On the Sojourn Time of the Brownian Process in a Multidimensional Sphere

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Abstract. We consider the Brownian motion process $B^m(s)$ in the m-space and the distribution

$$F^m(t,x,a) = P\big\{\sup_{0\le s\le t}\big|B^m(s)+x\big| < a\big\},\quad \text{where } a>0,\ x\in R^m,\ |x|<a.$$

There is a probability that a particle starting from the point x on the sphere S_r^m with the radius $r=|x|<a$ will not be absorbed by the sphere S_a^m with a radius a before the epoch t.

Keywords: Brownian motion process, distribution, random variable.

1 Introduction

The most important stochastic process is a Brownian or Wiener process. It was first discussed by Louis Bachelier (1900), who was interested in modelling fluctuations of prices in financial markets, and by Albert Einstein (1905), who gave a mathematical model for the irregular motion of colloidal particles, first observed by the Scottish botanist, Robert Brown, in 1827.

Let there be an m-dimensional Euclidean space and $e_1, e_2, \dots, e_m$ be a fixed basis in R^m, where $x_1, x_2, \dots, x_m$, are coordinates of the vector from R^m in the basis. A scalar product of the elements x and $y \in R^m$ is the number $(x\cdot y) = \sum_{i=1}^m x_i\cdot y_i$, and the norm of the element $x\in R^m$ is a (non-negative) number $|x| = \sqrt{(x\cdot x)}$. Let S_a^m be an m-dimensional sphere with the center at the beginning of coordinates and the radius a.

Distribution of the random variable $B^m(s)$ is defined by density of the distribution

$$p(s,x) = (2\pi s)^{\frac{m}{2}}\exp\left(-\frac{|x|^2}{2s}\right),$$

so for every Borel set $A \in R^m$ we get

$$P\{B^m(s) \in A\} = (2\pi s)^{\frac{m}{2}} \int\limits_A \exp\left(-\frac{|x|^2}{2s}\right) \mathrm{d}x. \tag{1}$$

We have examined the distribution

$$F^m(t,x,a) = P\{\sup_{0\le s\le t} \left|B^m(s)+x\right| < a\}, \tag{2}$$

where $a > 0$, $x \in R^m$ and $|x| < a$.

There is a probability that a particle starting from the point x on the sphere S_r^m with the radius $r = |x| < a$ will not be absorbed by the sphere S_a^m with a radius a before the epoch t.

In a one-dimensional case, the probability distribution function

$$F^1(t,0,a) = P\{\sup_{0\le s\le t} |B(s)| < a\}$$

has a complicated expression and different authors obtained several forms of this function in [1–10]. The author [11] has proved that all the expressions are equivalent.

P. Levy [7] examined one-dimensional Brownian motion starting at the point x $(-a_1 < x < a_2)$, impeded by two absorbing barriers at $-a_1 < 0 < a_2$, and obtained the general formula

$$P\{-a_1 < B(s)+x < a_2,\ 0 \le s \le t\}$$
$$= \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \int\limits_{-a_1}^{a_2} \left[e^{-\frac{(x-x_k'-y)^2}{2t}} - e^{-\frac{(x-x_k''+y)^2}{2t}}\right] \mathrm{d}y, \tag{3}$$

where $x_k' = 2dk,\ x_k'' = 2a_2 - 2dk,\ d = a_1 + a_2$ and $k = \ldots,-1,0,1,\ldots$.

If $a_1 = a_2 = a,\ d = 2a$, then it follows that

$$F^1(t,x,a) = P\{\sup_{0\le s\le t} |B(s)+x| < a\}$$
$$= \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \int\limits_{-a}^{a} \left(e^{-\frac{(x-4ka-y)^2}{2t}} - e^{-\frac{(x+4ka-2a+y)^2}{2t}}\right) \mathrm{d}y. \tag{4}$$

W. Feller [4] considered one-dimensional Brownian motion starting at the point $0 < x < a$, impeded by two absorbing barriers at 0 and $a > 0$ and has obtained two very different representations for the same distribution function $\lambda_a(t,x)$ (see [4, Chapter X]):

$$\lambda_a(t,x) = P\{0 < B(s)+x < a,\ 0 \le s \le t\}$$
$$= \sum_{k=-\infty}^{\infty} \left\{\Phi\left(\frac{2ka+a-x}{\sqrt{t}}\right) - \Phi\left(\frac{2ka-x}{\sqrt{t}}\right) - \Phi\left(\frac{2ka+a+x}{\sqrt{t}}\right) + \Phi\left(\frac{2ka+x}{\sqrt{t}}\right)\right\} \tag{5}$$

and

$$\lambda_a(t,x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2 t}{2a^2}\right)\sin\left(-\frac{(2k+1)\pi x}{a}\right), \tag{6}$$

where $\Phi(x)$ is standard normal distribution function.

Fortunately, the series in (5) converges reasonably only when t is small, whereas (6) is applicable to large t.

In [11], the author derived an other different representation for the same distribution function (4) $F^1(t,x,a)$

$$F^1(t,x,a) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2 t}{8a^2}\right)\cos\left(\frac{(2k+1)\pi x}{2a}\right), \tag{7}$$

where $-a < x < a$. This formula gives a probability that the Brownian motion leaving the point x, will not be absorbed till the moment t.

The authors in [12, 13] examined the distribution $F^m(t,0,a)$. They considered the Brownian motion $B^m(t)$ starting from the origin. Definition of such probabilities is one of the most important problems in the theory of random processes. Following the results of A.V. Skorokhod [9], the probability $F^m(t,x,a)$, we are interested in, satisfies a differential equation of diffusion. In the case of an m-dimensional Brownian motion, we impose a condition of a circular symmetry which leads to the equation

$$2\frac{\partial F^m(t,x,a)}{\partial t} = \frac{\partial^2 F^m(t,x,a)}{\partial x_1^2} + \ldots + \frac{\partial^2 F^m(t,x,a)}{\partial x_m^2} \tag{8}$$

under the boundary condition $F^m(t,x,a)|_{|x|=a} = 0$ and the initial condition $F^m(t,x,a)|_{t=0} = 1$.

Passing to spherical coordinates, we shall transform equation (8) into the following shape:

$$2\frac{\partial v^m(t,r,a)}{\partial t} = \frac{\partial^2 v^m(t,r,a)}{\partial r^2} + \frac{m-1}{r}\,\frac{\partial v^m(t,r,a)}{\partial r} \tag{9}$$

under the boundary condition

$$v^m(t,r,a)|_{r=a} = 0 \tag{10}$$

and the initial condition

$$v^m(t,r,a)|_{t=0} = 1. \tag{11}$$

This paper is meant for studying the properties of distribution functions $F^m(t,x,a) = v^m(t,r,a)$, where $a > r = |x| > 0$.

2 Statement of the basic results

We consider the Brownian motion process $B^m(t)$ in an m-space starting from the point x on the sphere S_r^m with the radius $r = |x| < a$. We shall prove the following theorem.

Theorem 1. *Let $B^m(s),\ 0 \le s \le t$, be an m-dimensional Brownian motion, starting from the point x on the sphere S_r^m with the radius $r = |x| < a$. Then*

$$v^m(t,r,a) = \sum_{n=1}^{\infty} \frac{2a^\nu J_\nu(\mu_n r/a)}{r^\nu \mu_n J_{\nu+1}(\mu_n)} \exp\left(-\frac{\mu_n^2 t}{2a^2}\right), \tag{12}$$

where $\mu_n, n = 1, 2, \ldots$, are the positive roots of the Bessel function $J_\nu(z)$ with $\nu = m/2 - 1$.

Proof. We find the solution to this differential diffusion equation (9) by the standard Fourier method. We try to find a solution of the form

$$v^m(t,r,a) = T(t)R(r), \tag{13}$$

where $T(t)$ is a function only of the variable t and $R(r)$ is a function only of the variable r. Substituting the proposed form of solution (13) into equation (9) and dividing both sides of the equality by $T(t)R(r)$, we obtain

$$2\,\frac{T'(t)}{T(t)} = \frac{R''(r) + \frac{m-1}{r}R'(r)}{R(r)} = -\lambda^2. \tag{14}$$

Then, from equality (14) we obtain two ordinary equations

$$2T'(t) + \lambda^2 T(t) = 0, \tag{15}$$

$$R''(r) + \frac{m-1}{r}R'(r) + \lambda^2 R(r) = 0. \tag{16}$$

Boundary condition (10) yields $R(a) = 0$. Thus, in view of the found function $R(r)$, we derive the simplest problem on eigenvalues: find the values of the parameter λ at which there exist nontrivial solutions of equation (16) and the boundary condition $R(a) = 0$.

Set

$$R(r) = \frac{u(r)}{r^\nu} \tag{17}$$

in equation (16). Then $u(r)$ satisfies the Bessel equation

$$r^2 u''(r) + ru'(r) + \left(\lambda^2 r^2 - \nu^2\right)u(r) = 0, \quad \text{where } \nu = \frac{m}{2} - 1. \tag{18}$$

The general solution of equation (18) is of the shape:

$$u(r) = c_1 J_\nu(\lambda r) + c_2 Y_\nu(\lambda r), \tag{19}$$

where $J_\nu(\lambda r)$ is the Bessel function of the first kind of order ν and $Y_\nu(\lambda r)$ is the Bessel function of the second kind. It follows from (17) and (19) that

$$R(r) = \frac{c_1 J_\nu(\lambda r) + c_2 Y_\nu(\lambda r)}{r^\nu}. \tag{20}$$

Since $Y_\nu(\lambda r) \to \infty$ as $r \to 0$, most probably $c_2 = 0$. Under the boundary condition (8) we get the following equation

$$J_\nu(\lambda a) = 0, \tag{21}$$

that has infinitely many positive zeros $\mu_1, \mu_2, \mu_3, \ldots$ (see [14]).

Hence we derive that λ_k is defined by the formulas

$$\lambda_k = \frac{\mu_k}{a},$$

and

$$R_k(r) = \frac{J_\nu(\frac{\mu_k r}{a})}{r^\nu}, \;\; T_n(t) = c_n \exp\left(-\frac{\mu_n^2 t}{2a^2}\right), \quad k = 1, 2, 3, \ldots, \infty. \tag{22}$$

Now, in view of equations (13), (15) and (22), we find that the functions

$$v^m(t, r, a) = c_n \exp\left(-\frac{\mu_n^2 t}{2a^2}\right) \frac{J_\nu(\frac{\mu_n r}{a})}{r^\nu} \tag{23}$$

satisfy equation (9) and the boundary condition (10) for any c_n.

Let us compose a series

$$v^m(t, r, a) = \sum_{n=1}^{\infty} c_n \frac{J_\nu(\frac{\mu_n r}{a})}{r^\nu} \exp\left(-\frac{\mu_n^2 t}{2a^2}\right). \tag{24}$$

To satisfy the initial condition (11), we need to fulfil the equality

$$\sum_{n=1}^{\infty} c_n J_\nu\left(\frac{\mu_n r}{a}\right) = r^\nu. \tag{25}$$

The written series represents an expansion of the function r^ν in Bessel functions in the interval $(0, a)$. The coefficients of expansions are defined by the formula

$$c_n = \frac{2}{a^2 J_{\nu+1}^2(\mu_n)} \int_0^a r^{\nu+1} J_\nu\left(\frac{\mu_n r}{a}\right) \mathrm{d}r. \tag{26}$$

Let $y = \frac{\mu_n r}{a}$, then

$$c_n = \frac{2}{a^2 J_{\nu+1}^2(\mu_n)} \left(\frac{a}{\mu_n}\right)^{\nu+2} \int_0^{\mu_n} y^{\nu+1} J_\nu(y) \, \mathrm{d}y. \tag{27}$$

Making use of the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}y} y^{\nu+1} J_{\nu+1}(y) = y^{\nu+1} J_\nu(y),$$

it is easy to find that

$$\int_0^{\mu_n} y^{\nu+1} J_\nu(y)\,\mathrm{d}y = \int_0^{\mu_n} \mathrm{d}\big(y^{\nu+1} J_{\nu+1}(y)\big) = \mu_n^{\nu+1} J_{\nu+1}(\mu_n). \tag{28}$$

It follows from (27) and (28) that

$$c_n = \frac{2a^\nu}{\mu_n J_{\nu+1}(\mu_n)}. \tag{29}$$

Formulae (24) and (29) complete the proof of Theorem 1. □

Let us mention some corollaries.

Corollary 1. *Let $B^m(s)$ be an m-dimensional Brownian motion, starting from the origin. Then, passing to the limit from Theorem 1 as $r \to 0$, we obtain*

$$P\big\{\sup_{0 \le s \le t} |B^m(s)| < a\big\} = \sum_{n=1}^{\infty} \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{\mu_n^{\nu-1}}{J_{\nu+1}(\mu_n)} \exp\left(-\frac{\mu_n^2 t}{2a^2}\right), \tag{30}$$

where $a > 0$.

Proof. We obtain the limit from formula (4.14.4) in [15]

$$\lim_{r \to 0} \frac{J_\nu(\mu_n r/a)}{(\mu_n r/a)^\nu} = \frac{1}{2^\nu \Gamma(\nu+1)}$$

and

$$\lim_{r \to 0} \frac{2a^\nu J_\nu(\mu_n r/a)}{r^\nu \mu_n J_{\nu+1}(\mu_n)} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{\mu_n^{\nu-1}}{J_{\nu+1}(\mu_n)}.$$

Hence we derive the result [12]. The proof is complete. □

We can easily find positive roots of the Bessel functions $J_\nu(z)$ in formula (12) only for one-dimensional and three-dimensional cases. Therefore, only for that cases we present the following corollaries:

Corollary 2. *Let $B(s)$ be a one-dimensional Brownian motion, starting from the point $x \in [-a, a]$. Then*

$$\begin{aligned} F^1(t, x, a) &= P\big\{\sup_{0 \le s \le t} |B(s) + x| < a\big\} \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2 \pi^2 t}{8a^2}\right) \cos\left(\frac{(2k+1)\pi x}{2a}\right), \end{aligned} \tag{31}$$

where $-a < x < a$.

Proof. It is easy to see, that if $m = 1$, then $\nu = -\frac{1}{2}$, $J_\nu(x) = J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$, $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$. The positive roots of the Bessel function $J_{-\frac{1}{2}}(x)$ are $\mu_n = \frac{\pi}{2}(1 + 2n)$, $n = 0, 1, 2, \ldots$.

Thus, we have

$$\frac{2a^\nu J_\nu(\mu_n r/a)}{r^\nu \mu_n J_{\nu+1}(\mu_n)} = \frac{2}{\mu_n} \frac{\cos(\frac{\mu_n r}{a})}{\sin(\mu_n)} = \frac{4}{\pi(2n+1)} \cos\left(\frac{(2k+1)\pi x}{2a}\right)(-1)^n.$$

Applying this formula and (12), we get the proof of Corollary 2. The proof is complete. □

This formula gives a probability that the one-dimensional Brownian motion leaving the point x, will not be absorbed till the moment t. Hence we derive the result [11].

Corollary 3. *Let $B^3(s)$, $0 \le s \le t$, be a three-dimensional Brownian motion, starting from the point x on the sphere S_r^3 with the radius $r = |x| < a$. Then*

$$v^3(t, r, a) = -2 \sum_{n=1}^{\infty} (-1)^n \frac{a}{\pi r n} \sin\left(\frac{\pi r n}{a}\right) \exp\left(-\frac{n^2 \pi^2 t}{2a^2}\right). \tag{32}$$

Proof. If $m = 3$, then $\nu = \frac{m}{2} - 1 = \frac{1}{2}$ and $J_\nu(x) = J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$, $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}(\frac{\sin(x)}{x} - \cos(x))$. The positive roots of the Bessel function $J_{\frac{1}{2}}(x)$ are $\mu_n = \pi n$, $n = 1, 2, \ldots$.

Consequently

$$\frac{2a^\nu J_\nu(\mu_n r/a)}{r^\nu \mu_n J_{\nu+1}(\mu_n)} = -\frac{2a}{\pi r n} \sin\left(\frac{\pi r n}{a}\right)(-1)^n.$$

The proof is complete. □

Corollary 4. *Let $B^3(s)$ be a three-dimensional Brownian movement, starting from the beginning of coordinates, then passing to the limit as $r \to 0$, we obtain.*

$$v^3(t, 0, a) = -2 \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{n^2 \pi^2 t}{2a^2}\right). \tag{33}$$

Proof. It is obvious, that the limit:

$$\lim_{r \to 0} \frac{a}{\pi r n} \sin\left(\frac{\pi r n}{a}\right) = 1$$

It proves (33). The proof is complete. □

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