State-feedback stabilization for stochastic high-order nonlinear systems with a ratio of odd integers power*

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Received: 2009-04-02 **Revised:** 2009-10-09 **Published online:** 2010-03-04

Abstract. This paper investigates the problem of globally asymptotically stable in probability by state-feedback for a class of stochastic high-order nonlinear systems with a ratio of odd integers power. By extending the adding a power integrator technique and choosing an appropriate Lyapunov function, a linear smooth state-feedback controller is explicitly constructed to render the system globally asymptotically stable in probability. Furthermore, we address the problem of state-feedback inverse optimal stabilization in probability. A simulation example is provided to show the effectiveness of the proposed approach.

Keywords: stochastic high-order nonlinear systems, state-feedback control, inverse optimal stabilization.

1 Introduction

Consider the following stochastic high-order nonlinear systems described by:

$$\begin{aligned} \mathrm{d}x_i &= \big(d_i(t)x_{i+1}^r + f_i(\bar{x}_i)\big)\,\mathrm{d}t + \phi_i(\bar{x}_i)^T\,\mathrm{d}\omega, \quad i = 1,\dots,n-1,\\ \mathrm{d}x_n &= \big(d_n(t)u^r + f_n(x)\big)\,\mathrm{d}t + \phi_n(x)^T\,\mathrm{d}\omega, \end{aligned} \tag{1}$$

where $x = (x_1,\dots,x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state, and control input, respectively. $\bar{x}_i = (x_1,\dots,x_i)^T$, $i = 1,\dots,n$, $\bar{x}_n = x$. $r \in \mathbb{R}^* \triangleq \{q \in \mathbb{R}\colon q \geq 1,\ q = \frac{n}{m} \geq 1$ with positive odd integers $n, m\}$. ω is an m-dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with Ω being a sample space, $\mathcal{F}$ being a filtration, and P being a probability measure. $f_i\colon \mathbb{R}^i \to \mathbb{R}$, and $\phi_i\colon \mathbb{R}^i \to \mathbb{R}^m$,

*This work was supported by National Natural Science Foundation of China (No. 60774010, 10971256, 60974028), Natural Science Foundation of Jiangsu Province (No. BK2009083), Program for Fundamental Research of Natural Sciences in Universities of Jiangsu Province (No. 07KJB510114), Shandong Provincial Natural Science Foundation of China.

$i = 1, \ldots, n$, are assumed to be at least $\mathcal{C}^1$ functions with $f_i(0) = 0$ and $\phi_i(0) = 0$. $d_i(t)$ ($i = 1, \ldots, n$) is a $\mathcal{C}^1$ function of time t, which represents an unknown time-varying parameter.

When $r = d_1 = \ldots = d_n \equiv 1$, system (1) reduces to the well-known normal form, whose design of globally asymptotically stable state-feedback controller was firstly given by [1]. Since then, by adopting different approaches, much research work has been focused on the state-feedback for more general stochastic nonlinear systems under various structures or growth conditions, e.g., [2–8] and references therein.

In the case of r being positive odd integer and $r > 1$, similar to its deterministic counterpart in [9] and the related papers, some interesting features of (1) are that the Jacobian linearization of the system is neither controllable nor feedback linearizable, so the existing design tools are hardly applicable to (1). Recently, [10] addressed state-feedback stabilization for high-order stochastic nonlinear systems with stochastic inverse dynamics for the first time, [11–13] considered respectively the state-feedback stabilization problem for more general systems with different system structures. All the existing results on state-feedback stabilization are achieved under the assumption that the power of stochastic nonlinear system is positive odd integer. While for more general stochastic high-order nonlinear system in which system's power is only a ratio of odd integers (i.e. $r \in \mathbb{R}^*$), to the best of authors' knowledge, the problem of state-feedback stabilization has not yet been considered.

In this paper, by extending the adding a power integrator technique and choosing an appropriate Lyapunov function, we develop a systematic design algorithm that achieves a smooth state-feedback controller, which ensures that the equilibrium at the origin of the closed-loop system is globally asymptotically stable in probability. Furthermore, we also address the problem of state-feedback inverse optimal stabilization in probability.

Notations. The following notations will be used throughout the paper. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers and $\mathbb{R}^n$ denotes the real n-dimensional space. For a given vector or matrix X, X^T denotes its transpose, $\mathrm{Tr}\{\mathrm{X}\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of a vector X. C^i denotes the set of all functions with continuous ith partial derivatives. $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which are of $\mathcal{K}$ for each fixed t, and decrease to zero as $t \to \infty$ for each fixed s. For a class $\mathcal{K}_\infty$ function γ whose derivative exists and is also a class $\mathcal{K}_\infty$ function, ℓ_γ denotes the transform $\ell_\gamma(s) = s(\dot{\gamma})^{-1}(s) - \gamma((\dot{\gamma})^{-1}(s))$, where $(\dot{\gamma})^{-1}(s)$ stands for the inverse function of $\frac{d\gamma(s)}{ds}$ for any variable s, $L_f V(x) \triangleq \frac{\partial V}{\partial x} f(x)$.

2 Preliminary results and useful lemmas

Consider the following stochastic nonlinear system

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)^T \mathrm{d}\omega, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state of the system, ω is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$. The Borel measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $g^T\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz in $x \in \mathbb{R}^n$.

The following definitions and lemmas will be used throughout the paper.

Definition 1 ([14]). For any given $V(x) \in \mathcal{C}^2$ associated with stochastic system (2), the differential operator $\mathcal{L}$ is defined as:

$$\mathcal{L}V(x) \triangleq \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\operatorname{Tr}\left\{ g(x)\frac{\partial^2 V}{\partial x^2} g(x)^T \right\}. \tag{3}$$

Definition 2 ([14]). For the stochastic system (2) with $f(0) = 0$, $g(0) = 0$, the equilibrium $x(t) = 0$ of (2) is globally asymptotically stable (GAS) in probability if for any $\varepsilon > 0$, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that $P\big\{|x(t)| < \beta(|x_0|, t)\big\} \geq 1-\varepsilon$ for any $t \geq 0$ and $x_0 \in \mathbb{R}^n \setminus \{0\}$.

Lemma 1 ([15]). *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $p \geq 1$ is a constant, the following inequality hold:*

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|,$$

if $p \in \mathbb{R}^*$, then

$$|x-y|^p \leq 2^{p-1}|x^p - y^p|.$$

Lemma 2 ([15]). *Let c, d be positive constants, given any positive number $\gamma > 0$, the following inequality holds:*

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}.$$

Lemma 3 ([16]). *Let $x_1, \ldots, x_n, p$ be positive real numbers, then*

$$(x_1 + \ldots + x_n)^p \leq \max\big\{n^{p-1}, 1\big\}\big(x_1^p + \ldots + x_n^p\big).$$

Lemma 4 ([15]). *Let $p \in \mathbb{R}^*$ and x, y be real-valued functions, then for a constant $c > 0$*

$$\begin{aligned} |x^p - y^p| &\leq p|x-y|\big(x^{p-1} + y^{p-1}\big) \\ &\leq c|x-y|\big|(x-y)^{p-1} + y^{p-1}\big|. \end{aligned}$$

Lemma 5 ([14]). *Consider the stochastic system* (2)*, if there exist a $\mathcal{C}^2$ function $V(x)$, class $\mathcal{K}_\infty$ functions α_1 and α_2, constants $c_1 > 0$ and $c_2 \geq 0$, and a nonnegative function $W(x)$ such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \mathcal{L}V \leq -c_1 W(x) + c_2,$$

then

(i) *For* (2)*, there exists an almost surely unique solution on* $[0, \infty)$*;*

(ii) *When* $c_2 = 0$*,* $f(0) = 0$*,* $g(0) = 0$*, and* $W(x) = \alpha_3(|x|)$*, where* $\alpha_3(\cdot)$ *is a class* $\mathcal{K}$ *function, the equilibrium* $x = 0$ *is GAS in probability and* $P\{\lim_{t\to\infty} |x(t)| = 0\} = 1$.

Consider the following stochastic nonlinear system

$$\mathrm{d}x = \hat{f}(x)\,\mathrm{d}t + \hat{g}_1(x)\,\mathrm{d}\omega + \hat{g}_2(x)u^r\mathrm{d}t, \quad x_0 \in \mathbb{R}^n, \tag{4}$$

where x and ω have the same definitions as those in (2). $\hat{f}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\hat{g}_1\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\hat{g}_2\colon \mathbb{R}^n \to \mathbb{R}^n$ are some locally Lipschitz Borel measurable functions, and u is the input. We give the result on the problem of inverse optimal stabilization in probability.

Lemma 6. *Consider the control law*

$$u = \alpha(x) = -\left[R(x)^{-1}(L_{\hat{g}_2}V)^T \frac{\ell_\gamma(|(L_{\hat{g}_2}V)R(x)^{-\frac{1}{2}}|)}{|(L_{\hat{g}_2}V)R(x)^{-\frac{1}{2}}|^2}\right]^{\frac{1}{r}}, \tag{5}$$

where $V(x)$ *is a Lyapunov function candidate,* $\gamma(\cdot)$ *is a class* $\mathcal{K}_\infty$ *function whose derivative exists and is also a class* $\mathcal{K}_\infty$ *function, and* $R(x) = R(x)^T > 0$ *is a matrix-valued function. If the control law* (5) *achieves GAS in probability for* (4) *with respect to* $V(x)$*, then the control law*

$$u^* = \alpha^*(x) = -\left[\frac{\beta}{2}R(x)^{-1}(L_{\hat{g}_2}V)^T \frac{(\dot{\gamma})^{-1}(|(L_{\hat{g}_2}V)R(x)^{-\frac{1}{2}}|)}{|(L_{\hat{g}_2}V)R(x)^{-\frac{1}{2}}|}\right]^{\frac{1}{r}}, \quad \beta \geq 2 \tag{6}$$

solves the problem of inverse optimal stabilization in probability for (4) *by minimizing the cost function*

$$J(u) = E\left\{\int_0^\infty \left[l(x) + \beta^2\gamma\left(\frac{2}{\beta}\left|R(x)^{\frac{1}{2}}u^r\right|\right)\right]\mathrm{d}\tau\right\}, \tag{7}$$

where

$$l(x) = 2\beta\left[\ell_\gamma\left(\left|(L_{\hat{g}_2}V)R(x)^{-\frac{1}{2}}\right|\right) - L_{\hat{f}}V - \frac{1}{2}\,\mathrm{Tr}\left\{\hat{g}_1(x)^T\frac{\partial^2 V(x)}{\partial x^2}\hat{g}_1(x)\right\}\right] + \beta(\beta-2)\ell_\gamma\left(\left|(L_{\hat{g}_2}V)R(x)^{-\frac{1}{2}}\right|\right).$$

Proof. Choosing $\hat{u} = u^r$, (4) becomes the same form as (3.66) in Theorem 3.9 of [14], hence this lemma can be proved easily. □

3 Controller design and analysis

The objective of this paper is to design a state-feedback controller for system (1) such that the closed-loop system is GAS in probability at the origin and the controller is also optimal in probability.

In this paper, we need the following Assumptions.

Assumption 1. *For each* $d_i(t)$, $i = 1, \dots, n$, *there are positive real numbers* λ_i *and* μ_i *such that*

$$0 < \lambda_i \le d_i(t) \le \mu_i.$$

Assumption 2. *Given* r *defined in* (1), *there are nonnegative constants* a_1 *and* a_2 *such that*

$$|f_i(\bar{x}_i)| \le a_1 \sum_{m=1}^{i} |x_m|^r, \quad |\phi_i(\bar{x}_i)| \le a_2 \sum_{m=1}^{i} |x_m|^{\frac{1+r}{2}}.$$

Remark 1. Assumption 2 is similar to Assumption 1 in [13], whose significance and necessity is illustrated in that paper.

Define $\lambda \triangleq \min\{\lambda_1, \dots, \lambda_n\}$, $\mu \triangleq \max\{\mu_1, \dots, \mu_n\}$. We give the design procedure of controller as follows.

Step 1. Introducing $\xi_1 = x_1$ and constructing the first Lyapunov function $V_1(x_1) = \frac{1}{4}k_1\xi_1^4$, where $k_1 > 0$ is a constant, with the help of (1), (3) and Assumption 2, it can be verified that

$$\begin{aligned}
\mathcal{L}V_1 &= k_1\xi_1^3\big(d_1(t)x_2^r + f_1(x_1)\big) + \frac{3}{2}k_1\xi_1^2\mathrm{Tr}\big\{\phi_1(x_1)\phi_1(x_1)^T\big\} \\
&= d_1(t)k_1\xi_1^3x_2^r + k_1\xi_1^3f_1(x_1) + \frac{3}{2}k_1\xi_1^2|\phi_1(x_1)|^2 \\
&\le k_1\xi_1^3\big(d_1(t)x_2^r - \lambda x_2^{*r}\big) + \lambda k_1\xi_1^3x_2^{*r} + \left(a_1 + \frac{3}{2}a_2^2\right)k_1\xi_1^{3+r}.
\end{aligned} \tag{8}$$

Choosing the first smooth virtual controller

$$x_2^* = -b_1\xi_1, \quad b_1 = \left(\frac{c_{1,1} + (a_1 + \frac{3}{2}a_2^2)k_1}{\lambda k_1}\right)^{\frac{1}{r}}, \quad c_{1,1} > 0, \tag{9}$$

and noting that $-\xi_1^3x_2^{*r} \ge 0$, $0 < \lambda \le d_1(t) \le \mu$, one gets

$$\begin{aligned}
\mathcal{L}V_1 &\le -c_{1,1}\xi_1^{3+r} + k_1\xi_1^3\big(d_1(t)x_2^r - \lambda x_2^{*r}\big) \\
&\le -c_{1,1}\xi_1^{3+r} + k_1d_1(t)\xi_1^3(x_2^r - x_2^{*r}) \\
&\le -c_{1,1}\xi_1^{3+r} + \mu k_1|\xi_1|^3|x_2^r - x_2^{*r}|.
\end{aligned} \tag{10}$$

Step 2. Define $\xi_2 = x_2 - x_2^* = x_2 + b_1x_1$. From (1) it follows that

$$\begin{aligned}
\mathrm{d}\xi_2 &= \big(d_2(t)x_3^r + b_1d_1(t)x_2^r + f_2(\bar{x}_2) + b_1f_1(x_1)\big)\mathrm{d}t \\
&\quad + \big(\phi_2(\bar{x}_2) + b_1\phi_1(x_1)\big)^T\mathrm{d}\omega.
\end{aligned} \tag{11}$$

Choosing Lyanunov function $V_2(x_1,x_2)=V_1(x_1)+\frac{1}{4}k_2\xi_2^4$, where $k_2>0$ is a constant, by (3), (10) and (11), one has

$$\begin{aligned}\mathcal{L}V_2\le&-c_{1,1}\xi_1^{3+r}+d_2(t)k_2\xi_2^3x_3^r+\mu k_1|\xi_1|^3|x_2^r-x_2^{*r}|+b_1d_1(t)k_2\xi_2^3x_2^r\\&+k_2\xi_2^3\big(f_2(\bar{x}_2)+b_1f_1(x_1)\big)+\frac{3}{2}k_2\xi_2^2\left|\phi_2(\bar{x}_2)+b_1\phi_1(x_1)\right|^2.\end{aligned}\tag{12}$$

Using Lemmas 1, 2, 4, Assumptions 1, 2, one obtains

$$\begin{aligned}&\mu k_1|\xi_1|^3|x_2^r-x_2^{*r}|\\&\quad\le\mu ck_1|\xi_1|^3|x_2-x_2^*|\big|(x_2-x_2^*)^{r-1}+x_2^{*r-1}\big|\\&\quad\le\mu ck_1|\xi_1|^3|\xi_2|^r+\mu ck_1b_1^{r-1}|\xi_1|^{2+r}|\xi_2|\\&\quad\le(b_{2,1,1}+b_{2,2,1})\xi_1^{3+r}+\rho_{2,1}\xi_2^{3+r},\end{aligned}\tag{13}$$

$$\begin{aligned}&b_1d_1(t)k_2\xi_2^3x_2^r\\&\quad\le\mu b_1k_2|\xi_2|^3|\xi_2-b_1\xi_1|^r\\&\quad\le2^{r-1}\mu b_1k_2|\xi_2|^3\big(|\xi_2|^r+|b_1\xi_1|^r\big)\\&\quad=2^{r-1}\mu k_2b_1^{1+r}|\xi_1|^r|\xi_2|^3+2^{r-1}\mu b_1k_2\xi_2^{3+r}\\&\quad\le b_{2,1,2}\xi_1^{3+r}+\rho_{2,2}\xi_2^{3+r},\end{aligned}\tag{14}$$

$$\begin{aligned}&k_2\xi_2^3\big(f_2(\bar{x}_2)+b_1f_1(x_1)\big)\\&\quad\le k_2|\xi_2|^3\big((a_1+a_1b_1)|\xi_1|^r+a_1|x_2|^r\big)\\&\quad\le k_2|\xi_2|^3\big((a_1+a_1b_1+2^{r-1}a_1b_1^r)|\xi_1|^r+2^{r-1}a_1|\xi_2|^r\big)\\&\quad=a_1k_2\big(1+b_1+2^{r-1}b_1^r\big)|\xi_1|^r|\xi_2|^3+2^{r-1}a_1k_2\xi_2^{3+r}\\&\quad\le b_{2,1,3}\xi_1^{3+r}+\rho_{2,3}\xi_2^{3+r},\end{aligned}\tag{15}$$

$$\begin{aligned}&\frac{3}{2}k_2\xi_2^2|\phi_2(\bar{x}_2)+b_1\phi_1(x_1)|^2\\&\quad\le3k_2\xi_2^2\big(\phi_2^2(\bar{x}_2)+b_1^2\phi_1^2(x_1)\big)\\&\quad\le3k_2a_2^2\xi_2^2\big((2+b_1^2)\xi_1^{1+r}+2x_2^{1+r}\big)\\&\quad\le3k_2a_2^2\xi_2^2\big((2+b_1^2+2^{1+r}b_1^{1+r})\xi_1^{1+r}+2^{1+r}\xi_2^{1+r}\big)\\&\quad=3k_2a_2^2\big(2+b_1^2+2^{1+r}b_1^{1+r}\big)\xi_1^{1+r}\xi_2^2+3\cdot2^{1+r}k_2a_2^2\xi_2^{3+r}\\&\quad\le b_{2,1,4}\xi_1^{3+r}+\rho_{2,4}\xi_2^{3+r},\end{aligned}\tag{16}$$

where $\rho_{2,1}$, $\rho_{2,2}$, $\rho_{2,3}$, $\rho_{2,4}$, $b_{2,1,1}$, $b_{2,2,1}$, $b_{2,1,2}$, $b_{2,1,3}$, $b_{2,1,4}$ are some designed positive constants. One substitutes (13)–(16) into (12) yields

$$\mathcal{L}V_2\le-c_{2,1}\xi_1^{3+r}+k_2\xi_2^3\big(d_2(t)x_3^r-\lambda x_3^{*r}\big)+\lambda k_2\xi_2^3x_3^{*r}+\rho_2\xi_2^{3+r},\tag{17}$$

where $\rho_2=\rho_{2,1}+\rho_{2,2}+\rho_{2,3}+\rho_{2,4}$, $c_{2,1}=c_{1,1}-b_{2,1,1}-b_{2,2,1}-b_{2,1,2}-b_{2,1,3}-b_{2,1,4}>0$, which together with the smooth virtual controller

$$x_3^*=-b_2\xi_2,\quad b_2=\left(\frac{c_{2,2}+\rho_2}{\lambda k_2}\right)^{\frac{1}{r}},\quad c_{2,2}>0,\tag{18}$$

and $-\xi_2^3 x_3^{*r} \geq 0$, $0 < \lambda \leq d_2(t) \leq \mu$, leads to

$$\begin{aligned} \mathcal{L}V_2 &\leq -c_{2,1}\xi_1^{3+r} - c_{2,2}\xi_2^{3+r} + k_2\xi_2^3\big(d_2(t)x_3^r - \lambda x_3^{*r}\big) \\ &\leq -c_{2,1}\xi_1^{3+r} - c_{2,2}\xi_2^{3+r} + k_2 d_2(t)\xi_2^3\big(x_3^r - x_3^{*r}\big) \\ &\leq -c_{2,1}\xi_1^{3+r} - c_{2,2}\xi_2^{3+r} + \mu k_2|\xi_2|^3\big|x_3^r - x_3^{*r}\big|. \end{aligned} \tag{19}$$

Step i ($i = 3, \ldots, n$). Suppose that at step $i-1$, there exist a set of virtual controllers $x_1^*, \ldots, x_i^*$ defined by

$$\begin{aligned} &x_1^* = 0, \;\; \xi_1 = x_1 - x_1^* = x_1, \\ &x_k^* = -b_{k-1}\xi_{k-1}, \;\; \xi_k = x_k - x_k^* = x_k + b_{k-1}\xi_{k-1}, \quad k = 2, \ldots, i, \end{aligned} \tag{20}$$

such that the $(i-1)$th Lyapunov function candidate $V_{i-1}(x_1, \ldots, x_{i-1}) = \frac{1}{4}\sum_{j=1}^{i-1} k_j\xi_j^4$ satisfies

$$\begin{aligned} \mathcal{L}V_{i-1} \leq &-c_{i-1,1}\xi_1^{3+r} - c_{i-1,2}\xi_2^{3+r} - \ldots - c_{i-1,i-1}\xi_{i-1}^{3+r} \\ &+ \mu k_{i-1}|\xi_{i-1}|^3|x_i^r - x_i^{*r}|, \end{aligned} \tag{21}$$

where $b_1, \ldots, b_{i-1} > 0$ are designed parameters, $c_{i-1,j}, k_j, (j = 1, \ldots, i-1)$ are positive constants. In the sequel, we will prove that (21) still holds for the ith Lyapunov function candidate

$$V_i(x_1, \ldots, x_i) = V_{i-1}(x_1, \ldots, x_{i-1}) + \frac{1}{4}k_i\xi_i^4. \tag{22}$$

By (20) and (1), one has

$$\xi_i = x_i + b_{i-1}x_{i-1} + \ldots + b_{i-1}\ldots b_1 x_1, \tag{23}$$

and

$$\begin{aligned} \mathrm{d}\xi_i = &\left(d_i(t)x_{i+1}^r + \sum_{k=1}^{i-1} b_{i-1}\ldots b_k d_k(t)x_{k+1}^r + f_i(\bar{x}_i) + \sum_{k=1}^{i-1} b_{i-1}\ldots b_k f_k(\bar{x}_k)\right)\mathrm{d}t \\ &+ \left(\phi_i(\bar{x}_i) + \sum_{k=1}^{i-1} b_{i-1}\ldots b_k\phi_k(\bar{x}_k)\right)^T \mathrm{d}\omega. \end{aligned} \tag{24}$$

From (21), (22) and (24), one gets

$$\begin{aligned} \mathcal{L}V_i \leq &-\sum_{j=1}^{i-1} c_{i-1,j}\xi_j^{3+r} + d_i(t)k_i\xi_i^3 x_{i+1}^r + \mu k_{i-1}|\xi_{i-1}|^3|x_i^r - x_i^{*r}| \\ &+ k_i\xi_i^3\sum_{k=1}^{i-1} b_{i-1}\ldots b_k d_k(t)x_{k+1}^r + k_i\xi_i^3\left(f_i(\bar{x}_i) + \sum_{k=1}^{i-1} b_{i-1}\ldots b_k f_k(\bar{x}_k)\right) \\ &+ \frac{3}{2}k_i\xi_i^2\left|\phi_i(\bar{x}_i) + \sum_{k=1}^{i-1} b_{i-1}\ldots b_k\phi_k(\bar{x}_k)\right|^2. \end{aligned} \tag{25}$$

We concentrate on the last four terms in (25). Using (20), Lemmas 1–4 and Assumptions 1, 2, one gets

$$
\begin{aligned}
&\mu k_{i-1}|\xi_{i-1}|^3|x_i^r - x_i^{*r}| \\
&\quad \le \mu c k_{i-1}|\xi_{i-1}|^3|\xi_i|\big|(x_i - x_i^*)^{r-1} + x_i^{*r-1}\big| \\
&\quad \le (b_{i,i-1,1} + b_{i,i,1})\xi_{i-1}^{3+r} + \rho_{i,1}\xi_i^{3+r},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&k_i\xi_i^3\sum_{k=1}^{i-1} b_{i-1}\dots b_k d_k(t) x_{k+1}^r \\
&\quad \le \mu k_i|\xi_i|^3\sum_{k=1}^{i-1} b_{i-1}\dots b_k|\xi_{k+1} - b_k\xi_k|^r \\
&\quad \le 2^{r-1}\mu k_i|\xi_i|^3\sum_{k=1}^{i-1} b_{i-1}\dots b_k\big(|\xi_{k+1}|^r + |b_k\xi_k|^r\big) \\
&\quad = 2^{r-1}\mu k_i|\xi_i|^3\big(b_{i-1}\dots b_2 b_1^{1+r}|\xi_1|^r + \big(b_{i-1}\dots b_1 + b_{i-1}\dots b_3 b_2^{1+r}\big)|\xi_2|^r \\
&\qquad\qquad + \dots + \big(b_{i-1}b_{i-2} + b_{i-1}^{1+r}\big)|\xi_{i-1}|^r + b_{i-1}|\xi_i|^r\big) \\
&\quad \le b_{i,1,2}\xi_1^{3+r} + \dots + b_{i,i-1,2}\xi_{i-1}^{3+r} + \rho_{i,2}\xi_i^{3+r},
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&k_i\xi_i^3\left(f_i(\bar{x}_i) + \sum_{k=1}^{i-1} b_{i-1}\dots b_k f_k(\bar{x}_k)\right) \\
&\quad \le a_1 k_i|\xi_i|^3\left(\sum_{m=1}^{i}|x_m|^r + \sum_{k=1}^{i-1} b_{i-1}\dots b_k\sum_{j=1}^{k}|x_j|^r\right) \\
&\quad = a_1 k_i|\xi_i|^3\big(d_{i,1,1}|x_1|^r + d_{i,2,1}|x_2|^r + \dots + d_{i,i-1,1}|x_{i-1}|^r + |x_i|^r\big) \\
&\quad \le a_1 k_i|\xi_i|^3\big(\big(d_{i,1,1} + 2^{r-1}d_{i,2,1}b_1^r\big)|\xi_1|^r + 2^{r-1}\big(d_{i,2,1} + d_{i,3,1}b_2^r\big)|\xi_2|^r \\
&\qquad\qquad + \dots + 2^{r-1}\big(d_{i,i-1,1} + b_{i-1}^r\big)|\xi_{i-1}|^r + |\xi_i|^r\big) \\
&\quad \le b_{i,1,3}\xi_1^{3+r} + \dots + b_{i,i-1,3}\xi_{i-1}^{3+r} + \rho_{i,3}\xi_i^{3+r},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&\frac{3}{2}k_i\xi_i^2\left|\phi_i(\bar{x}_i) + \sum_{k=1}^{i-1} b_{i-1}\dots b_k\phi_k(\bar{x}_k)\right|^2 \\
&\quad \le 3k_i\xi_i^2\left(|\phi_i(\bar{x}_i)|^2 + \left(\sum_{k=1}^{i-1} b_{i-1}\dots b_k|\phi_k(\bar{x}_k)|\right)^2\right) \\
&\quad \le 3k_i a_2^2\xi_i^2\big(d_{i,1,2}x_1^{1+r} + d_{i,2,2}x_2^{1+r} + \dots + d_{i,i-1,2}x_{i-1}^{1+r} + i x_i^{1+r}\big) \\
&\quad \le 3k_i a_2^2\xi_i^2\big(\big(d_{i,1,2} + 2^r d_{i,2,2}b_1^{1+r}\big)\xi_1^{1+r} + 2^r\big(d_{i,2,2} + d_{i,3,2}b_2^{1+r}\big)\xi_2^{1+r} \\
&\qquad\qquad + \dots + 2^r\big(d_{i,i-1,2} + i b_{i-1}^{1+r}\big)\xi_{i-1}^{1+r} + 2^r i\xi_i^{1+r}\big) \\
&\quad \le b_{i,1,4}\xi_1^{3+r} + \dots + b_{i,i-1,4}\xi_{i-1}^{3+r} + \rho_{i,4}\xi_i^{3+r},
\end{aligned} \tag{29}
$$

where $\rho_{i,1}$, $\rho_{i,2}$, $\rho_{i,3}$, $\rho_{i,4}$, $b_{i,i-1,1}$, $b_{i,i,1}$, $b_{i,1,2}, \dots$, $b_{i,i-1,2}$, $b_{i,1,3}, \dots$, $b_{i,i-1,3}$, $b_{i,1,4}, \dots$,

$b_{i,i-1,4}$ are positive constants with

$$\begin{aligned} c_{i,1} &= c_{i-1,1} - b_{i,1,2} - b_{i,1,3} - b_{i,1,4} > 0, \\ &\vdots \\ c_{i,i-2} &= c_{i-1,i-2} - b_{i,i-2,2} - b_{i,i-2,3} - b_{i,i-2,4} > 0, \\ c_{i,i-1} &= c_{i-1,i-1} - b_{i,i-1,1} - b_{i,i,1} - b_{i,i-1,2} - b_{i,i-1,3} - b_{i,i-1,4} > 0, \end{aligned} \tag{30}$$

and $d_{i,1,1} = 1+\sum_{k=1}^{i-1} b_{i-1}\dots b_k,\ d_{i,2,1} = 1+\sum_{k=2}^{i-1} b_{i-1}\dots b_k, \dots,\ d_{i,i-1,1} = 1+b_{i-1}$, $d_{i,1,2} = (i-1)\sum_{k=1}^{i-1} k(b_{i-1}\dots b_k)^2 + i,\ d_{i,2,2} = (i-1)\sum_{k=2}^{i-1} k(b_{i-1}\dots b_k)^2 + i, \dots$, $d_{i,i-1,2} = (i-1)^2 b_{i-1}^2 + i$. Substituting (26)–(30) into (25) and noting that $-\xi_i^3 x_{i+1}^{*r} \geq 0$ and $0 < \lambda \leq d_i(t) \leq \mu$, the virtual controller

$$x_{i+1}^* = -b_i \xi_i, \quad b_i = \left(\frac{c_{i,i} + \rho_i}{\lambda k_i}\right)^{\frac{1}{r}}, \quad c_{i,i} > 0, \tag{31}$$

leads to

$$\begin{aligned} \mathcal{L}V_i &\leq -c_{i,1}\xi_1^{3+r} - \dots - c_{i,i-1}\xi_{i-1}^{3+r} - c_{i,i}\xi_i^{3+r} + k_i\xi_i^3\big(d_i(t)x_{i+1}^r - \lambda x_{i+1}^{*r}\big) \\ &\leq -c_{i,1}\xi_1^{3+r} - \dots - c_{i,i-1}\xi_{i-1}^{3+r} - c_{i,i}\xi_i^{3+r} + k_i d_i(t)\xi_i^3\big(x_{i+1}^r - x_{i+1}^{*r}\big) \\ &\leq -\sum_{j=1}^{i} c_{i,j}\xi_j^{3+r} + \mu k_i |\xi_i|^3 \big|x_{i+1}^r - x_{i+1}^{*r}\big|, \end{aligned} \tag{32}$$

where $\rho_i = \rho_{i,1} + \rho_{i,2} + \rho_{i,3} + \rho_{i,4}$ is a positive real number.

When $i = n$, by choosing the actual control law

$$u = x_{n+1}^* = -b_n \xi_n, \quad b_n = \left(\frac{c_{n,n} + \rho_n}{\lambda k_n}\right)^{\frac{1}{r}}, \quad c_{n,n} > 0, \tag{33}$$

one has

$$\mathcal{L}V_n \leq -\sum_{i=1}^{n} c_{n,i}\xi_i^{3+r}, \tag{34}$$

where

$$V_n(x_1, \dots, x_n) = \frac{1}{4}\sum_{i=1}^{n} k_i \xi_i^4 \tag{35}$$

and $c_{n,i} (i = 1, \dots, n)$ are positive real numbers.

Remark 2. For general systems, in the design procedure of controller, we can only give the existence of $\rho_{i,1}$, $\rho_{i,2}$, $\rho_{i,3}$, and $\rho_{i,4}$ $(i = 2, \dots, n)$ obtained by using Lemmas 1–4 rather than their explicit definitions. While for a practical example, by appropriately choosing design parameters, $\rho_{i,1}$, $\rho_{i,2}$, $\rho_{i,3}$, and $\rho_{i,4}$ $(i = 2, \dots, n)$ can be concretely obtained, so the state-feedback controller (33) can be implemented, see Section 4 for the details.

We are now in a position to state the main result in this paper.

Theorem 1. *If Assumptions* 1, 2 *hold for the stochastic nonlinear systems* (1)*, under the smooth state-feedback controller* (33)*, then*

(i) *The closed-loop system consisting of* (1), (9), (18), (20), (31) *and* (33) *has an almost surely unique solution on* $[0, \infty)$ *for any initial value* x_0*;*

(ii) *The equilibrium at the origin of the closed-loop system is GAS in probability and the states can be regulated to the origin almost surely, more precisely,*

$$P\left\{\lim_{t\to\infty}\sum_{i=1}^{n}|x_i(t)| = 0\right\} = 1;$$

(iii) *Specially, when* $d_i(t) \equiv 1, i = 1, \ldots, n$*, the control law*

$$u^* = -\xi_n\left(\frac{r+3}{6}\beta b_n^r\right)^{\frac{1}{r}}, \quad \beta \geq 2 \tag{36}$$

guarantees that the equilibrium at the origin of the closed-loop system is GAS in probability and also minimizes the cost functional

$$J(u) = E\left\{\int_0^\infty \left[l(x) + k_n b_n^{-3}\beta^2\frac{r}{r+3}\left(\frac{r+3}{3}\right)^{-\frac{3}{r}}\left(\frac{2}{\beta}\right)^{\frac{r+3}{r}} u^{r+3}\right]\mathrm{d}\tau\right\}, \tag{37}$$

where $l(x)$ *is defined in Lemma* 6.

Proof. Using (1), (20), (34), (35) and Lemma 5, it is obvious that (i) and (ii) hold.

Now, we prove conclusion (iii). By (1), one gets

$$\begin{aligned}\mathrm{d}x &= \begin{bmatrix} x_2^r + f_1(x_1) \\ \vdots \\ x_n^r + f_{n-1}(\bar{x}_{n-1}) \\ f_n(x)\end{bmatrix}\mathrm{d}t + \begin{bmatrix}\phi_1(x_1)^T \\ \vdots \\ \phi_{n-1}(\bar{x}_{n-1})^T \\ \phi_n(x)^T\end{bmatrix}\mathrm{d}\omega + \begin{bmatrix}0\\ \vdots \\ 0 \\ 1\end{bmatrix}u^r\mathrm{d}t \\ &\triangleq \hat{f}(x)\,\mathrm{d}t + \hat{g}_1(x)\,\mathrm{d}\omega + \hat{g}_2(x)u^r\,\mathrm{d}t.\end{aligned} \tag{38}$$

Using (5), (35) and (38), one has

$$L_{\hat{g}_2}V_n = \frac{\partial V_n}{\partial x}\hat{g}_2 = k_n\xi_n^3, \tag{39}$$

and

$$u = -\left(k_n^{-1}\xi_n^{-3}\ell_\gamma\left(\left|k_n\xi_n^3 R(x)^{-\frac{1}{2}}\right|\right)\right)^{\frac{1}{r}}, \tag{40}$$

where $R(x) > 0$ is a scalar-valued function. Choosing

$$\gamma(s) = \frac{r}{r+3} s^{\frac{r+3}{r}}, \tag{41}$$

one gets $(\dot{\gamma})^{-1}(s) = s^{r/3}$, which one substitutes into the definition of $\ell\gamma(s)$ to obtain

$$\ell_\gamma(s) = s s^{\frac{r}{3}} - \frac{r}{r+3} s^{\frac{r+3}{3}} = \frac{3}{r+3} s^{\frac{r+3}{3}}. \tag{42}$$

Choosing

$$R(x) = \left(\frac{r+3}{3} k_n^{-\frac{r}{3}} b_n^r \right)^{-\frac{6}{r+3}}, \tag{43}$$

by (40), (42), one has

$$\begin{aligned} u &= -\left(k_n^{-1} \xi_n^{-3} \ell_\gamma \left(\left| k_n \xi_n^3 \left(\frac{r+3}{3} k_n^{-\frac{r}{3}} b_n^r \right)^{\frac{3}{r+3}} \right| \right) \right)^{\frac{1}{r}} \\ &= -\left(\xi_n^{-3} \frac{3}{r+3} \, \xi_n^{r+3} \, \frac{r+3}{3} b_n^r \right)^{\frac{1}{r}} \\ &= -b_n \xi_n, \end{aligned} \tag{44}$$

which has exactly the same form as (33). Since (44) achieves GAS in probability, by (6), (39), (41) and (43), one can get the inverse optimal controller (36). From (7), (41) and (43), one can obtain the cost function (37). □

Remark 3. Let us compare the main contributions in this paper with [13] from the following aspects: (i) We further to address the problem of state-feedback inverse optimal stabilization in probability, which was not considered by [13]. (ii) The system's power r in this paper is a ratio of odd integers, which is more general than $p_1 = \ldots = p_n = p$ in [13], where p is a positive odd integer. (iii) All inequalities in [13] are only suitable for the case of r being positive integer, while for r being any positive real number, these inequalities need to be reproved. (iv) Compared with [13], the operations of most of inequalities in the design procedure of controller, whose powers involve more operations between fraction and integer, are much more complicated.

4 A simulation example

Consider the following system

$$\begin{aligned} \mathrm{d}x_1 &= x_2^{\frac{5}{3}} \,\mathrm{d}t + \frac{1}{10} x_1^{\frac{4}{3}} \,\mathrm{d}\omega, \\ \mathrm{d}x_2 &= \left((6 + \cos t) u^{\frac{5}{3}} + \frac{1}{5} x_1^{\frac{5}{3}} \right) \mathrm{d}t + \frac{1}{20} x_2 \sin x_2 \,\mathrm{d}\omega, \end{aligned} \tag{45}$$

where $d_1(t)=1, d_2(t)=6+\cos t$, $f_1(x_1)=0$, $\phi_1(x_1)=\frac{1}{10}x_1^{4/3}$, $f_2(\bar{x}_2)=\frac{1}{5}x_1^{5/3}$, $\phi_2(\bar{x}_2)=\frac{1}{20}x_2\sin x_2$. Obviously, $\lambda_1=\mu_1=1$, $\lambda_2=5$, $\mu_2=6$. Next, we need to prove the following inequality:

$$\left|\frac{1}{20}x_2\sin x_2\right|\le\frac{1}{10}x_2^{\frac{4}{3}}. \tag{46}$$

When $|x_2|=0$, one has $|\frac{1}{20}x_2\sin x_2|=\frac{1}{10}x_2^{4/3}$; when $0<|x_2|<1$, one has $|\frac{\frac{1}{20}\sin x_2}{x_2}|\le\frac{1}{20}<\frac{1}{10}\le\frac{1}{10}|x_2|^{-\frac{2}{3}}$, so $|\frac{1}{20}x_2\sin x_2|\le\frac{1}{10}x_2^{4/3}$; when $|x_2|\ge1$, one has $|\frac{1}{20}x_2\sin x_2|\le\frac{1}{20}|x_2|\le\frac{1}{10}|x_2|^{\frac{4}{3}}$.

From (46), we get $a_1=\frac{1}{5}, a_2=\frac{1}{10}$ in Assumption 2.

Next, we apply the above design procedure to (45). Introducing $\xi_1=x_1-x_1^*$ with $x_1^*=0$ and choosing $V_1(x_1)=\frac{1}{4}k_1\xi_1^4$, it is easy to deduce from (45) that $\mathcal{L}V_1\le-c_1\xi_1^{\frac{14}{3}}+k_1\xi_1^3(x_2^{\frac{5}{3}}-x_2^{*\frac{5}{3}})$ with $x_2^*=-b_1\xi_1=-(\frac{c_1}{k_1}+\frac{3}{200})^{\frac{3}{5}}x_1$.

Next, define $\xi_2=x_2-x_2^*=x_2+b_1x_1$, obviously, $\mathrm{d}\xi_2=((6+\cos t)u^{5/3}+\frac{1}{5}x_1^{5/3}+b_1x_2^{5/3})\,\mathrm{d}t+(\frac{1}{20}x_2\sin x_2+\frac{b_1}{10}x_1^{4/3})\,\mathrm{d}\omega$. By lemmas 1,2,4, one obtains

$$\begin{aligned}
&\left|k_1\xi_1^3\big(x_2^{\frac{5}{3}}-x_2^{*\frac{5}{3}}\big)\right|\\
&\quad\le(d_1+d_2)\xi_1^{\frac{14}{3}}+\frac{5}{14}\left(\frac{9}{14d_1}\right)^{\frac{9}{5}}\left(\frac{5k_1}{3}\right)^{\frac{14}{5}}\xi_2^{\frac{14}{3}}+\frac{3}{14}\left(\frac{11}{14d_2}\right)^{\frac{11}{3}}\left(\frac{10k_1}{3}b_1^{\frac{2}{3}}\right)^{\frac{14}{3}}\xi_2^{\frac{14}{3}}\\
&\quad\triangleq(d_1+d_2)\xi_1^{\frac{14}{3}}+\rho_{2,1}\xi_2^{\frac{14}{3}},\\
&\left|k_2\xi_2^3\left(\frac{1}{5}x_1^{\frac{5}{3}}+b_1x_2^{\frac{5}{3}}\right)\right|\\
&\quad\le k_2|\xi_2|^3\left(\left(2^{\frac{2}{3}}b_1^{\frac{8}{3}}+\frac{1}{5}\right)|\xi_1|^{\frac{5}{3}}+2^{\frac{2}{3}}b_1|\xi_2|^{\frac{5}{3}}\right)\\
&\quad\le d_3\xi_1^{\frac{14}{3}}+\left(\frac{9}{14}\left(\frac{5}{14d_3}\right)^{\frac{5}{9}}\left(k_2\left(2^{\frac{2}{3}}b_1^{\frac{8}{3}}+\frac{1}{5}\right)\right)^{\frac{14}{9}}+2^{\frac{2}{3}}b_1k_2\right)\xi_2^{\frac{14}{3}}\\
&\quad\triangleq d_3\xi_1^{\frac{14}{3}}+\rho_{2,2}\xi_2^{\frac{14}{3}},\\
&\frac{3}{2}k_2\xi_2^2\left|\frac{1}{20}x_2\sin x_2+\frac{b_1}{10}x_1^{\frac{4}{3}}\right|^2\\
&\quad\le0.03k_2\xi_2^2\big(\big(2^{\frac{5}{3}}b_1^{\frac{8}{3}}+b_1^2\big)\xi_1^{\frac{8}{3}}+2^{\frac{5}{3}}\xi_2^{\frac{8}{3}}\big)\\
&\quad\le d_4\xi_1^{\frac{14}{3}}+\left(\frac{3}{7}\left(\frac{4}{7d_4}\right)^{\frac{4}{3}}\big(0.03k_2\big(2^{\frac{5}{3}}b_1^{\frac{8}{3}}+b_1^2\big)\big)^{\frac{7}{3}}+0.03k_2\cdot2^{\frac{5}{3}}\right)\xi_2^{\frac{14}{3}}\\
&\quad\triangleq d_4\xi_1^{\frac{14}{3}}+\rho_{2,3}\xi_2^{\frac{14}{3}}.
\end{aligned}$$

Choosing $V_2(x_1, x_2) = V_1(x_1) + \frac{1}{4}k_2\xi_2^4$, a direct calculation leads to

$$\begin{aligned}
\mathcal{L}V_2 \leq & -c_1\xi_1^{\frac{14}{3}} + k_1\xi_1^3\big(x_2^{\frac{5}{3}} - x_2^{*\frac{5}{3}}\big) + k_2\xi_2^3\bigg((6+\cos t)u^{\frac{5}{3}} + \frac{1}{5}x_1^{\frac{5}{3}} + b_1 x_2^{\frac{5}{3}}\bigg) \\
& + \frac{3}{2}k_2\xi_2^2\left|\frac{1}{20}x_2\sin x_2 + \frac{b_1}{10}x_1^{\frac{4}{3}}\right|^2 \\
\leq & -(c_1 - d_1 - d_2 - d_3 - d_4)\xi_1^{\frac{14}{3}} + (6+\cos t)k_2\xi_2^3 u^{\frac{5}{3}} \qquad (47) \\
& + (\rho_{2,1} + \rho_{2,2} + \rho_{2,3})\xi_2^{\frac{14}{3}}, \qquad (48)
\end{aligned}$$

where $k_1, k_2, d_1, d_2, d_3, d_4$ are positive design constants. In simulation, we choose $c_1 = 2, c_2 = d_1 = d_2 = d_4 = 0.25$, $d_3 = 1$, $k_1 = k_2 = 0.1$ to obtain $b_1 = 6.0369$, $\rho_{2,1} = 22.7717$, $\rho_{2,2} = 36.9291$, $\rho_{2,3} = 2.2228$, and the control law

$$u = -b_2\xi_2 = -18.0627(6.0369x_1 + x_2). \qquad (49)$$

Substituting (49) into (48) leads to $\mathcal{L}V_2 \leq -\frac{1}{4}(\xi_1^{\frac{14}{3}} + \xi_2^{\frac{14}{3}})$.

In simulation, we choose the initial values $x_1(0) = -0.3$ and $x_2(0) = 1.6$. Fig. 1 gives the response of the closed-loop system (45) and (49), which demonstrates the effectiveness of the control scheme.

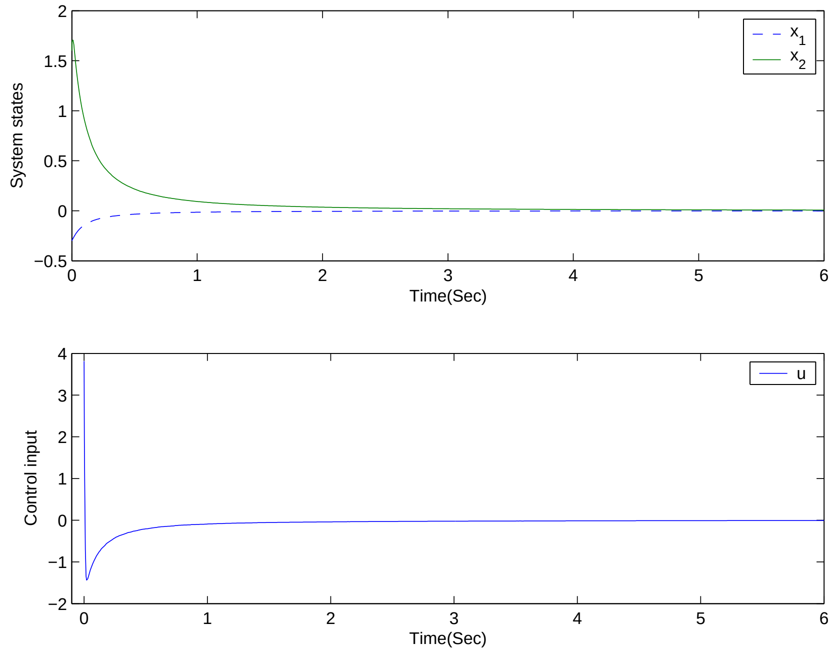

Fig. 1. The response of the closed-loop system (45) and (49).

5 Conclusions

This paper deals with the state-feedback stabilization problem for a class of stochastic nonlinear systems with a ratio of odd integers power for the first time. The designed smooth state-feedback controller ensures that the equilibrium at the origin of the closed-loop system is GAS in probability and the states can be regulated to the origin almost surely. Furthermore, the problem of inverse optimal stabilization in probability is also solved.

Some issues under current investigation are how to generalize the result in this paper to more general class of stochastic nonlinear systems with a ratio of odd integers power; how to design an output-feedback controller for system (1).

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