Global Existence of Periodic Solutions in a Delayed Kaldor-Kalecki Model

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Received: 2009-02-25 **Revised:** 2009-07-08 **Published online:** 2009-11-11

Abstract. This paper is concerned with a delayed Kaldor-Kalecki non-linear business cycle model in income. By applying a global Hopf bifurcation result due to Wu, the global existence of periodic solutions is investigated. Numerical examples will be given in the end, to illustrate our theoretical results.

Keywords: Kaldor-Kalecki business cycle, delayed differential equations, Hopf bifurcation, periodic solutions.

1 Introduction and mathematical models

For long time the analysis of Hopf bifurcations phenomena in business cycle models has attracted much attention due to it's practical significance (see for example, [1–5]). It has demonstrated that the existence of a branch of nonconstant periodic solutions identified a mechanism that gives rise to business cycles in economic activities. However, due to the nature of the Hopf bifurcations theory these periodic solutions are generally local, whereas business cycle behavior concerns global dynamics (see for example, [6–8]). Therefore, it is an important subject to investigate if nonconstant periodic solutions exist globally.

For determining the global existence of periodic solutions, there exist two methods, one is based on [9] ejective fixed point argument, the other is based on [10, 11] purely topological argument.

In this paper we use a method on the global existence of periodic solutions given by Wu [11] to study the following delayed Kaldor-Kalecki model of business cycle (see [5, 12, 14, 15]):

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t), K(t)\big) - S\big(Y(t), K(t)\big)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau), K(t-\tau)\big) - \delta K(t), \end{cases} \tag{1}$$

where Y is the income, K is the capital stock, α is the adjustment coefficient in the goods market, δ is the depreciation rate of capital stock, I is the investment function, S is the saving and τ is the time delay needed for new capital to be installed (see [1]).

In [12, 2008], we investigated the local Hopf bifurcation, that is proven to exist as the delay cross some critical value τ_0. In [13, 2009], we established an explicit algorithm for determining the direction of Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions using the methods presented by Diekmann et al. in [16, 1995].

In this paper, we would like to extend the analysis of the local Hopf bifurcation and to present some new results concerning global Hopf bifurcation of system (1). Also some numerical simulations are given to illustrate the theoretical analysis.

2 Local existence of periodic solutions

As in [12], we consider some assumptions on the investment and saving functions:

$$I(Y, K) = I(Y) - \delta_1 K,$$

and

$$S(Y, K) = \gamma Y,$$

where $\delta_1 > 0$ and $\gamma \in (0, 1)$.

Then system (1) becomes:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t)\big) - \delta_1 K(t) - \gamma Y(t)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau)\big) - \delta_1 K(t-\tau) - \delta K(t). \end{cases} \tag{2}$$

In the following proposition, we give a sufficient conditions for the existence and uniqueness of positive equilibrium E^* of the system (2).

Proposition 1 ([12]). *Suppose that*

(H01) *There exists a constant* $L > 0$ *such that* $|I(Y)| \leq L$ *for all* $Y \in \mathbb{R}$ *;*

(H02) $I(0) > 0$*;*

(H03) $I'(Y) - \gamma < \frac{\gamma\delta_1}{\delta}$ *for all* $Y \in \mathbb{R}$.

Then there exists a unique equilibrium $E^* = (Y^*, K^*)$ *of system* (2), *where* Y^* *is the positive solution of*

$$I(Y) - \frac{(\delta_1 + \delta)\gamma}{\delta} Y = 0 \tag{3}$$

and K^* *is determined by*

$$K^* = \frac{\gamma}{\delta} Y^*. \tag{4}$$

Let $y = Y - Y^*$ and $k = K - K^*$. Then by linearizing system (2) around (Y^*, K^*) we have

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \alpha\big(I'(Y^*) - \gamma\big)y(t) - \alpha\delta_1 k(t), \\ \dfrac{\mathrm{d}k}{\mathrm{d}t} = I'(Y^*)y(t-\tau) - \delta_1 k(t-\tau) - \delta k(t). \end{cases} \tag{5}$$

The characteristic equation associated to system (5) is

$$\lambda^2 + a\lambda + b\lambda\exp(-\lambda\tau) + c + d\exp(-\lambda\tau) = 0, \tag{6}$$

where

$$\begin{aligned} a &= \delta - \alpha(I'(Y^*) - \gamma), \quad b = \delta_1, \\ c &= -\alpha\delta(I'(Y^*) - \gamma), \quad d = \alpha\delta_1\gamma. \end{aligned}$$

Theorem 1 ([12])**.** *Let the hypotheses:*

(H1) $|I'(Y^*) - \gamma| < \frac{\gamma\delta_1}{\delta}$,

(H2) $I'(Y^*) - \gamma < \frac{\delta+\delta_1}{\alpha}$.

Then there exists $\tau_0 > 0$ such that, when $\tau \in [0, \tau_0)$ the steady state E^* is locally asymptotically stable, when $\tau > \tau_0$, E^* is unstable and when $\tau = \tau_n$, $n = 0, 1, 2, \dots$. Equation (6) *has a pair of purely imaginary roots $\pm i\omega_0$, with*

$$\begin{aligned} \omega_0^2 = &-\frac{1}{2}\big[\alpha^2\big(I'(Y^*) - \gamma\big)^2 + \delta^2 - \delta_1^2\big] \\ &+ \frac{1}{2}\Big\{\big[\alpha^2\big(I'(Y^*)-\gamma\big)^2 + \delta^2 - \delta_1^2\big]\Big\}^2 - 4\Big\{\alpha^2\big[\delta^2\big(I'(Y^*)-\gamma\big)^2 - \delta_1^2\gamma^2\big]\Big\}^{1/2} \end{aligned} \tag{7}$$

and

$$\tau_n = \frac{1}{\omega_0}\arctan\frac{\alpha[\gamma\delta - (\alpha\gamma - \delta)(I'(Y^*) - \gamma)]\omega_0 + \omega_0^3}{(\alpha I'(Y^*) - \delta)\omega_0^2 + \alpha^2\gamma\delta(I'(Y^*) - \gamma)} + \frac{2n\pi}{\omega_0}. \tag{8}$$

2.1 Local Hopf bifurcation occurrence

According to the Hopf bifurcation theorem [17], we establish sufficient conditions for the local existence of periodic solutions.

Theorem 2. *Under hypotheses* (H1) *and* (H2) *of Theorem* 1*, there exists a continuous function $\tau(\varepsilon)$ with $\tau(0) = \tau_n$, $n = 0, 1, 2, \dots,$ and for all sufficiently small values of $\varepsilon \neq 0$ there exists a continuous family of nonconstant periodic solutions for the system* (2)*, which collapses to the positive equilibrium point E^* as $\varepsilon \to 0$.*

Remark 1. From Theorem 2, periodic solutions bifurcating from the positive equilibrium point E^* occurs when the time delay τ is close to τ_n, $n = 0, 1, 2, \dots$.

Proof. For the proof of this theorem we apply the Hopf bifurcation theorem introduced in [17]. From Theorem 1, the characteristic equation (6) has a pair of imaginary roots $\pm i\omega_0$ at $\tau = \tau_n$. In the first, lets show that $i\omega_0$ is simple: Consider the branch of characteristic roots $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$, of equation (6) bifurcating from $i\omega_0$ at $\tau = \tau_n$. By derivation of (6) with respect to the delay τ, we obtain

$$\begin{aligned}&\bigl\{2\lambda + \delta - \alpha\bigl(I'(Y^*) - \gamma\bigr) + \bigl(\delta_1 - \tau(\delta_1\lambda + \alpha\delta_1\gamma)\bigr)\exp(-\lambda\tau)\bigr\}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\\ &\qquad = (\delta_1\lambda + \alpha\delta_1\gamma)\lambda\exp(-\lambda\tau).\end{aligned} \tag{9}$$

If we suppose, by contradiction, that $i\omega_0$ is not simple, the right hand side of (9) gives

$$\alpha\gamma + i\omega_0 = 0,$$

and leads a contradiction with the fact that α and γ are positive.

Lastly we need to verify the transversally condition,

$$\frac{\mathrm{d}\,\mathrm{Re}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_n \neq 0}.$$

From (9), we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{(2\lambda + \delta - \alpha(I'(Y^*) - \gamma))\exp(\lambda\tau) + \delta_1}{\lambda(\delta_1\lambda + \alpha\delta_1\gamma)} - \frac{\tau}{\lambda}.$$

As,

$$\mathrm{Sign}\frac{\mathrm{d}\,\mathrm{Re}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_n} = \mathrm{Sign}\,\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau_n}.$$

Then

$$\mathrm{Sign}\frac{\mathrm{d}\,\mathrm{Re}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_n} = \mathrm{Sign}\,\mathrm{Re}\frac{(-2i\omega_0 + \delta_1 + \delta + \alpha(I'(Y^*) - \gamma))\exp(i\omega_0\tau_n)}{-i\alpha\delta_1 I'(Y^*)\omega_0}.$$

From (6), we have

$$\exp(\lambda\tau) = -\frac{\delta_1\lambda + \alpha\delta_1\gamma}{\lambda^2 + (\delta - \alpha(I'(Y^*) - \gamma)\lambda - \alpha\delta(I'(Y^*) - \gamma)}. \tag{10}$$

So, by (H1) and (7) we obtain

$$\begin{aligned}\mathrm{Sign}\frac{\mathrm{d}\,\mathrm{Re}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_n} &= \mathrm{Sign}\,\bigl(\bigl\{\bigl[\alpha^2\bigl(I'(Y^*) - \gamma\bigr)^2 + \delta^2 - \delta_1^2\bigr]^2\\ &\qquad - 4\alpha^2\bigl[\delta^2\bigl(I'(Y^*) - \gamma\bigr)^2 - \delta_1^2\gamma^2\bigr]\bigr\}^{1/2}\bigr).\end{aligned}$$

Consequently,

$$\frac{\mathrm{d}\,\mathrm{Re}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_n} > 0.$$

□

2.2 Direction of Hopf bifurcation

In this subsection we use a formula on the direction of the Hopf bifurcation given by Diekman in [16] to formulate an explicit algorithm about the direction and the stability of the bifurcating branch of periodic solutions of system (2).

Let

$$\begin{aligned}\mathrm{Re}(c) = {} & \frac{\alpha\tau_0}{2} I'''(Y^*) + \frac{\tau_0\alpha\delta I''(Y^*)^2}{(\delta_1+\delta)\gamma - \delta I'(Y^*)} \\ & + \frac{\tau_0^2\alpha^2 I''(Y^*)^2}{2(B^2+C^2)}[(B\delta\tau_0 + 2C\omega_0)],\end{aligned} \tag{11}$$

where

$$\begin{aligned}B &= -4\omega_0^2 - \alpha\delta\big(I'(Y^*)-\gamma\big)\tau_0^2 + 2\delta_1\tau_0\omega_0\sin(2\omega_0) + \alpha\delta_1\gamma\tau_0^2\cos(2\omega_0),\\ C &= 2\delta\tau_0\omega_0 - 2\alpha\big(I'(Y^*)-\gamma\big)\tau_0\omega_0 - \alpha\delta_1\gamma\tau_0^2\sin(2\omega_0) + 2\delta_1\tau_0\omega_0\cos(2\omega_0).\end{aligned}$$

Theorem 3 ([13])**.** *Assume* (H1), *and* (H2)*. Then,*

(i) *the Hopf bifurcation occurs as τ crosses τ_0 to the right (supercritical Hopf bifurcation) if* $\mathrm{Re}(c) > 0$ *and to the left (subcritical Hopf bifurcation) if* $\mathrm{Re}(c) < 0$*; and*

(ii) *the bifurcating periodic solutions is stable if* $\mathrm{Re}(c) > 0$ *and unstable if* $\mathrm{Re}(c) < 0$*; where* $\mathrm{Re}(c)$ *is given by* (11)*.*

Remark 2. Note that, Theorem 3 provides an explicit algorithm for computing an indicator $\mathrm{Re}(c)$ of stability or instability of the bifurcating branch of periodic solutions of system (2).

3 Global existence of periodic solutions

For determining the global existence of periodic solutions, there exist two methods, one is based on ejective fixed point argument, see for example [9], the other is based on purely topological argument, see for example [10, 11]. In this section, we investigate the global continuation of periodic solutions bifurcated from the point (E^*, τ_n), $n = 0, 1, 2 \ldots$. for system (2), by applying the method given by Wu [11].

For simplification of notations, setting $z = (Y, K)$, we may rewrite system (2) as the following functional differential equation:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = F\big(z(t), z(t-\tau)\big). \tag{12}$$

Note that F satisfies the hypotheses (A1) and (A2) in [11, p. 4813–4814].

Following the work of Wu [11], we need to define

$$\begin{aligned}X &= C\big([-\tau, 0], \mathbb{R}^2\big),\\ \Sigma &= Cl\big\{(z,\tau,p) \in X \times \mathbb{R} \times \mathbb{R}^+ \colon z \text{ is a } p\text{-periodic solution of system (12)}\big\},\end{aligned}$$

and let $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ denote the connected component of $(E^*, \tau_n, \frac{2\pi}{\omega_0})$ in Σ, where ω_0 and τ_n are defined respectively in (7) and (8).

Lemma 1. *Under hypotheses* (H1) *and* (H2) *of Theorem* 1*, the periodic solutions of system* (12) *are uniformly bounded.*

Proof. Let $(Y(t), K(t))$ be a nonconstant periodic solution of system (12). Then we obtain

$$\begin{cases} Y(t) = \exp(-\alpha\gamma t) \\ \qquad \times \left\{ Y(0) + \alpha \int\limits_0^t \left[I\big(Y(s-\tau)\big) - \delta_1 K(s-\tau)\right] \exp(\alpha\gamma s)\,\mathrm{d}s \right\}, \\ K(t) = \exp(-\delta t) \\ \qquad \times \left\{ K(0) + \int\limits_0^t \left[I\big(Y(s-\tau)\big) - \delta_1 K(s-\tau)\right] \exp(\delta s)\,\mathrm{d}s \right\}. \end{cases} \tag{13}$$

By using the generalized Gronwall Lemma, we get

$$|K(t)| \leq \left[\frac{L}{\delta} + K(0)\right] \exp\left(\frac{\delta_1}{\delta} e^{\delta\tau}\right), \tag{14}$$

where L is defined in proposition 1.

We set $M1 := [\frac{L}{\delta} + K(0)] \exp(\frac{\delta_1}{\delta} e^{\delta\tau})$. Thus, we have

$$|Y(t)| \leq \frac{L + \delta_1 M1}{\gamma} + Y(0). \tag{15}$$

Choosing $M = \max(M1, M2)$, where $M2 := \frac{L+\delta_1 M1}{\gamma} + Y(0)$. Then a nonconstant periodic solution $(Y(t), K(t))$ of system (12) is uniformly bounded for bounded M. □

Remark 3. Although the boundaries $M1$ and $M2$ of $Y(t)$ and $K(t)$ depend on the value of τ, they are uniformly bounded on τ when τ is bounded.

Lemma 2.

$$I'(Y) - \gamma < \frac{\delta + \delta_1}{\alpha}, \quad \textit{for all } \ Y \in \mathbb{R}.$$

Under hypothesis (H1) *and* (H2) *of Theorem* 1*, system* (12) *has no nontrivial τ-periodic solution.*

Proof. For a contradiction, suppose that system (12) has a nontrivial τ-periodic solution. Then the following system of ordinary differential equations has periodic solution

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\left[I\big(Y(t)\big) - \delta_1 K(t) - \gamma Y(t)\right], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t)\big) - \delta_1 K(t) - \delta K(t). \end{cases} \tag{16}$$

Denote $P(Y,K) = \alpha(I'(Y) - \delta_1 K - \gamma Y)$, and $Q(Y,K) = I(Y) - (\delta_1 + \delta)K$, then we have

$$\frac{\partial P}{\partial Y} + \frac{\partial Q}{\partial K} = \alpha\big(I'(Y) - \gamma\big) - (\delta_1 + \delta).$$

By (H2) in Theorem 1, we have that

$$I'(Y) < \frac{\alpha\gamma + \delta + \delta_1}{\alpha},$$

which leads to

$$\frac{\partial P}{\partial Y} + \frac{\partial Q}{\partial K} < 0.$$

Due to the Bendixson's criterion [18, p. 373], we conclude that the system (16) has no periodic solution. The conclusion follows. ☐

Lemma 3. $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ *is unbounded.*

Proof. We regard τ as a parameter. By Theorem 1, we have that (E^*, τ) is the only stationary solution of (12) and the corresponding characteristic matrix

$$\Delta_{(E^*,\tau)}(\lambda) = \begin{pmatrix} \lambda - \alpha(I'(Y^*) - \gamma) & \alpha\delta_1 \\ -I'(Y^*)\exp(-\lambda\tau) & \lambda + \delta_1 \exp(-\lambda\tau) + \delta \end{pmatrix}, \tag{17}$$

is clearly continuous in $(\lambda, \tau) \in \mathbb{C} \times \mathbb{R}_+$. This justifies hypothesis $(A3)$ in [11, p. 4814], for the considered system (12).

A stationary solution $(E^*, \overline{\tau})$ is called a center if $\det(\Delta_{(E^*,\overline{\tau})}(im\frac{2\pi}{\omega_0})) = 0$ for some positive integer m. A center $(E^*, \overline{\tau})$ is said to be isolated if it is the only center in some neighborhood of $(E^*, \overline{\tau})$.

It follows from theorem 1 that (E^*, τ_n) is an isolated center and from the implicit function theorem, there exist $\varepsilon > 0$, $\nu > 0$ and a smooth curve $\lambda\colon (\tau - \nu, \tau + \nu) \to \mathbb{C}$ such that $\det(\Delta_{(E^*,\tau_n,\frac{2\pi}{\omega_0})}(\lambda) = 0$, $|\lambda(\tau_n) - \omega_0| < \varepsilon$ for all $\tau \in (\tau - \nu, \tau + \nu)$ and $\lambda(\tau_n) = i\omega_0$, $\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau_n} > 0$.

Let

$$\Omega_\varepsilon := \left\{(u,p)\colon\ 0 < u < \varepsilon,\ \left|p - \frac{2\pi}{\omega_0}\right| < \varepsilon\right\}.$$

Clearly, if $|\tau - \tau_n| < \nu$ and $(u,p) \in \partial\Omega_\varepsilon$ such that $\det(\Delta_{(E^*,\tau,p)}(u + i\frac{2\pi}{\omega_0}))) = 0$, then $\tau = \tau_n$, and $u = 0$. This justifies hypothesis (A4) in [11, p. 4814], for $m = 1$. Moreover, if we put

$$H_1^{\pm}\left(E^*, \tau_n, \frac{2\pi}{\omega_0}\right)(u,p) := \det\left(\Delta_{(0,\tau_n \pm \nu, p)}\left(u + im\frac{2\pi}{p}\right)\right),$$

then at $m = 1$, we have the crossing number of (E^*, τ, p)

$$\Gamma\left(E^*, \tau_n, \frac{2\pi}{\omega_0}\right) = \deg_B\big(H_1^-, \Omega_\varepsilon, 0\big) - \deg_B\big(H_1^+, \Omega_\varepsilon, 0\big) = -1,$$

where $\deg_B$ denotes the classical Brouwer degree. By Theorem 3.3 in [11], we conclude that the connected component $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ is unbounded. □

Theorem 4. *Under hypotheses* (H1) *and* (H2) *of Theorem* 2.1*, for each* $\tau > \tau_n$, $n = 1, 2 \ldots$, *system* (2) *has at least* $n + 1$ *periodic solutions.*

Proof. It is sufficient to verify that the projection of $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ onto τ-space is $[\tau^*, +\infty)$, where $\tau^* \le \tau_n$, $n = 1, 2, 3 \ldots$. Lemma 1 implies that the projection of $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ onto z-space is bounded. Also, note that the proof of Lemma 2 implies that the system (12) with $\tau = 0$ has no nonconstant periodic solution. Therefore, the projection of $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ onto the τ-space is bounded below.

By the definition of τ_n, we know that $\tau_n > \frac{2\pi}{\omega_0}$ for each $n \ge 1$. For a contradiction, we assume that the projection of $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ onto the τ-space is bounded. Then there exists $\overline{\tau} > \tau_n$ such that the projection of $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ on to the τ-space is included in the interval $[0, \overline{\tau})$. $\frac{2\pi}{\omega_0} < \tau_n$ and Lemma 2 imply that $0 < p < \overline{\tau}$ for $(z(t), \tau, p)$ belonging to $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$. Applying Lemma 1 we have that $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ is bounded. This a contradiction and hence that the projection of $l(E^*, \tau_n, \frac{2\pi}{\omega_0})$ into the τ-space is $[\tau^*, +\infty)$, where $\tau^* \le \tau_n$, $n = 1, 2, 3$... The proof is complete. □

4 Numerical examples

Consider the following Kaldor-type investment function:

$$I(Y) = \frac{\exp(Y)}{1 + \exp(Y)}.$$

4.1 Stability of the bifurcating branch of periodic solutions

Theorems 1 and 2 implie:

Proposition 2. *If* $\alpha = 3$; $\delta_1 = 0.2$; $\delta = 0.1$; $\gamma = 0.2$. *Then system* (2) *has the following positive equilibrium*

$$E^* = (1.31346, 2.62699).$$

Furthermore, the critical delay, the period of oscillations and the indicator of stability corresponding to system (2) *are* $\tau_0 = 2.9929$, *and* $P_0 = 48.2646$ *and* $\mathrm{Re}(c) = 0.2133$.

The following numerical simulations are given for system (2) for $E_0 = (1, 1)$, $E_0 = (4, 1)$, $E_0 = (2, 1)$, and $E_0 = (0.5, 0)$. By the previous proposition and Theorem 3, if we increase the value of τ, then a stable periodic solution occurs at $\tau_0 = 2.9929$ (see Fig. 1).

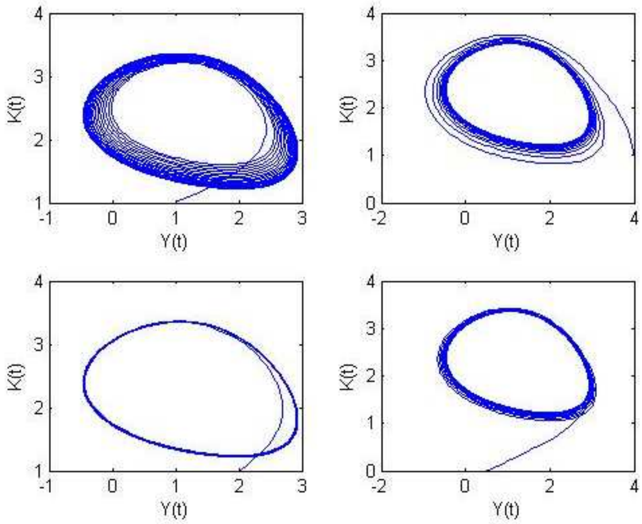

Fig. 1. A family of stable periodic solutions bifurcating from E^* occurs when $\tau = 2.9929$.

4.2 Global existence of periodic solutions

From section 3, we have:

Proposition 3. *If* $\alpha = 3$; $\delta_1 = 0.2$; $\delta = 0.1$; $\gamma = 0.2$ *Then a family of periodic solutions bifurcating from* E^* *occurs when* $\tau > 2.9929$ *(see Fig.* 2 *).*

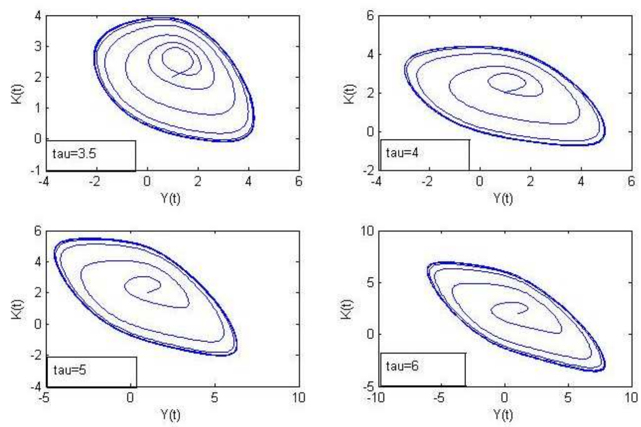

Fig. 2. A family of periodic solutions bifurcating from E^* occurs when $\tau > 2.9929$.

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