Local Hopf Bifurcation and Stability of Limit Cycle in a Delayed Kaldor-Kalecki Model

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Received: 2008-09-24 **Revised:** 2009-01-11 **Published online:** 2009-09-11

Abstract. We consider a delayed Kaldor-Kalecki business cycle model. We first consider the existence of local Hopf bifurcation, and we establish an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions using the methods presented by O. Diekmann et al. in [1]. In the end, we conclude with an application.

Keywords: Kaldor-Kalecki business cycle, delayed differential equations, Hopf bifurcation, periodic solutions.

1 Introduction and mathematical models

In a recent paper [2], we formulate a delayed Kaldor-Kalecki business cycle model by introducing the Kalecki's time delay [3] in the Kaldor model [4] as follows:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t),K(t)\big) - S\big(Y(t),K(t)\big)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau),K(t-\tau)\big) - \delta K(t), \end{cases} \tag{1}$$

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, δ is the depreciation rate of capital stock, $I(Y,K)$ is the investment function, $S(Y,K)$ is the saving and τ is the time delay needed for new capital to be installed.

The dynamics are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exists as the delay (taken as a parameter of bifurcation) cross some critical value.

In this paper, we reconsider the model (1) and we establish an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions using the methods presented by O. Diekmann et al. in [1].

The first model in this optic is proposed by Kalecki in [3, 1935]. The main characteristic feature of his model is the distinction between investment decisions and implementation, i.e. there is a time delay after which capital equipment is available for production.

Besides the influence of Keynes in [5, 1936] and Kalecki in [6, 1937], Kaldor in [4, 1940] presented a nonlinear model of business cycle by an ordinary differential equations as follows:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t), K(t)\big) - S\big(Y(t), K(t)\big)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t), K(t)\big). \end{cases} \tag{2}$$

In this model the nonlinearity of investment and saving function leads to limit cycle solution (see also [7–9] for more information).

Based on the Kaldor model of business cycle and the Kalecki's idea on time delay, Krawiec and Szydłowski in [10, 1999] proposed the following Kaldor-Kalecki model of business cycle:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I(Y(t-\tau), K(t)) - \delta K(t). \end{cases} \tag{3}$$

The fundamental characteristics of this model is the nonlinearity of investment function and the inclusion of time delay into the gross product in capital accumulation equation.

In [10, 11, 2000], Krawiec and Szydblowski investigated the stability and Hopf bifurcation of a positive equilibrium E^* of system (3) in the special case of small time delay. In [12, 2001], they showed that for a small time delay parameter the Kaldor-Kalecki model assumes the form of the Lienard equation. In [13, 2005], they investigate the stability of limit cycle. Zhang and Wei [14, 2004], investigated local and global existence of Hopf bifurcation for (3).

In this work, the dynamics of the system (1) are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exist as the delay (taken as a parameter of bifurcation) cross some critical value. Additionally we establish an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions using the methods presented by O. Diekmann et al. in [1]. In the end, we give a numerical illustrations.

2 Steady state and stability analysis

As in [2, 11], we consider some assumptions on the investment and saving functions:

$$I(Y, K) = I(Y) - \beta K,$$

and

$$S(Y, K) = \gamma Y,$$

where $\beta > 0$ and $\gamma \in (0,1)$. Then system (1) becomes:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t)\big) - \beta K(t) - \gamma Y(t)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau)\big) - \beta K(t-\tau) - \delta K(t). \end{cases} \tag{4}$$

2.1 Steady state

In the following proposition, we give a sufficient conditions for the existence and uniqueness of positive equilibrium E^* of the system (4).

Proposition 1 ([2]). *Suppose that:*

(i) *there exists a constant* $L > 0$ *such that* $|I(Y)| \leq L$ *for all* $Y \in \mathbb{R}$*;*

(ii) $I(0) > 0$*;*

(iii) $I'(Y) - \gamma < \frac{\gamma\beta}{\delta}$ *for all* $Y \in \mathbb{R}$.

Then there exists a unique equilibrium $E^* = (Y^*, K^*)$ *of system* (4)*, where* Y^* *is the positive solution of*

$$I(Y) - \frac{(\beta+\delta)\gamma}{\delta}Y = 0 \tag{5}$$

and K^* *is determined by*

$$K^* = \frac{\gamma}{\delta}Y^*. \tag{6}$$

2.2 Local stability and local Hopf bifurcation analysis

Let $y = Y - Y^*$ and $k = K - K^*$. Then by linearizing system (4) around (Y^*, K^*) we have

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \alpha\big(I'(Y^*) - \gamma\big)y(t) - \alpha\beta k(t), \\ \dfrac{\mathrm{d}k}{\mathrm{d}t} = I(Y^*)y(t-\tau) - \beta k(t-\tau) - \delta k(t). \end{cases} \tag{7}$$

The characteristic equation associated to system (7) is

$$\lambda^2 + a\lambda + b\lambda\exp(-\lambda\tau) + c + d\exp(-\lambda\tau) = 0, \tag{8}$$

where

$$\begin{aligned} a &= \delta - \alpha\big(I'(Y^*) - \gamma\big), \\ b &= \beta, \\ c &= -\alpha\delta\big(I'(Y^*) - \gamma\big), \end{aligned}$$

and

$$d = \alpha\beta\gamma.$$

The local stability of the steady state E^* is a result of the localization of the roots of the characteristic equation (8). In order to investigate the local stability of the steady state, we begin by considering the case without delay $\tau = 0$. In this case the characteristic equation (8) reads as

$$\lambda^2 + (a+b)\lambda + c + d = 0, \tag{9}$$

hence, according to the Hurwitz criterion, we have the following lemma.

Lemma 1. *For $\tau = 0$, the equilibrium E^* is locally asymptotically stable if and only if $I'(Y^*) - \gamma < \min(\frac{\gamma\beta}{\delta}, \frac{\delta+\beta}{\alpha})$.*

We now return to the study of equation (8) with $\tau > 0$.

Theorem 1 ([2])**.** *Let the hypotheses*

$$\text{(H1)}\ \ |I'(Y^*) - \gamma| < \frac{\gamma\beta}{\delta}$$

and

$$\text{(H2)}\quad I'(Y^*) - \gamma < \frac{\delta+\beta}{\alpha}.$$

Then there exists $\tau_0 > 0$ such that, when $\tau \in [0, \tau_0)$ the steady state E^* is locally asymptotically stable, when $\tau > \tau_0$, E^* is unstable and when $\tau = \tau_0$, equation (8) *has a pair of purely imaginary roots $\pm i\omega_0$, with*

$$\omega_0^2 = -\frac{1}{2}\big(\alpha^2\big(I'(Y^*)-\gamma\big)^2+\delta^2-\beta^2\big) + \frac{1}{2}\big[\big(\alpha^2\big(I'(Y^*)-\gamma\big)^2+\delta^2-\beta^2\big)^2-4\big(\alpha^2\delta^2\big(I'(Y^*)-\gamma\big)^2-\beta^2\gamma^2\big)\big]^{1/2} \tag{10}$$

and

$$\tau_0 = \frac{1}{\omega_0}\arctan\frac{\alpha[\gamma\delta-(\alpha\gamma-\delta)(I'(Y^*)-\gamma)]\omega_0+\omega_0^3}{(\alpha I'(Y^*)-\delta)\omega_0^2+\alpha^2\gamma\delta(I'(Y^*)-\gamma)}. \tag{11}$$

Theorem 2 ([2])**.** *Assume that*

$$\text{(H3)}\ \ I'(Y^*) - \gamma \le \min\bigg(-\frac{\beta\gamma}{\delta}, \frac{\delta^2-\beta^2}{\alpha^2}\bigg).$$

Then E^* is locally asymptotically stable for all $\tau \ge 0$.

According to the Hopf bifurcation theorem [15], we establish sufficient conditions for the local existence of periodic solutions.

Theorem 3 ([2])**.** *Under hypotheses* (H1) *and* (H2) *of Theorem* 1*, there exists $\varepsilon_0 > 0$ such that, for each $0 \le \varepsilon < \varepsilon_0$, system* (4) *has a family of periodic solutions $p(\varepsilon)$ with period $T = T(\varepsilon)$, for the parameter values $\tau = \tau(\varepsilon)$ such that $p(0) = 0, T(0) = \frac{2\pi}{\omega_0}$ and $\tau(0) = \tau_0$, where τ_0 and ω_0 are stated in Theorem* 1.

3 Direction of Hopf bifurcation

In this section we use a formula on the direction of the Hopf bifurcation given by Diekman in [1] to formulate an explicit algorithm about the direction and the stability of the bifurcating branch of periodic solutions of (4).

Normalizing the delay τ by scaling $t \to \frac{t}{\tau}$ and effecting the change $U(t) = Y(\tau t)$ and $V(t) = K(\tau t)$, the system (4) is transformed into

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = \alpha\tau\big[I\big(U(t)\big) - \beta V(t) - \gamma U(t)\big], \\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = \tau\big[I\big(U(t-1)\big) - \beta V(t-1) - \delta V(t)\big]. \end{cases} \tag{12}$$

By the translation $Z(t) = (U, V) - (Y^*, K^*)$, system (12) is written as a functional differential equation in $C := C([-1, 0], \mathbb{R}^2)$,

$$\dot{Z}(t) = L(\tau)Z_t + h(Z_t, \tau), \tag{13}$$

where $L(\tau)\colon C \to \mathbb{R}^2$ the linear operator and $h\colon C \times \mathbb{R} \to \mathbb{R}^2$ the nonlinear part of (13) are given respectively by:

$$L(\tau)\varphi = \tau \begin{pmatrix} \alpha[(I'(Y^*) - \gamma)\varphi_1(0) - \beta\varphi_2(0)] \\ I'(Y^*)\varphi_1(-1) - \beta\varphi_2(-1) - \delta\varphi_2(0) \end{pmatrix}$$

$$h(\varphi, \tau) = \tau \begin{pmatrix} \alpha[I(\varphi_1(0) + Y^*) - I'(Y^*)\varphi_1(0) - \beta K^* - \gamma Y^*] \\ I(\varphi_1(-1) + Y^*) - I'(Y^*)\varphi_1(0) - (\beta + \delta)K^* \end{pmatrix}$$

Let

$$L := L(\tau_0)\colon C\big([-1, 0], \mathbb{R}^2\big) \to \mathbb{R}^2.$$

Using the Riesz representation theorem (see [15]), we obtain

$$L\varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\varphi(\theta), \tag{14}$$

where

$$d\eta(\theta) = \tau_0 \begin{pmatrix} \alpha(I'(Y^*) - \gamma)\delta(\theta) & -\alpha\beta\delta(\theta) \\ -I'(Y^*)\delta(\theta + 1) & \beta\delta(\theta + 1) + \delta\delta(\theta) \end{pmatrix}, \tag{15}$$

$\delta(.)$ denotes the Dirac function.

Let $A(\tau)$ denotes the generator of semigroup generated by the linear part of (13) and $A = A(\tau_0)$.

Then,

$$A\varphi(\theta) = \begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\theta}(\theta) & \text{for } \theta \in [-1, 0), \\ L\varphi & \text{for } \theta = 0 \end{cases} \tag{16}$$

for $\varphi = (\varphi_1, \varphi_2) \in C$.

From Theorem 1, a Hopf bifurcation occurs at the critical value $\tau = \tau_0$. By the Taylor expansion of the time delay function $\tau(\varepsilon)$ near the critical value τ_0, we have

$$\tau(\varepsilon) = \tau_0 + \tau_2\varepsilon^2 + o(\varepsilon^2). \tag{17}$$

The sign of τ_2 determines either the bifurcation is supercritical (if $\tau_2 > 0$) and periodic orbits exist for $\tau > \tau_0$, or it is subcritical (if $\tau_2 < 0$) and periodic orbits exist for $\tau < \tau_0$. The term τ_2 may be calculated (see [1]) using the formula,

$$\tau_2 = \frac{\mathrm{Re}(c)}{\mathrm{Re}(qD_2M_0(i\zeta_0, \tau_0)p)}, \tag{18}$$

where M_0 is the characteristic matrix of the linear part of (13),

$$M_0(\lambda, \tau) = \begin{pmatrix} \lambda - \tau\alpha(I'(Y^*) - \gamma) & \tau\alpha\beta \\ -\tau I'(Y^*)\exp(-\lambda) & \lambda + \tau\beta\exp(-\lambda) + \tau\delta \end{pmatrix}, \tag{19}$$

$D_2M_0(i\omega_0, \tau_0)$ denotes the derivative of M_0 with respect to τ at $\tau = \tau_0$, the constant c is defined as follows

$$\begin{aligned} c = {} & \frac{1}{2}qD_1^3h(0, \tau_0)\big(P^2(\theta), \overline{P}(\theta)\big) \\ & + qD_1^2h(0, \tau_0)\big(e^0M_0^{-1}(0, \tau_0)D_1^2h(0, \tau_0)\big(P(\theta), \overline{P}(\theta)\big), P(\theta)\big) \\ & + \frac{1}{2}qD_1^2h(0, \tau_0)\big(e^{2i\omega_0}M_0^{-1}(2i\omega_0, \tau_0)D_1^2h(0, \tau_0)\big(P(\theta), P(\theta)\big), \overline{P}(\theta)\big), \end{aligned}$$

where $D_1^ih, i = 2, 3$, denotes the $i - th$ derivative of h with respect to φ, $P(\theta)$ denotes the eigenvector of A, $\overline{P}(\theta)$ denotes it conjugate eigenvector and p, q are defined later.

To study the direction of Hopf bifurcation, one needs to calculate the second and third derivatives of nonlinear part of (13) with respect to φ,

$$D_1^2h(\varphi, \tau)\psi\chi = \tau\begin{pmatrix} \alpha I''(\varphi_1(0) + Y^*)\psi_1(0)\chi_1(0) \\ I''(\varphi_1(-1) + Y^*)\psi_1(-1)\chi_1(-1) \end{pmatrix} \tag{20}$$

and

$$D_1^3h(\varphi, \tau)\psi\chi\upsilon = \tau\begin{pmatrix} \alpha I'''(\varphi_1(0) + Y^*)\psi_1(0)\chi_1(0)\upsilon_1(0) \\ I'''(\varphi_1(-1) + Y^*)\psi_1(-1)\chi_1(-1)\upsilon_1(-1) \end{pmatrix} \tag{21}$$

Then

$$D_1^2 h(0,\tau_0)\psi\chi = \Big[\tau_0 \alpha I''(Y^*)\psi_1(0)\chi_1(0)\begin{pmatrix}1\\0\end{pmatrix} + \tau_0 I''(Y^*)\psi_1(-1)\chi_1(-1)\begin{pmatrix}0\\1\end{pmatrix}\Big] \tag{22}$$

and

$$D_1^3 f_0(0,\tau_0)\psi\chi\upsilon = \Big[\tau_0 \alpha I'''(Y^*)\psi_1(0)\chi_1(0)\upsilon_1(0)\begin{pmatrix}1\\0\end{pmatrix} + \tau_0 I'''(Y^*)\psi_1(-1)\chi_1(-1)\upsilon_1(-1)\begin{pmatrix}0\\1\end{pmatrix}\Big], \tag{23}$$

$$\psi = (\psi_1,\psi_2), \quad \chi = (\chi_1,\chi_2), \quad \upsilon = (\upsilon_1,\upsilon_2) \in C\big([-1,0],\mathbb{R}^2\big).$$

As $i\omega_0$ is a solution of (8) at $\tau = \tau_0$, then $i\omega_0$ is an eigenvalue of A and there exist a corresponding eigenvector of the form $P(\theta) = pe^{i\omega_0\theta}$ where $p = (p_1, p_2) \in \mathbb{C}^2$, satisfy the equations:

$$Mp = 0$$

with

$$M = M_0(i\omega_0,\tau_0). \tag{24}$$

Then one may assume

$$p_1 = 1,$$

and calculate

$$p_2 = \frac{-i\omega_0 + \tau_0\alpha(I'(Y^*) - \gamma)}{\tau_0\alpha\beta}.$$

So, from (22) and (23), we have

$$D_1^2 h(0,\tau_0)\big(P(\theta),\overline{P}(\theta)\big) = \tau_0 I''(Y^*)\begin{pmatrix}\alpha\\1\end{pmatrix} \tag{25}$$

$$D_1^2 h(0,\tau_0)\big(P(\theta),P(\theta)\big) = \tau_0 I''(Y^*)\begin{pmatrix}\alpha\\ \exp(-2i\omega_0)\end{pmatrix} \tag{26}$$

and

$$D_1^3 h(0,\tau_0)\big(P^2(\theta),\overline{P}(\theta)\big) = \tau_0 I'''(Y^*)\begin{pmatrix}\alpha\\ \exp(-i\omega_0)\end{pmatrix}. \tag{27}$$

Now, consider A^*, a conjugate operator of A, $A^*\colon C([0,1],\mathbb{R}^2)\to\mathbb{R}^2$, defined by,

$$A^*\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi}{\mathrm{d}s}(s) & \text{for } s\in(0,1],\\ -\displaystyle\int_{-1}^{0}\psi(-s)\,\mathrm{d}\eta(s) & \text{for } s=0,\end{cases}\tag{28}$$

$\psi=(\psi_1,\psi_2)\in C([0,1],\mathbb{R}^2)$.

Let $Q(s)=qe^{i\omega_0 s}$ be the eigenvector for A^* associated to the eigenvalue $i\omega_0$, $q=(q_1,q_2)^T$. One needs to choose q such that the inner product (see [15]),

$$<Q,P>=1,$$

where

$$<Q,P>=Q(0)\overline{P}(0)-\int_{-1}^{0}\int_{0}^{\theta}Q(\xi-\theta)\,\mathrm{d}\eta(\theta)\overline{P}(\xi)\,\mathrm{d}\xi.$$

If we take $q_2=0$, then $q_1=1$ and from (27), we have

$$\frac{1}{2}qD_1^3h(0,\tau_0)\big(P^2(\theta),\overline{P}(\theta)\big)=\frac{\alpha\tau_0}{2}I'''(Y^*).\tag{29}$$

From the expression of M_0 in (19), we have

$$M_0^{-1}(0,\tau_0)=\frac{1}{\alpha\tau_0^2[(\beta+\delta)\gamma-\delta I'(Y^*)]}\begin{pmatrix}\tau_0(\beta+\delta) & -\alpha\beta\tau_0\\ \tau_0I'(Y^*) & -\alpha\tau_0(I'(Y^*)-\gamma)\end{pmatrix}\tag{30}$$

and

$$\begin{aligned}M_0^{-1}(2i\omega_0,\tau_0)&=\frac{1}{\det M_0(2i\omega_0,\tau_0)}\\ &\quad\times\begin{pmatrix}2i\omega_0+\tau_0\beta\exp(-2i\omega_0)+\delta\tau_0 & -\tau_0\alpha\beta\\ \tau_0I'(Y^*)\exp(-2i\omega_0) & 2i\omega_0-\tau_0\alpha(I'(Y^*)-\gamma)\end{pmatrix}.\end{aligned}\tag{31}$$

From (25), (26), (30), (31), we deduce,

$$\begin{aligned}&qD_1^2h(0,\tau_0)\big(e^0M_0^{-1}(0,\tau_0)D_1^2h(0,\tau_0)\big(P(\theta),\overline{P}(\theta)\big),P(\theta)\big)\\ &\quad=\frac{\tau_0\alpha\delta I''(Y^*)^2}{(\beta+\delta)\gamma-\delta I'(Y^*)}\end{aligned}\tag{32}$$

and

$$\begin{aligned}&\frac{1}{2}qD_1^2h(0,\tau_0)\big(e^{2i\omega_0}M_0^{-1}(2i\omega_0,\tau_0)D_1^2h(0,\tau_0)\big(P(\theta),P(\theta)\big),\overline{P}(\theta)\big)\\ &\quad=\frac{\tau_0^2\alpha^2I''(Y^*)^2}{2(B^2+C^2)}[(B\delta\tau_0+2C\omega_0)+i(2B\omega_0-C\delta\tau_0)],\end{aligned}\tag{33}$$

where

$$B = -4\omega_0^2 - \alpha\delta\big(I'(Y^*) - \gamma\big)\tau_0^2 + 2\beta\tau_0\omega_0 \sin(2\omega_0) + \alpha\beta\gamma\tau_0^2 \cos(2\omega_0),$$
$$C = 2\delta\tau_0\omega_0 - 2\alpha\big(I'(Y^*) - \gamma\big)\tau_0\omega_0 - \alpha\beta\gamma\tau_0^2 \sin(2\omega_0) + 2\beta\tau_0\omega_0 \cos(2\omega_0).$$

Then

$$\mathrm{Re}(c) = \frac{\alpha\tau_0}{2} I'''(Y^*) + \frac{\tau_0\alpha\delta I''(Y^*)^2}{(\beta+\delta)\gamma - \delta I'(Y^*)} + \frac{\tau_0^2\alpha^2 I''(Y^*)^2}{2(B^2+C^2)}[(B\delta\tau_0 + 2C\omega_0)]. \quad (34)$$

Now, from (19) we have

$$\mathrm{Re}\big(qD_2M_0(i\omega_0, \tau_0)p\big) = \mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)(\tau_0),$$

and from the proof of Theorem 1, we have

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)(\tau_0) > 0.$$

Consequently we deduce the following result:

Theorem 4. *Assume* (H1) *and* (H2)*. Then,*

(i) *the Hopf bifurcation occurs as τ crosses τ_0 to the right (supercritical Hopf bifurcation) if $\mathrm{Re}(c) > 0$ and to the left (subcritical Hopf bifurcation) if $\mathrm{Re}(c) < 0$; and*

(ii) *the bifurcating periodic solutions is stable if $\mathrm{Re}(c) > 0$ and unstable if $\mathrm{Re}(c) < 0$; where $\mathrm{Re}(c)$ is given by* (34)*.*

Note that, Theorem 4 provides an explicit algorithm for detecting the direction and stability of Hopf bifurcation.

4 Application

Consider the following Kaldor-type investment function:

$$I(Y) = \frac{\exp(Y)}{1 + \exp(Y)}.$$

Theorems 1 and 4 imply:

Proposition 2. *If*

$$\alpha = 3, \quad \beta = 0.2, \quad \delta = 0.1, \quad \gamma = 0.2,$$

then system (4) *have the following positive equilibrium*

$$E^* = (1.31346, 2.62699).$$

Furthermore, the critical delay corresponding to (4) *is $\tau_0 = 2.9929$ and $\mathrm{Re}(c) = 0.2133$.*

By the previous proposition, we have if $\tau < 2.9929$, then system (4) have a stable equilibrium point E^*. Fig. 1 shows that behavior of system (4) is stable for $\tau = 2$. If we increase the value of τ, then we find a stable periodic solution occurs at $\tau_0 = 2.9929$ and E^* becomes unstable for $\tau > 2.9929$. Fig.2 show that is E^* unstable for $\tau > 2.9929$.

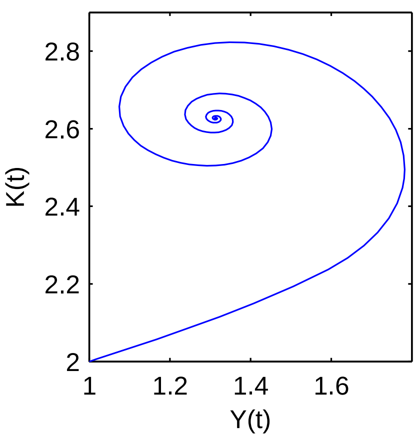

Fig. 1. The steady state E^* of (4) is stable when $\tau = 2$.

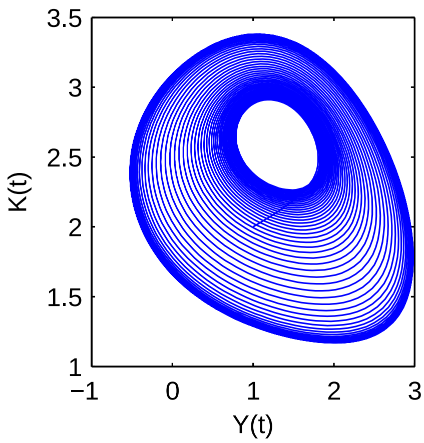

Fig. 2. The steady state E^* of (4) is unstable when $\tau = 3$.

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