Taylor–Couette flow of a generalized second grade fluid due to a constant couple

M. Athar[1], **M. Kamran**[1], **C. Fetecau**[2]

[1]Abdus Salam School of Mathematical Sciences, GC University
Lahore, Pakistan

[2]Technical University "Gh. Asachi" of Iasi
Romania
fetecau_constantin@yahoo.com

Received: 2009-08-09 **Revised:** 2009-10-14 **Published online:** 2010-03-04

Abstract. The velocity field and the adequate shear stress, corresponding to the flow of a generalized second grade fluid in an annular region between two infinite coaxial cylinders, are determined by means of Laplace and finite Hankel transforms. The motion is produced by the inner cylinder which is rotating about its axis due to a constant torque f per unit length. The solutions that have been obtained satisfy all imposed initial and boundary conditions. For $\beta \to 1$ or $\beta \to 1$ and $\alpha_1 \to 0$, the corresponding solutions for an ordinary second grade fluid, respectively, for the Newtonian fluid, performing the same motion, are obtained as limiting cases.

Keywords: generalized second grade fluid, velocity field, shear stress, exact solutions.

1 Introduction

Among the many constitutive assumptions that have been employed to study the non-Newtonian fluid behavior, one class that has gained support from both the experimentalists and the theoreticians is that of Rivlin-Ericksen fluids of second grade. The Cauchy stress tensor $\mathbf{T}$ for such fluids is related to the fluid motion by [1–3]

$$\mathbf{T} = -p\,\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $-p$ is the hydrostatic pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$, $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T, \quad \mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A}_1. \tag{2}$$

In the above relations, $\mathbf{v}$ is the velocity, $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the material time derivative and grad is the gradient operator. Since the fluid is incompressible, it can undergo only isochoric motions.

The flows to be here considered have the velocity field of the form [4–6]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\,\mathbf{e}_\theta, \tag{3}$$

where $\mathbf{e}_\theta$ is the unit vector along the θ-direction of the cylindrical coordinate system r, θ and z. For such flows the constraint of incompressibility is automatically satisfied. Introducing (3) into the constitutive equation (1), we find that

$$\tau(r,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left[\frac{\partial w(r,t)}{\partial r} - \frac{w(r,t)}{r}\right], \tag{4}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the tangential shear stress that is different of zero. In the absence of a pressure gradient in the flow direction and neglecting the body forces, the balance of the linear momentum leads to the relevant equation

$$\rho\,\frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t). \tag{5}$$

Eliminating $\tau(r,t)$ between equations (4) and (5), we get the governing equation

$$\frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t), \tag{6}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid, ρ is its constant density and $\alpha = \alpha_1/\rho$.

In the last time, the fractional calculus has encountered much success in the description of visco-elasticity [5, 7–12]. Especially, the rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of the polymer solutions and melts. Generally, these equations are derived from those for non-Newtonian fluids by replacing the inner time derivatives of an integer order with the so called Riemann-Liouville operator [13, 14]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\,\mathrm{d}\tau, \quad 0 \le \beta < 1, \tag{7}$$

where $\Gamma(.)$ is the Gamma function.

Consequently, the governing equations corresponding to the motion (3) of a generalized second grade fluid are (cf. [5, Eqs. (2), (4)]) or [10, Eqs. (7), (9)]

$$\frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha D_t^\beta\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t), \tag{8a}$$

$$\tau(r,t) = \left(\mu + \alpha_1 D_t^\beta\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t). \tag{8b}$$

where the new material constant α_1 (for simplicity, we are keeping the same notation) goes to the initial α_1 for $\beta \to 1$.

In this paper, we are interested into the motion of a generalized second grade fluid whose governing equations are given by equations (8). More exactly, we would like to extend the results of the Section 5 from [4] to a larger class of fluids. The fractional partial differential equations (8), with adequate initial and boundary conditions, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. The Laplace transform will be used to eliminate the time variable and the finite Hankel transform to eliminate the spatial variable.

2 Taylor–Couette flow between two infinite cylinders

Consider an incompressible generalized second grade fluid at rest in the annular region between two infinitely long co-axial cylinders. At time $t = 0^+$, let the inner cylinder of radius R_1 be set in rotation about its axis by a constant torque per unit length $2\pi R_1 f$ and let the outer cylinder of radius R_2 be held stationary. Owing to the shear, the fluid between cylinders is gradually moved, its velocity being of the form (3). The governing equations are given by equations (8) and the appropriate initial and boundary conditions are (see also [4, Eqs. (5.2), (5.3)]

$$w(r,0) = 0, \quad r \in [R_1, R_2], \tag{9}$$

$$\begin{aligned} \tau(R_1,t) &= \left(\mu + \alpha_1 D_t^{\beta}\right)\left(\frac{\partial w(r,t)}{\partial r} - \frac{w(r,t)}{r}\right)\Big|_{r=R_1} = f, \\ w(R_2,t) &= 0, \quad t > 0, \end{aligned} \tag{10}$$

where f is a constant.

3 Calculation of the velocity field

Applying the Laplace transform to the equations (8a) and (10), we get

$$q\overline{w}(r,q) = (\nu + \alpha q^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w}(r,q), \tag{11}$$

$$\overline{\tau}(R_1,q) = (\mu + \alpha_1 q^{\beta})\left(\frac{\partial}{\partial r} - \frac{1}{R_1}\right)\overline{w}(R_1,q) = \frac{f}{q}, \quad \overline{w}(R_2,q) = 0, \tag{12}$$

where $\overline{w}(r,q)$, and $\overline{\tau}(R_1,q)$ are the Laplace transforms of the functions $w(r,t)$ and $\tau(R_1,t)$ respectively. We denote by

$$\overline{w}_H(r_n,q) = \int_{R_1}^{R_2} r\overline{w}(r,q)B(rr_n)\,dr, \tag{13}$$

the finite Hankel transform of the function $\overline{w}(r,q)$, where

$$B(rr_n) = J_1(rr_n)Y_2(R_1r_n) - J_2(R_1r_n)Y_1(rr_n), \tag{14}$$

r_n are the positive roots of the equation $B(R_2r) = 0$ and $J_p(.)$, $Y_p(.)$ are the Bessel functions of the first and second kind of order p.

By means of equations (12) and of the identity

$$J_1(z)Y_2(z) - J_2(z)Y_1(z) = -\frac{2}{\pi z}, \tag{15}$$

we can easily prove that

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w}(r,q)B(rr_n)\mathrm{d}r$$
$$= -r_n^2\,\overline{w}_H(r_n,q) + \frac{2}{\pi r_n}\left(\frac{\partial}{\partial r} - \frac{1}{R_1}\right)\overline{w}(R_1,q). \tag{16}$$

Combining (11), (12) and (16), we find that

$$\overline{w}_H(r_n,q) = \frac{2f}{\pi r_n}\,\frac{1}{q}\,\frac{1}{\rho q + \alpha_1 q^\beta r_n^2 + \mu\, r_n^2}, \tag{17}$$

or equivalently

$$\overline{w}_H(r_n,q) = \frac{2f}{\pi r_n^3}\,\frac{1}{q(\mu+\alpha_1 q^\beta)} - \frac{2f}{\pi r_n^3}\,\frac{1}{(\mu+\alpha_1 q^\beta)(q+\alpha q^\beta r_n^2 + \nu r_n^2)}$$
$$= \overline{w}_{1H}(r_n,q) + \overline{w}_{2H}(r_n,q), \tag{18}$$

where

$$\overline{w}_{1H}(r_n,q) = \frac{2f}{\pi r_n^3}\frac{1}{q(\mu+\alpha_1 q^\beta)}, \tag{19a}$$

$$\overline{w}_{2H}(r_n,q) = -\frac{2f}{\pi r_n^3}\frac{1}{(\mu+\alpha_1 q^\beta)(q+\alpha q^\beta r_n^2 + \nu r_n^2)}. \tag{19b}$$

The inverse Hankel transforms of $\overline{w}_{1H}(r_n,q)$ and $\overline{w}_{2H}(r_n,q)$, are given by (see (A.1) from Appendix)

$$\overline{w}_1(r,q) = \frac{R_1^2 f(r^2 - R_2^2)}{2R_2^2 r}\frac{1}{q(\mu+\alpha_1 q^\beta)}, \tag{20a}$$

$$\overline{w}_2(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_1^2(R_2r_n)B(rr_n)}{J_2^2(R_1r_n) - J_1^2(R_2r_n)}\,\overline{w}_{2H}(r_n,q). \tag{20b}$$

The inverse Laplace transform of the last factor $H(q) = \frac{1}{\mu+\alpha_1 q^\beta} = \frac{1}{\alpha_1}\frac{1}{q^\beta+\frac{\mu}{\alpha_1}}$, from equation (20a) is

$$h(t) = L^{-1}[H(q)] = \frac{1}{\alpha_1} G_{\beta,0,1}\left(-\frac{\mu}{\alpha_1}, t\right) = \frac{1}{\alpha_1}\sum_{k=0}^{\infty}\left(-\frac{\mu}{\alpha_1}\right)^k \frac{t^{(k+1)\beta-1}}{\Gamma[(k+1)\beta]}, \quad (21)$$

where the generalized function $G_{a,b,c}(.\,,.)$ is defined by [15, Eqs. (97), (101)]

$$G_{a,b,c}(d,t) = L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = \sum_{k=0}^{\infty}\frac{d^k\Gamma(c+k)}{\Gamma(c)\Gamma(k+1)}\frac{t^{(c+k)a-b-1}}{\Gamma[(c+k)a-b]},$$
$$\mathrm{Re}(ac-b) > 0, \quad \left|\frac{d}{q^a}\right| < 1. \quad (22)$$

By taking the inverse Laplace transform of equation (20a) and using (21), we find that

$$\begin{aligned} w_1(r,t) &= \frac{R_1^2 f(r^2-R_2^2)}{2R_2^2 r} L^{-1}\left[\frac{1}{q}\,H(q)\right] = \frac{R_1^2 f(r^2-R_2^2)}{2R_2^2 r}\int_0^t h(s)\mathrm{d}s \\ &= \frac{R_1^2 f(r^2-R_2^2)}{2R_2^2 r\alpha_1}\sum_{k=0}^{\infty}\left(-\frac{\mu}{\alpha_1}\right)^k \frac{t^{(k+1)\beta}}{\Gamma[(k+1)\beta+1]} \\ &= \frac{R_1^2 f(r^2-R_2^2)}{2R_2^2 r\alpha_1} G_{\beta,\,-1,\,1}\left(-\frac{\mu}{\alpha_1}, t\right). \end{aligned} \quad (23)$$

In order to determine the inverse Laplace transform of the function $\overline{w}_2(r,q)$, we rewrite the function $\overline{w}_{2H}(r_n,q)$ in the form

$$\overline{w}_{2H}(r_n,q) = -\frac{2f}{\pi r_n^3}\,H(q).H_1(r_n,q), \quad H_1(r_n,q) = \frac{1}{q+\alpha q^\beta r_n^2+\nu r_n^2}. \quad (24)$$

Using again equation (22) and the following expansion of the function $H_1(r_n,q)$

$$H_1(r_n,q) = \frac{q^{-\beta}}{(q^{1-\beta}+\alpha r_n^2)+\nu r_n^2 q^{-\beta}} = \sum_{k=0}^{\infty}\frac{(-\nu r_n^2)^k\, q^{-\beta(k+1)}}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}, \quad (25)$$

we get

$$h_1(r_n,t) = L^{-1}[H_1(r_n,q)] = \sum_{k=o}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta(k+1),k+1}\left(-\alpha r_n^2, t\right). \quad (26)$$

Applying the Laplace transform to equation (20b) and using equations (21), (24), (26) and the property

$$L^{-1}[H(q)H_1(r_n,q)] = h(t) * h_1(r_n,t) = \int_0^t h(t-s)h_1(r_n,s)\mathrm{d}s,$$

we find that

$$w_2(r,t)=L^{-1}[\overline{w}_2(r,q)]=-\frac{\pi f}{\alpha_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(rr_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}$$
$$\times\sum_{k=o}^{\infty}(-\nu r_n^2)^k\int_0^t G_{\beta,0,1}\Big(-\frac{\mu}{\alpha_1},t-s\Big)G_{1-\beta,-\beta(k+1),k+1}\big(-\alpha r_n^2,s\big)\mathrm{d}s, \quad (27)$$

Consequently, the velocity field $w(r,t)$ is given by

$$w(r,t)=\frac{R_1^2f(r^2-R_2^2)}{2R_2^2r\alpha_1}G_{\beta,-1,1}\Big(-\frac{\mu}{\alpha_1},t\Big)-\frac{\pi f}{\alpha_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(rr_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}$$
$$\times\sum_{k=o}^{\infty}\big(-\nu r_n^2\big)^k\int_0^t G_{\beta,0,1}\Big(-\frac{\mu}{\alpha_1},t-s\Big)G_{1-\beta,-\beta(k+1),\,k+1}\big(-\alpha r_n^2,s\big)\mathrm{d}s. \quad (28)$$

4 Calculation of the shear stress

Applying the Laplace transform to equation (8b), we find that

$$\overline{\tau}(r,q)=\big(\mu+\alpha_1q^\beta\big)\Big(\frac{\partial}{\partial r}-\frac{1}{r}\Big)\overline{w}(r,q)$$
$$=\big(\mu+\alpha_1q^\beta\big)\Big[\Big(\frac{\partial}{\partial r}-\frac{1}{r}\Big)\overline{w}_1(r,q)+\Big(\frac{\partial}{\partial r}-\frac{1}{r}\Big)\overline{w}_2(r,q)\Big], \quad (29)$$

Using equation (20), we obtain

$$\overline{\tau}(r,q)=\frac{R_1^2f}{r^2}\,\frac{1}{q}+\pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\,\frac{1}{(q+\alpha q^\beta r_n^2+\nu r_n^2)}. \quad (30)$$

where $B_1(rr_n)=J_2(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_2(rr_n)$. Now taking the inverse Laplace transform of both sides of equation (30) and using (25), we get

$$\tau(r,t)=\frac{R_1^2f}{r^2}+\pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B_1(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}$$
$$\times\sum_{k=o}^{\infty}\big(-\nu r_n^2\big)^kG_{1-\beta,-\beta k-\beta,k+1}\big(-\alpha r_n^2,t\big). \quad (31)$$

5 The special case $\beta \to 1$

Making $\beta \to 1$ into equations (28) and (31), we obtain the similar solutions

$$w(r,t) = \frac{R_1^2 f(r^2 - R_2^2)}{2R_2^2 r\alpha_1} G_{1,-1,1}\left(-\frac{\mu}{\alpha_1}, t\right) - \frac{\pi f}{\alpha_1} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=o}^{\infty} \left(-\nu r_n^2\right)^k \int_0^t G_{1,0,1}\left(-\frac{\mu}{\alpha_1}, t-s\right) G_{0,-k-1,k+1}\left(-\alpha r_n^2, s\right) \mathrm{d}s \quad (32)$$

and

$$\tau(r,t) = \frac{R_1^2 f}{r^2} + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=o}^{\infty} \left(-\nu r_n^2\right)^k G_{0,-k-1,k+1}\left(-\alpha r_n^2, t\right), \quad (33)$$

for a second grade fluid performing the same motion. Now, using the identities (A.2) and (A.3), $w(r,t)$ and $\tau(r,t)$ can be written in the simplified forms

$$w(r,t) = \frac{R_1^2 f(r^2 - R_2^2)}{2R_2^2 r\mu} \left[1 - \exp\left(-\frac{\mu t}{\alpha_1}\right)\right] - \frac{\pi f}{\alpha_1} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \int_0^t \exp\left(-\frac{\mu}{\alpha_1}(t-s)\right) \frac{1}{1+\alpha r_n^2} \exp\left(-\frac{\nu r_n^2 s}{1+\alpha r_n^2}\right) \mathrm{d}s, \quad (34)$$

$$\tau(r,t) = \frac{R_1^2 f}{r^2} + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \frac{1}{1+\alpha r_n^2} \exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right). \quad (35)$$

The expression (34) of $w(r,t)$ can be further processed to give the simpler form

$$w(r,t) = \frac{R_1^2 f(r^2 - R_2^2)}{2R_2^2 r\mu} - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right). \quad (36)$$

which is identical to equation (5.17) from [4], obtained by a different technique.

Making α_1 and then $\alpha \to 0$ into equations (36) and (35), the velocity field

$$w(r,t) = \frac{R_1^2 f(r^2 - R_2^2)}{2R_2^2 r\mu} - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \exp\left(-\nu r_n^2 t\right) \quad (37)$$

and the associated shear stress

$$\tau(r,t) = \frac{R_1^2 f}{r^2} + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B_1(r r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \exp\left(-\nu r_n^2 t\right), \quad (38)$$

corresponding to a Newtonian fluid are recovered.

6 Conclusions

The aim of this paper is to provide exact solutions for the velocity field and shear stress corresponding to the flow of a generalized second grade fluid between two infinite coaxial cylinders, the inner one being subject to a constant torque. These solutions, obtained by means of the Laplace and finite Hankel transforms, are presented under series form in terms of the generalized $G_{a,b,c}(.,.)$ functions. They satisfy all imposed initial and boundary conditions. Indeed, making $r = R_1$ into (31) and having in mind the definition of the transcendental function $B_1(r r_n)$, it immediately results $\tau(R_1, t) = f$. As regards the second boundary condition $(10)_2$, it can be easily proved using the expansion (22) for $G_{\beta,-1,1}(.,t)$ and the known relation

$$D^\alpha(t^\beta) = \frac{t^{\beta-\alpha}\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}.$$

In the special case when $\beta \to 1$, the solutions that have been obtained take the simplified forms (32) and (33) corresponding to an ordinary second grade fluid performing the same motion. Of course, these last solutions can be further processed to give the simpler forms (36) and (35), the first of them being identical to equation (5.17) obtained in [4] by a different technique. Finally, making α_1 and then $\alpha \to 0$ into (36) and (35), the similar solutions for a Newtonian fluid are recovered. Furthermore, making $t \to \infty$ into equations (37) and (38), the solutions

$$w(r) = \frac{R_1^2 f(r^2 - R_2^2)}{2R_2^2 r\mu}, \quad \tau(r) = \frac{R_1^2 f}{r^2}, \quad (39)$$

corresponding to the steady motion are obtained. They are the same for both types of fluids, Newtonian or second grade.

Finally, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity and the shear stress are presented for different values of the time *t* and of the fractional parameter β. Figs. 1 and 2 clearly show that the velocity $v(r,t)$ and the shear stress $\tau(r,t)$ (in absolute value) are increasing functions of *t*. From Figs. 3 and 4 it results that the velocity and the shear stress increases for increasing β. For $\beta \to 1$ their diagrams tend to those for an ordinary second grade fluid. The units of the material parameters are SI units and the roots r_n have been approximated by $\frac{(2n-1)\pi}{[2(R_2-R_1)]}$.

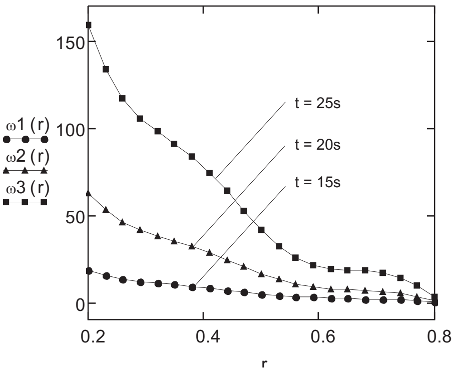

Fig. 1. Profiles of the velocity $w(r,t)$ given by (28), for $f=-1$, $\nu=0.0011746$, $\mu=1.48$, $r_1 = 0.2$, $r_2 = 0.8$, $\alpha_1 = 2$, $\beta = 0.8$ and different values of t.

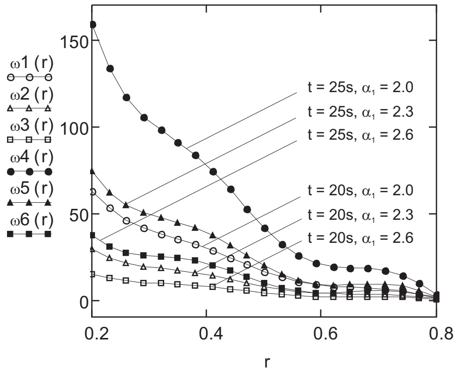

Fig. 2. Profiles of the velocity $w(r,t)$ given by (28), for $f=-1$, $\nu=0.0011746$, $\mu=1.48$, $r_1 = 0.2$, $r_2 = 0.8$, $\alpha_1 = 2$, $\beta = 0.8$ and different values of t and α_1.

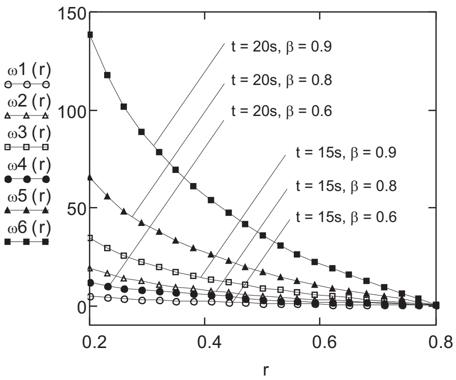

Fig. 3. Profiles of the velocity $w(r,t)$ given by (28), for $f=-1$, $\nu=0.0011746$, $\mu=1.48$, $r_1 = 0.2$, $r_2 = 0.8$, $\alpha = 0.0015$ and different values of t and β.

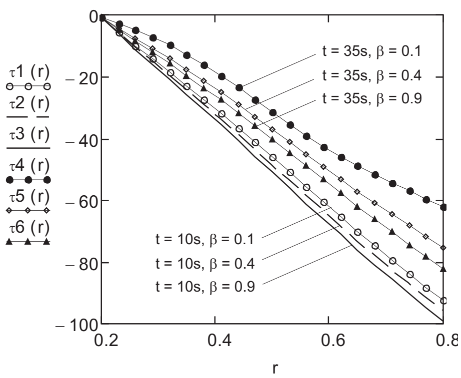

Fig. 4. Profiles of the shear stress $\tau(r,t)$ given by (31), for $f=-1$, $\nu=0.0011746$, $\mu=1.48$, $r_1 = 0.2$, $r_2 = 0.8$, $\alpha_1 = 2$, $\beta = 0.8$ and different values of t.

Appendix

$$\frac{R_1^2(r^2-R_2^2)}{2R_2^2 r} = \pi \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]}, \tag{A.1}$$

$$\begin{aligned} &G_{1,\,0,\,1}\left(-\frac{\mu}{\alpha_1}, t\right) = \exp\left(-\frac{\mu\, t}{\alpha_1}\right), \\ &G_{1,-1,\,1}\left(-\frac{\mu}{\alpha_1}, t\right) = \left(\frac{\alpha_1}{\mu}\right)\left[1-\exp\left(-\frac{\mu\, t}{\alpha_1}\right)\right], \end{aligned} \tag{A.2}$$

$$\sum_{k=o}^{\infty}\big(-\nu r_n^2\big)^k G_{0,-k-1,\,k+1}\big(-\alpha r_n^2, t\big) = \frac{1}{1+\alpha r_n^2}\,\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right). \tag{A.3}$$

Acknowledgement

C. Fetecau acknowledges support from the Ministry of Education and Research, CNCSIS through PN II-Ideas, Grant No. 26/28-09-2007, cod CNSIS ID_593.

References

1. K.R. Rajagopal, A note on unsteady unidirectional flows of a non-Newtonian fluid, *Int. J. Nonlinear Mech.*, **17**, pp. 369–373, 1982.

2. T. Hayat, M. Khan, M. Ayub, A.M. Siddique, The unsteady Couette flow of a second grade fluid in a layer of porous medium, *Arch. Mech.*, **57**, pp. 405–416, 2005.

3. C. Fetecau, Corina Fetecau, Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder, *Int. J. Eng. Sci.*, **44**, pp. 788–796, 2006.

4. R. Bandelli, K.R. Rajagopal, Start-up flows for second grade fluids in domains with one finite dimension, *Int. J. Nonlinear Mech.*, **30**, pp. 817–839, 1995.

5. D. Tong, Y. Liu, Exact solutions for the unsteady rotational flow of non-Newtonian fluid in an annular pipe, *Int. J. Eng. Sci.*, **43**, pp. 281–289, 2005.

6. C. Fetecau, A.U. Awan, Corina Fetecau, Taylor–Couette flow of an Oldroud-B fluid in a circular cylinder subject to a time-dependent rotation, *Bull. Math. Soc. Sci. Math. Roumanie Tome*, **52**(100), pp. 117–128, 2009.

7. M. Khan, S. Nadeem, T. Hayat, Unsteady motions of a generalized second grade fluid, *Math. Comput. Model.*, **41**, pp. 629–637, 2005.

8. M. Khan, K. Maqbool, T. Hayat, Influence of Hall current on the flows of a generalized Oldroyd-B fluid in a porous space, *Acta Mech.*, **184**, pp. 333–339, 2006.

9. H. Qi, M. Xu, Stokes' first problem for a viscoelastic fluid with the generalized Oldroyd-B model, *Acta Mech. Sinica*, **23**, pp. 463–469, 2007.

10. Corina Fetecau, C. Fetecau, M. Khan, D. Vieru, Decay of a potential vortex in a generalized Oldroyd-B fluid, *Appl. Math. Comput.*, **205**, pp. 497–506, 2008.

11. C. Fetecau, A. Mahmood, Corina Fetecau, D. Vieru, Some exact solutions for the helical flow of a generalized Oldroyd-B fluid in a circular cylinder, *Comput. Math. Appl.*, **56**, pp. 3096–3108, 2008.

12. C. Fetecau, Corina Fetecau, M. Kamran, D. Vieru, Exact solutions for the flow of a generalized Oldroyd-B fluid induced by a constantly accelerating plate between two side walls perpendicular to the plate, *J. Non-Newton. Fluid*, **156**, pp. 189–201, 2009.

13. I. Podlubny, *Fractional Differential Equations*, Academic press, San Diego 1999.

14. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

15. C.F. Lorenzo, T.T. Hartley, *Generalized Functions for the Fractional Calculus*, NASA/TP-1999-209424/REV1, 1999.