A note on "Taylor–Couette flow of a generalized second grade fluid due to a constant couple"

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Abstract. In this brief note, we show that the unsteady flow of a generalized second grade fluid due to a constant couple, as well as the similar flow of Newtonian and ordinary second grade fluids, ultimately becomes steady. For this, a new form of the exact solution for velocity is established. This solution is presented as a sum of the steady and transient components. The required time to reach the steady-state is obtained by graphical illustrations.

Keywords: constant couple, steady solution, transient solution.

1 Introduction

In a recent paper [1], the exact solutions corresponding to the flow of a generalized second grade fluid (GSGF) between two infinite coaxial cylinders, the inner one being subject to a constant couple, have been established using Laplace and finite Hankel transforms. These solutions, presented under integral and series form in terms of the generalized $G_{a,b,c}(\cdot,t)$ functions, have been easy specialized to give the similar solutions for Newtonian and ordinary second grade fluids performing the same motion. The last solutions, presented as a sum between the steady and transient solutions, describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, they tend to the steady solutions which are independent of the initial conditions.

The aim of this note is to show that the unsteady flow of a GSGF, ultimately becomes steady. In order to prove that, the exact solution corresponding to the velocity field is also presented as a sum between the steady and transient solutions. Finally, the required time to reach the steady-state for generalized fluids is determined by graphical illustration: This time, as it results from Fig. 1, is increasing with respect to β.

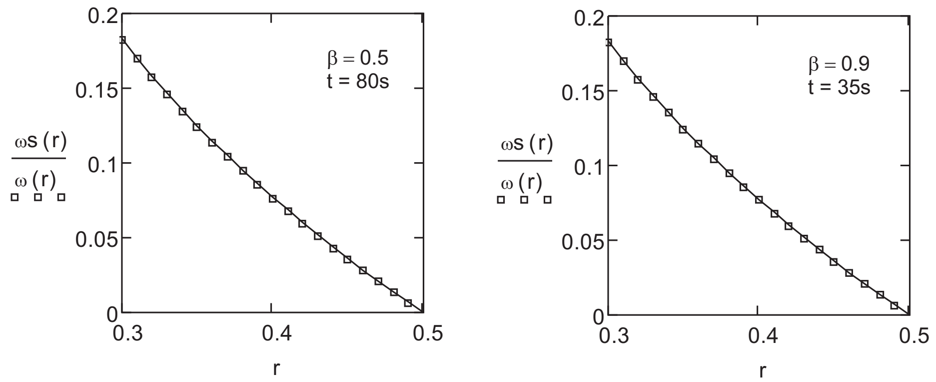

Fig. 1. The time after which the diagrams of $w(r,t)$ are almost identical to those of $w_s(r)$, for $f = -2$, $R_1 = 0.3$, $R_2 = 0.5$, $\nu = 0.001188$, $\mu = 1.05$, $\alpha = 0.002$, $\beta = 0.5$ and 0.9.

2 Statement and solution of the problem

According to equations (8a), (9) and (10) from [1], we must solve the problem

$$\frac{\partial w(r,t)}{\partial t} = \big(\nu + \alpha D_t^{\beta}\big)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t); \quad r \in (R_1, R_2), \ \ t > 0, \tag{1a}$$

$$w(r,0) = 0; \quad r \in (R_1, R_2], \tag{1b}$$

$$\big(\nu + \alpha D_t^{\beta}\big)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t)|_{r=R_1} = \frac{f}{\rho}, \quad w(R_2,t) = 0; \ \ t > 0, \tag{1c}$$

where $w(r,\,t)$ is the velocity of the fluid, ν is the kinematic viscosity, ρ the constant density, α a material constant and D_t^{β} $(0 \le \beta < 1)$ is the Riemann–Liouville operator. Applying the Laplace and finite Hankel transforms to equations (1a)–(1c) and using equation (16) from [1] we find that

$$\overline{w}_H(r_n,q) = \frac{2f}{\pi r_n}\frac{1}{q}\frac{1}{\rho q + \big(\mu + \alpha_1 q^{\beta}\big) r_n^2}, \tag{2}$$

where $r_n \, (n = 1,2,3,\ldots)$ are the positive roots of the transcendental equation $B(R_2 r) = 0$, $\overline{w}_H(r_n,q)$ is the mixed transform of $w(r,t)$ and

$$B(r r_n) = J_1(r r_n) Y_2(R_1 r_n) - J_2(R_1 r_n) Y_1(r r_n). \tag{3}$$

Writing $\overline{w}_H(r_n,q)$ under the equivalent forms

$$\begin{aligned}\overline{w}_H(r_n,q) &= \frac{2f}{\mu\pi r_n^3}\left[\frac{1}{q} - \frac{1 + \alpha r_n^2 q^{\beta-1}}{q + (\nu + \alpha q^{\beta}) r_n^2}\right] \\ &= \frac{2f}{\mu\pi r_n^3}\left[\frac{1}{q} - \frac{q^{-\beta} + \alpha r_n^2 q^{-1}}{(q^{1-\beta} + \alpha r_n^2) + \nu r_n^2 q^{-\beta}}\right]\end{aligned} \tag{4}$$

and applying the inverse transforms, we find the velocity field under the simple and suitable form

$$w(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=o}^{\infty}\big(-\nu r_n^2\big)^k\big[G_{1-\beta,-\beta k-\beta,k+1}\big(-\alpha r_n^2, t\big) + \alpha r_n^2 G_{1-\beta,-\beta k-1,k+1}\big(-\alpha r_n^2, t\big)\big], \tag{5}$$

where the generalized $G_{a,b,c}(\cdot,\ t)$ functions are defined by [1, Eq. (22)] or [2, Eq. (101)].

Making $\beta \to 1$ into the last relation and using equation (A3) from [1], we recover the solution (cf. [1, Eq. (36)])

$$w_{SG}(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right), \tag{6}$$

corresponding to a second grade fluid. Of course, by letting now $\alpha \to 0$ into above relation, the velocity field for a Newtonian fluid is recovered. Furthermore making $t \to \infty$ into equation (6), the last term which represents the transient solution tends to zero and

$$w_{SG}(r,t) \to w_{SG}(r,\infty) = w_s(r) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right). \tag{7}$$

3 Numerical results and conclusions

The exact solution $w(r,t)$, as well as $w_{SG}(r,t)$, is presented as a sum of two terms. Its first term, which is independent of t, is just the steady solution $w_s(r)$. In order to prove that the unsteady motion of a GSGF, as well as that of an ordinary fluid becomes steady, it is sufficient to show that the diagrams of $w(r,t)$ tend to superpose over those of $w_s(r)$ if t increases. Furthermore, by graphical illustrations, we can also determine the required time after which the fluid is flowing according to the steady solution. This time, as it results from Fig. 1, decreases if the fractional parameter β increases. Consequently, the required time to reach the steady-state for a generalized fluid, as it was to be expected, is greater in comparison with an ordinary fluid (10 s for the Newtonian fluid and 15 s for a second grade fluid with the same values of common parameters).

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