where $x(0) = x_0 \geq 0,\ U(0) = U_0 \geq 0,\ T(0) = T_0 \geq 0$;

$$r_0,\ r_1,\ f,\ k,\ l,\ m,\ k_1,\ h,\ l_1,\ Q,\ \rho,\ \theta,\ \theta_0,\ T_0 > 0;$$

$$u_{T_0}(T) = \begin{cases} 0 & \text{if } T < T_0, \\ 1 & \text{if } T > T_0, \end{cases} \quad \text{for } T_0 \geq 0.$$

In model (1), $x(t)$ is the density of biological species, $U(t)$ is the uptake concentration of the pollutant by the species, $T(t)$ is the concentration of pollutant in the environment and $I(t)$ is the environmental tax imposed on the emitters at time $t \geq 0$. r_0

is the intrinsic growth rate of the population in the environment without pollutant, r_1 is the decreasing rate of the intrinsic growth rate associated with the uptake of the pollutant. The first equation of (1) assumes that the population $x(t)$ satisfies the logistic equation and the pollutant causes the intrinsic growth rate of the population to decrease linearly. We can see that if $r_0 - r_1 U \leq 0$, then $x(t)$ will get extinct in the end, so we suppose

$$U < \frac{r_0}{r_1}. \tag{2}$$

k is the uptake rate coefficient of pollutant due to the biological population. l and m are the depletion rate coefficients of U due to egestion and depuration of pollutants, respectively. Constant Q represents the rate of introduction of pollutant into the environment, ρ is the tax repulsion coefficient and $g(I)$ is the function of I, the environmental tax, introduced to control the emission of pollutants, collected in unit period of time from various industries of the ecosystem as they emit pollutants. When there is no tax, this emission of the pollutants is considered as Q, which is constant at particular point of time but with the imposition of tax, this introduction is limited by the factor $\rho g(I)$. This indicates that with the increase in total environmental tax, concentration of pollutants in the environment decreases with time. For our analysis, we consider this function $g(I)$ as a very simple identity function, i.e. $g(I) = I$. k_1 is the depletion rate coefficient of T due to its uptake by the biological species, h is the loss rate of pollutant due to various natural processes including biological transformation, volatization, gravitational deposition on the ground leading to chemical hydrolysis etc., l_1 is the increase rate of pollutant due to egestion of the species. Tax is imposed only if T crosses the permissible limit T_0 (limit upto which there is no harm to the population). θ is the tax rate coefficient and the term $\theta_0 I$ in the fourth equation of system (1) is considered due to some practical difficulties on implementing the foolproof tax system. In every tax system, there are some pilferages, natural and administrative problems and faults of the system due to which the increase of the tax amount is not directly proportional to the difference of T and T_0. The unit step function $u_{T_0}(T)$ is introduced to ensure that when T is less than T_0, no tax is introduced to the system, change of I with t becomes zero. The factor θ_0 is small in comparison to θ and it can be safely considered that when I is small so that right hand side of fourth equation of (1) remains positive. For all practical purposes we are concerned with a system in which the concentration of pollutants has crossed the harmful limit and there is no need to use the step function. Therefore, in the rest of this paper, we consider the value of step function as 1 and mathematical model as given below:

$$\begin{aligned}
\dot{x}(t) &= x(t)\big(r_0 - r_1 U(t) - f x(t)\big),\\
\dot{U}(t) &= kT(t) - lU(t) - mU(t),\\
\dot{T}(t) &= Q - k_1 T(t) x(t) - hT(t) + l_1 U(t) x(t) - \rho I(t),\\
\dot{I}(t) &= \theta\big(T(t) - T_0\big) - \theta_0 I(t),
\end{aligned} \tag{3}$$

where $x(0) = x_0 \geq 0,\ U(0) = U_0 \geq 0,\ T(0) = T_0 \geq 0.$

3 Boundedness of solutions

To analyze the model (3), we need the bounds of dependent variables involved. For this we find the region of attraction in the following lemma.

Lemma 1. *If $h > k$ and $l + m - \frac{l_1 r_0}{f} > 0$. Then the set*

$$\Omega = \left\{(x, U, T, I) \colon\ 0 \le x \le \frac{r_0}{f},\ \ 0 \le U + T \le \frac{Q}{\delta},\ \ 0 \le I \le \frac{Q\theta}{\delta\theta_0}\right\},$$

where $\delta = \min\{l + m - \frac{l_1 r_0}{f}, h - k\}$, attracts all solutions initiating in the interior of the positive octant.

Proof. It is assumed here that all the initial values of variables considered in model (3) belongs to the region Ω and are positive.

From the first equation of system (3), we have

$$\dot{x} \le x(r_0 - fx).$$

This implies that, $\limsup_{t\to\infty} x(t) \le \frac{r_0}{f}$.

From the second and third equation of model (3), we get

$$\begin{aligned}\dot{U} + \dot{T} &\le Q + kT - lU - mU - hT + l_1 U x \\ &\le Q - \left(l + m - \frac{l_1 r_0}{f}\right)U - (h-k)T \\ &\le Q - \delta(U + T),\end{aligned}$$

where $\delta = \min\{l + m - \frac{l_1 r_0}{f}, h - k\}$. This implies that, $\limsup_{t\to\infty}(U(t) + T(t)) \le \frac{Q}{\delta}$.

Now from the last equation of model (3), we have

$$\dot{I} \le \theta T - \theta_0 I \le \frac{Q\theta}{\delta} - \theta_0 I.$$

This implies that, $\limsup_{t\to\infty} I(t) \le \frac{Q\theta}{\delta\theta_0}$. This completes the proof of the Lemma 1. □

Biological interpretation of conditions involved in Lemma 1

The condition $h > k$ implies that the loss rate of pollutants due to various natural processes should be greater than the uptake rate coefficient of pollutant due to the biological population for the boundedness of solutions. The condition $\frac{l+m}{l_1} > \frac{r_0}{f}$ implies that if the depletion rate coefficients of U due to egestion and depuration of pollutants i.e. l and m respectively are small and increase rate of pollutants due to egestion of the species i.e. l_1 is large then the solutions of the system (3) may not be bounded.

4 Equilibrium analysis

The system (3) has two non-negative equilibria in x, U, T, I space namely, $E_0(0, \bar{U}, \bar{T}, \bar{I})$ and $E_1(\hat{x}, \hat{U}, \hat{T}, \hat{I})$.

Existence of $E_0(0, \bar{U}, \bar{T}, \bar{I})$

In this case $\bar{U}, \bar{T}, \bar{I}$ are obtained by solving the following equations:

$$\begin{aligned}
&k\bar{T} - l\bar{U} - m\bar{U} = 0,\\
&Q - \rho\bar{I} - h\bar{T} = 0,\\
&\theta\big(\bar{T} - T_0\big) - \theta_0\bar{I} = 0.
\end{aligned}$$

Clearly, $\bar{T} = \frac{Q\theta_0 + \rho\theta T_0}{\rho\theta + h\theta_0} > 0$, $\bar{U} = \frac{k(Q\theta_0 + \rho\theta T_0)}{(l+m)(\rho\theta + h\theta_0)} > 0$, and $\bar{I} = \frac{\theta(Q - hT_0)}{\rho\theta + h\theta_0} > 0$ if and only if $Q > hT_0$.

Existence of $E_1(\hat{x}, \hat{U}, \hat{T}, \hat{I})$

In this case $\hat{x}, \hat{u}, \hat{T}, \hat{I}$ are the positive solutions of the following equations

$$r_0 - r_1 U - fx = 0, \tag{4}$$

$$U = \frac{A}{k_1 x + h - \frac{l_1 k x}{l+m} + \frac{\rho\theta}{\theta_0}}, \tag{5}$$

$$T = \frac{(l+m)A}{k\big(k_1 x + h - \frac{l_1 k x}{l+m} + \frac{\rho\theta}{\theta_0}\big)}, \tag{6}$$

$$I = \frac{\theta(l+m)A}{k\theta_0\big(k_1 x + h - \frac{l_1 k x}{l+m} + \frac{\rho\theta}{\theta_0}\big)} - \frac{\theta T_0}{\theta_0}, \tag{7}$$

where

$$A = \frac{k}{l+m}\left(Q + \frac{\rho\theta T_0}{\theta_0}\right).$$

It is noted from the equation (5) that U is a function of x only. In order to show the existence of E_1 we define a function $F(x)$ from equation (4), after using (5), as follows

$$F(x) = (r_0 - fx)\left(k_1 x + h - \frac{l_1 k x}{l+m} + \frac{\rho\theta}{\theta_0}\right) - r_1 A. \tag{8}$$

Now, from (8), we get

$$F(0) = r_0\left(h + \frac{\rho\theta}{\theta_0}\right) - r_1 A.$$

This implies $F(0) > 0$, iff $r_0(h + \frac{\rho\theta}{\theta_0}) > r_1 A$. Also from (8), we have

$$F\left(\frac{r_0}{f}\right) = -r_1 A < 0.$$

Now since $F(x)$ is quadratic in x and $F(0) > 0$ and $F(\frac{r_0}{f}) < 0$, so $F(x) = 0$ will have a unique root $\hat{x}$ in the interval $0 < \hat{x} < \frac{r_0}{f}$ which is obtained by solving

$$F(\hat{x}) = 0.$$

After knowing the value of $\hat{x}$, values of $\hat{U}, \hat{T}$ and $\hat{I}$ can be found from equations (5), (6) and (7), respectively.

5 Stability analysis

5.1 Local Stability

If $E^*(x^*, U^*, T^*, I^*)$ is an equilibrium, then the local stability can be determined from the eigenvalues of the variational matrix $V(E^*)$, whose entries are given by the differentiating the right hand side of system (3) with respect to x, U, T, I and evaluating at x^*, U^*, T^*, I^*, i.e.

$$V(E^*) = \begin{bmatrix} r_0 - r_1U^* - 2fx^* & -r_1x^* & 0 & 0 \\ 0 & -(l+m) & k & 0 \\ -k_1T^* + l_1U^* & l_1x^* & -(k_1x^* + h) & -\rho \\ 0 & 0 & \theta & -\theta_0 \end{bmatrix}.$$

Using the notation $V(E_0)$ for the variational matrix of equilibrium E_0, we get

$$V(E_0) = \begin{bmatrix} r_0 - r_1\bar{U} & 0 & 0 & 0 \\ 0 & -(l+m) & k & 0 \\ -k_1\bar{T} + l_1\bar{U} & 0 & -h & -\rho \\ 0 & 0 & \theta & -\theta_0 \end{bmatrix}.$$

The eigenvalues of $V(E_0)$ are $r_0 - r_1\bar{U}$, $-(l+m)$ and $\bar{\lambda}_\pm$, where

$$\bar{\lambda}_\pm = \frac{1}{2}\Big[-(\theta_0 + h) \pm \sqrt{(\theta_0 + h)^2 - 4(h\theta_0 + \rho\theta)} \Big].$$

The signs of the real parts of $\bar{\lambda}_\pm$ are negative. This implies that E_0 is locally asymptotically stable in the $U - T - I$ plane. As to the x-direction, E_0 is stable if $\bar{U} > \frac{r_0}{r_1}$ and unstable if $\bar{U} < \frac{r_0}{r_1}$ (necessary condition for E_1 to exists). Therefore E_0 is a saddle point if E_1 exists otherwise it is locally asymptotically stable. Here, we also note that E_0 is neutral at $\bar{U} = \frac{r_0}{r_1}$ because $\bar{U} = \frac{r_0}{r_1}$ is the bifurcation point i.e. at $\bar{U} = \frac{r_0}{r_1}$ stability change occurs.

The stability behavior of E_1 is not obvious from the corresponding variational matrix. Therefore by using Liapunov's method in the following theorem, we find sufficient conditions for E_1 to be locally asymptotically stable.

Theorem 1. *Let the following inequality holds*

$$\max\left\{ r_1^2\hat{x}^2, \left(\frac{2k(k_1\hat{T} - l_1\hat{U})}{h + k_1\hat{x}}\right)^2 \right\} < f\hat{x}(l+m), \tag{9}$$

then E_1 is locally asymptotically stable.

Proof. We first linearize system (3) around the positive equilibrium E_1 by taking the transformations where $x = x_1 + \hat{x},\ U = U_1 + \hat{U},\ T = T_1 + \hat{T},\ I = I_1 + \hat{I}$, where x_1, U_1, T_1 and I_1 are small perturbations about E_1. Then we consider the following positive definite function in the linearised system of model (3),

$$W_1 = \frac{1}{2}x_1^2 + \frac{1}{2}U_1^2 + \frac{c_1}{2}T_1^2 + \frac{c_2}{2}I_1^2,$$

where c_1, c_2 are some positive constants to be chosen appropriately.

Now, differentiating W_1 with respect to time t we can find $\dot{W}_1$ along the solutions of linearised system of (3) as follows

$$\begin{aligned}\dot{W}_1 = &\Big[-\frac{1}{2}f\hat{x}x_1^2 - r_1\hat{x}U_1x_1 - \frac{1}{2}(l+m)U_1^2\Big]\\ &+\Big[-\frac{1}{2}(l+m)U_1^2 + (k + c_1l_1\hat{x})T_1U_1 - \frac{1}{2}c_1(h+k_1\hat{x})T_1^2\Big]\\ &+\Big[-\frac{1}{2}c_1(h+k_1\hat{x})T_1^2 - c_1\big(k_1\hat{T} - l_1\hat{U}\big)x_1T_1 - \frac{1}{2}f\hat{x}x_1^2\Big]\\ &- c_1\rho I_1T_1 + c_2\theta I_1T_1 - c_2\theta_0 I_1^2.\end{aligned}$$

Now choosing $c_1 = \frac{k}{l_1\hat{x}}$ and $c_2 = \frac{c_1\rho}{\theta}$, we note that the sufficient conditions for $\dot{W}_1$ to be negative definite are that the following inequalities hold

$$\begin{aligned}&r_1^2\hat{x}^2 < f\hat{x}(l+m),\\ &4k^2 < c_1(l+m)(h+k_1\hat{x}),\\ &c_1\big(k_1\hat{T} - l_1\hat{U}\big)^2 < f\hat{x}(h+k_1\hat{x}).\end{aligned}$$

Summarizing above inequalities, we note that the $\dot{W}_1$ would be negative definite under the inequality (9) showing that W_1 is a Lyapunov's function. This completes the proof of the Theorem 1. ☐

5.2 Global stability

Theorem 2. *Let the following inequality holds in Ω*

$$\max\left\{r_1^2, \left(\frac{2kk_1\hat{T}}{h}\right)^2\right\} < f(l+m), \tag{10}$$

then E_1 is globally asymptotically stable.

Proof. Consider the following positive definite function about E_1

$$W_2 = \left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + \frac{1}{2}\big(U - \hat{U}\big)^2 + \frac{c_3}{2}\big(T - \hat{T}\big)^2 + \frac{c_4}{2}\big(I - \hat{I}\big)^2, \tag{11}$$

where c_3, c_4 are some positive constants to be chosen appropriately.

Computing the time derivative of (10) along solutions of system (3) and after doing some algebraic manipulations, we get

$$
\begin{aligned}
\dot{W}_2 = & \left[-\frac{1}{2} f(x-\hat{x})^2 - r_1(x-\hat{x})\big(U-\hat{U}\big) - \frac{1}{2}(l+m)\big(U-\hat{U}\big)^2 \right] \\
& + \left[-\frac{1}{2}(l+m)\big(U-\hat{U}\big)^2 + (k + c_3 l_1 \hat{x})\big(U-\hat{U}\big)\big(T-\hat{T}\big) \right. \\
& \quad \left. - \frac{1}{2} c_3 (h + k_1 x)\big(T-\hat{T}\big)^2 \right] \\
& + \left[-\frac{1}{2} c_3 (h + k_1 x)\big(T-\hat{T}\big)^2 - c_3\big(k_1 \hat{T} - l_1 U\big)(x-\hat{x})\big(T-\hat{T}\big) \right. \\
& \quad \left. - \frac{1}{2} f(x-\hat{x})^2 \right] \\
& - c_3 \rho \big(I-\hat{I}\big)\big(T-\hat{T}\big) + c_4 \theta \big(I-\hat{I}\big)\big(T-\hat{T}\big) - c_4 \theta_0 \big(I-\hat{I}\big)^2.
\end{aligned}
$$

Now choosing $c_3 = \frac{k}{l_1 \hat{x}}$ and $c_4 = \frac{c_3 \rho}{\theta}$ we note that the sufficient conditions for $\dot{W}_2$ to be negative definite are given by the following inequalities

$$
\begin{aligned}
& r_1^2 < (l+m) f, \\
& 4k^2 < c_3 (l+m)(h + k_1 x), \\
& c_3 \big(k_1 \hat{T} - l_1 U\big)^2 < f(h + k_1 x).
\end{aligned}
\tag{12}
$$

After maximizing the L.H.S. and minimizing the R.H.S. of the inequalities given in (12), the stability conditions can be obtained appropriately.

$$r_1^2 < (l+m) f, \tag{13a}$$

$$4k^2 < c_3 (l+m) h, \tag{13b}$$

$$c_3 k_1^2 \hat{T}^2 < f h. \tag{13c}$$

Now summarizing inequalities (13a)–(13c), we note that $\dot{W}_2$ would be negative definite under the inequality (10) showing that W_2 is a Liapunov's function and hence the Theorem 2 is proved. □

Remark 1. If r_1 are is large, then condition (10) may not be satisfied. This implies that if the decreasing rate of the intrinsic growth rate of the population due to the uptake of the pollutant is large, then it destabilizes the system.

Remark 2. If k and k_1 are large, then condition (10) may not be satisfied. This implies that if the uptake rate coefficient of pollutant due to the biological species and depletion rate coefficient of pollutant due to its uptake by the biological species are large, then it also destabilizes the system.

Remark 3. If h, l and m are small, then the condition (10) may not be satisfied. This implies that if the loss rate of pollutants due to various natural processes, depletion rate coefficients of uptake concentration of pollutants by the species due to egestion and depuration of pollutants, respectively are small, then it destabilizes the system.

6 Permanence of solutions

Biologically, persistence means the survival of all populations in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non-negative cone. Persistence may be defined mathematically as, a population $N(t)$ is said to persist (sometimes called strongly persist) if $N(0) > 0$ implies $N(t) > 0$ and $\liminf_{t\to\infty} N(t) > 0$. Further, a population $N(t)$ is said to be uniformly persistent (also known as permanence) if $N(t)$ persists and there exists $\zeta > 0$ independent of $N(0) > 0$ such that $\liminf_{t\to\infty} N(t) \geq \zeta$. Finally, we say that a system persists (uniformly) whenever each component persists (uniformly).

Theorem 3. *Assume that $Q > T_0(h + \frac{k_1 r_0}{f})$. Then the system* (3) *is uniformly persistent if*

$$\delta > \max\left\{\frac{r_1 Q}{r_0}, \frac{\rho\theta}{\theta_0}\left(\frac{Q}{Q - T_0\big(h + \frac{k_1 r_0}{f}\big)}\right)\right\}.$$

Proof. From the first equation of model (3), we get

$$\dot{x} \geq x\left(r_0 - r_1\frac{Q}{\delta} - fx\right).$$

This implies that,

$$\liminf_{t\to\infty} x(t) \geq \frac{1}{f}\left(r_0 - r_1\frac{Q}{\delta}\right).$$

From the third equation of model (3), we get

$$\dot{T} \geq Q\left(1 - \frac{\rho\theta}{\delta\theta_0}\right) - \left(h + \frac{k_1 r_0}{f}\right)T.$$

This implies that,

$$\liminf_{t\to\infty} T(t) \geq \frac{Q}{h + \frac{k_1 r_0}{f}}\left(1 - \frac{\rho\theta}{\delta\theta_0}\right).$$

Now from the second equation of model (3), we get

$$\dot{U} \geq \frac{kQ}{h + \frac{k_1 r_0}{f}}\left(1 - \frac{\rho\theta}{\delta\theta_0}\right) - (l + m)U.$$

This implies that,

$$\liminf_{t\to\infty} U(t) \geq \frac{kQ}{(l+m)(h+\frac{k_1 r_0}{f})}\left(1-\frac{\rho\theta}{\delta\theta_0}\right).$$

Lastly, from the fourth equation of model (3), we get

$$\dot{I} \geq \theta\left[\frac{Q\big(1-\frac{\rho\theta}{\delta\theta_0}\big)}{h+\frac{k_1 r_0}{f}} - T_0\right] - \theta_0 I.$$

This implies that,

$$\liminf_{t\to\infty} I(t) \geq \frac{\theta}{\theta_0}\left[\frac{Q\big(1-\frac{\rho\theta}{\delta\theta_0}\big)}{h+\frac{k_1 r_0}{f}} - T_0\right].$$

According to the above arguments and Lemma 1, we have

$$\begin{aligned}
\frac{1}{f}\left(r_0-\frac{r_1 Q}{\delta}\right) &\leq \liminf_{t\to\infty} x(t) \ \leq \limsup_{t\to\infty} x(t) \leq \frac{r_0}{f},\\
\frac{kQ}{(l+m)\big(h+\frac{k_1 r_0}{f}\big)}\left(1-\frac{\rho\theta}{\delta\theta_0}\right) &\leq \liminf_{t\to\infty} U(t) \leq \limsup_{t\to\infty} U(t) \leq \frac{Q}{\delta},\\
\frac{Q}{h+\frac{k_1 r_0}{f}}\left(1-\frac{\rho\theta}{\delta\theta_0}\right) &\leq \liminf_{t\to\infty} T(t) \leq \limsup_{t\to\infty} T(t) \leq \frac{Q}{\delta},\\
\frac{\theta}{\theta_0}\left[\frac{Q\big(1-\frac{\rho\theta}{\delta\theta_0}\big)}{h+\frac{k_1 r_0}{f}} - T_0\right] &\leq \liminf_{t\to\infty} I(t) \ \leq \limsup_{t\to\infty} I(t) \leq \frac{Q\theta}{\delta\theta_0}.
\end{aligned}$$

This completes the proof of the Theorem 3. □

7 Numerical simulations and discussion

To visualize the above analytical findings and the behavior of the system (3) for different rates of emission of pollutants as well as in the presence and absence of environmental tax, numerical simulation is done here. For this, the system (3) is integrated using fourth order Runge–Kutta Method under the following set of parameters

$$\begin{aligned}
&h = 12, \quad k = 3, \quad r_1 = 0.8, \quad l = 0.2, \quad m = 4, \quad k_1 = 0.1, \quad r_0 = 8,\\
&f = 4, \quad l_1 = 0.02, \quad \rho = 5, \quad \theta = 10, \quad \theta_0 = 5, \quad T_0 = 0.25, \quad Q = 10.
\end{aligned} \tag{14}$$

With the above values of parameters it is found that the interior equilibrium exists and is given by

$$\hat{x} = 1.9194, \quad \hat{U} = 0.4028, \quad \hat{T} = 0.5639, \quad \hat{I} = 0.6279.$$

Here, we note that the conditions of local stability, global stability and permanence are satisfied. Using MATLAB software package, graphs are plotted for different values of important parameters Q, ρ and θ in order to conclude and confirm some important points.

In Fig. 1, the biological species is plotted against time for different rate of emission of pollutants. From this plot, we note that as the emission rate of pollutants in the environment increases, the equilibrium density of biological population decreases.

Variations of the biological population (x) uptake concentration of pollutant by the population (U) and pollutant (T) with time in the presence and absence of the environmental tax are plotted in Figs. 2, 3 and 4, respectively. From these plots, we can infer that the presence of the environmental tax increases the endemic level of biological population and decreases the endemic levels of uptake concentration of pollutant by the population and pollutant, respectively for the same value of Q.

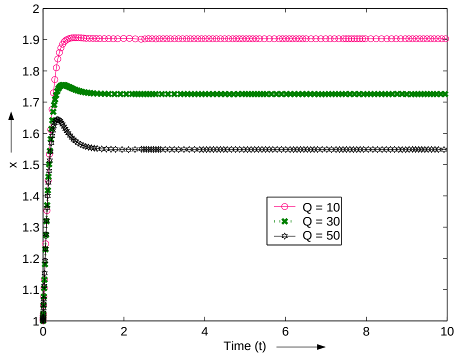

Fig. 1. Variation of x with time for different rate of emission of pollutants and other values of parameters are same as (14).

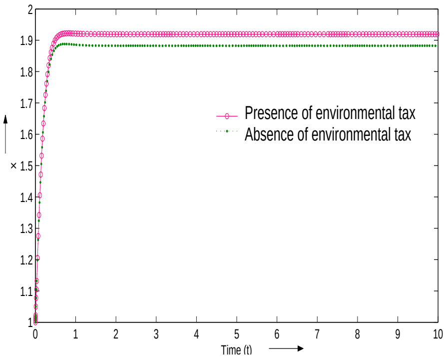

Fig. 2. Variation of x with time in the presence and absence of environmental tax for the set of parameter values given in (14).

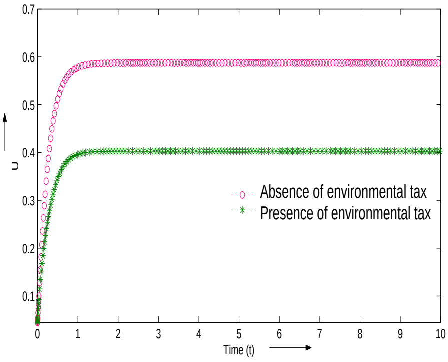

Fig. 3. Variation of U with time in the presence and absence of environmental tax for the set of parameter values given in (14).

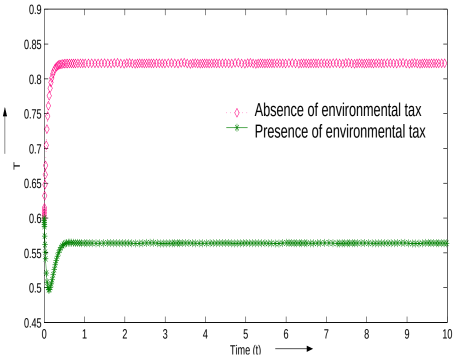

Fig. 4. Variation of T with time in the presence and absence of environmental tax for the set of parameter values given in (14).

In Figs. 5, 6, the variation of population with time is shown for different ρ and θ, respectively. It is concluded that with the increase of these parameters, the equilibrium density of population increases. It is observed that the equilibrium density of the population is more sensitive to the parameter ρ, the tax repulsion coefficient, in comparison to θ, the tax rate coefficient. Thus the tax repulsion coefficient, ρ, which characterizes the sensitivity of the industries which emit pollutants and pay the required tax on the basis of the amount of the emission is the key parameter which we need to choose very carefully in order to maintain the equilibrium level of population.

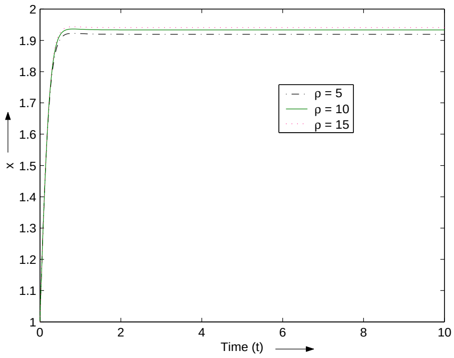

Fig. 5. Variation of x with time for different tax repulsion coefficients (ρ) and other values of parameters are same as in (14).

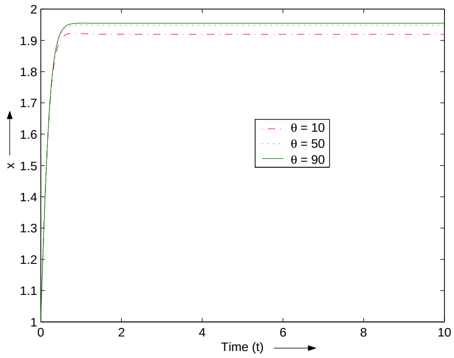

Fig. 6. Variation of x with time for different tax rate coefficients (θ) and other values of parameters are same as in (14).

Simulation is performed for different initial starts I, II, III, IV in Fig. 7 to graphically illustrate the global stability of the interior equilibrium point, E_1, in the x–U plane, where initial starts are

Initial start I: $[1 \ \ 0.045 \ \ 0.6 \ \ 1]$;
Initial start II: $[3 \ \ 0.045 \ \ 0.6 \ \ 1]$;
Initial start III: $[1 \ \ 0.6 \ \ 0.6 \ \ 1]$;
Initial start IV: $[3 \ \ 0.6 \ \ 0.6 \ \ 1]$.

It is depicted from the graph that the solutions of the system converge to equilibrium point E_1 for different value of initial starts, indicating that the system is globally asymptotically around this point. Now to depict the global stability of the interior equilibrium point E_1, in the $x - T$ plane, we have performed simulations for different initial starts I, II, III, IV, V in Fig. 8, where initial starts are

Initial start I: $[1 \ \ 0.45 \ \ 0.2 \ \ 1]$;
Initial start II: $[1 \ \ 0.45 \ \ 1.2 \ \ 1]$;
Initial start III: $[4 \ \ 0.45 \ \ 0.2 \ \ 1]$;

Initial start IV: $[4\ \ 0.45\ \ 1.2\ \ 1]$;
Initial start V: $[2.5\ \ 0.45\ \ 1.2\ \ 1]$.

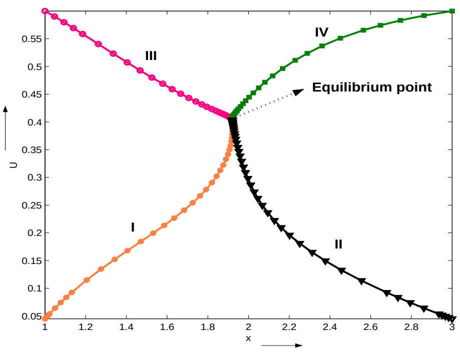

Fig. 7. Variation of population with uptake concentration of toxicant for different initial starts I, II, III and IV.

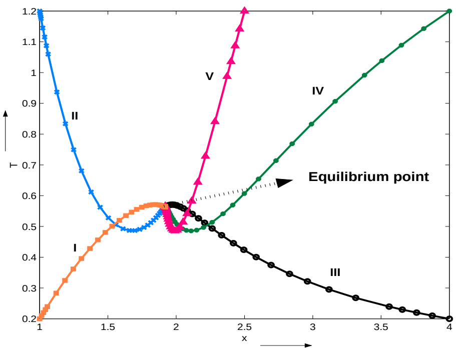

Fig. 8. Variation of population with toxicant for different initial starts I, II, III, IV and V.

8 Conclusions

The main focus of this paper is to model the process of the survival of a biological population, when the population is affected by a pollutant emitted into the environment by external sources. It is assumed that the biological population is growing logistically in the environment. It is further assumed that the introduction of the pollutant from external sources is dynamic in nature and its cumulative rate of emission is reduced due to the levy of taxes. Existence of all the equilibria and stabilities of the same have been carried out. The first equilibrium corresponds to the extinction of the population. When the first equilibrium is unstable, the second equilibrium exists and is locally as well as globally asymptotically stable under certain conditions. Conditions which influence the permanence of the system are also given. By numerical simulation, it is shown that as the cumulative rate of the emission of the pollutant from external sources increases, the endemic level of population decreases and may become extinct. So we need to control the emission rate of pollutants from external sources. Moreover, we note that when taxes are imposed on emitters of pollutants, the endemic level of population increases and shift more near to the level when the ecosystem is toxicant free.

The environmental taxes enhance the environmental quality and revenue. First, pollution taxes provide a measure of certainty to regulated firms. Second, pollution taxes raise revenue for the federal budget. These revenues could help finance some of the tax reform initiatives and to reduce rates on distorting taxes or fund cleanup programs. Also, the strategy of charging tax on the basis of emission of pollutants proves as a disincentive to the industries and the regulator may attempt to increase efficiency, maintain a fair distributional impact, minimize the costs of administration and compliance, and implement an efficient use of revenue.

Acknowledgement

Second author thankfully acknowledges the Council of Scientific and Industrial Research, New Delhi, India for the financial assistance in the form of Senior Research Fellowship (09/107(0317)/2007-EMR-I).

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