Nonlinear Analysis: Modelling and Control, Vol. 24, No. 5, 755–774
https://doi.org/10.15388/NA.2019.5.5

ISSN: 1392-5113
eISSN: 2335-8963

Spectrum curves for a discrete Sturm–Liouville problem with one integral boundary condition*

Kristina Bingelė[a], **Agnė Bankauskienė**[b], **Artūras Štikonas**[a]

[a]Institute of Applied Mathematics, Vilnius University,
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
kristina.bingele@mif.vu.lt; arturas.stikonas@mif.vu.lt

[b]Bioinformatics and Biostatistics Centre, Vilnius University
Santariškių str. 2, LT-08661 Vilnius, Lithuania
agne.skucaite@mii.vu.lt

Received: May 28, 2019 / **Revised:** July 3, 2019 / **Published online:** September 27, 2019

Abstract. This paper presents new results on the spectrum on complex plane for discrete Sturm–Liouville problem with one integral type nonlocal boundary condition depending on three parameters: γ, ξ_1 and ξ_2. The integral condition is approximated by the trapezoidal rule. The dependence on parameter γ is investigated by using characteristic function method and analysing spectrum curves which gives qualitative view of the spectrum for fixed $\xi_1 = m_1/n$ and $\xi_2 = m_2/n$, where n is discretisation parameter. Some properties of the spectrum curves are formulated and illustrated in figures for various ξ_1 and ξ_2.

Keywords: Sturm–Liouville problem, finite difference sheme, nonlocal boundary condition, complex eigenvalues, spectrum curves.

1 Introduction

Problems with integral Nonlocal Boundary Conditions (NBC) arise in various fields of mathematical physics, biology, biotechnology [5, 14]. Cannon investigated Boundary Value Problem (BVP) with integral type NBC [1]. A parabolic problem with integral boundary condition was investigated by Kamynin in [13]. Differential equations with various types of NBCs were investigated by many scientists. Problems with two-points and multi-points NBCs were investigated by Ionkin and Moiseev in [8]. Čiegis investigated elliptic and parabolic problems with integral and Bitsadze–Samarskii type NBCs and Finite Difference Schemes (FDS) for them in [2,3]. Sapagovas with co-authors began to investigate eigenvalues for problems with NBCs [4, 12, 17–19]. They showed that there exist eigenvalues which do not depend on parameter γ in boundary conditions and complex eigenvalues may exist. Sapagovas with co-authors investigated the spectrum

*The research was partially supported by the Research Council of Lithuania (grant No. MIP-047/2014).

of discrete SLP, too. Recent results in the field of problems with NBC can be found in survey [24] and [23].

Sturm–Liouville Problem (SLP) is very important for investigation of existence and uniqueness of solutions for classical stationary problems. Such problems are complicated, not self-adjoint and spectrum for such problems may be not positive (or real). Ionkin and Valikova investigated SLP with NBCs [9]

$$u(0) = u(1), \qquad u_x(1) = 0.$$

They prove that all nonzero eigenvalues are not simple, i.e., for each such eigenvalue, there exist eigenfunction and generalized eigenfunction. Gulin et al. investigated a spectrum for one-dimensional SLPs and proved stability for FDS

$$u(0,t) = \alpha u(1,t), \qquad u_x(1,t) = \gamma u_x(0,t).$$

Gulin and Mokin have similar results for NBCs [7]

$$u(0,t) = 0, \qquad u_x(1,t) = u_x(0,t) + \alpha u(1,t).$$

We investigate spectrum of discrete SLP with NBC using the Characteristic Function (CF) method [25]. In [25], this method was used for SLP with NBC

$$u(0) = 0, \qquad u(1) = \gamma u(\xi), \quad \xi \in (0,1),$$

The CF method is used for investigation of spectrum and its qualitative dependence on parameter γ (while parameter ξ is fixed). New results on a spectrum in a complex plane for the second-order stationary differential equation with one Bitsadze–Samarskii type NBC were proved. A definition of CF was introduced for the SLP with general NBCs, Constant Eigenvalues (CE) points, Poles and Critical Points (CP) of Complex-Real CF were investigated.

The differential SLP

$$-u'' = \lambda u, \quad t \in (0,1),\ \lambda \in \mathbb{C}_\lambda := \mathbb{C}, \tag{1}$$

with one classical boundary condition and another integral type NBC

$$u(0) = 0, \qquad u(1) = \gamma \int\limits_{\xi_1}^{\xi_2} u(t)\,\mathrm{d}t, \tag{2}$$

where NBC's parameter $\gamma \in \mathbb{R}$ and $\boldsymbol{\xi} \in S_{\boldsymbol{\xi}} := \{(\xi_1,\xi_2) \in [0,1]^2\colon\ 0 \leqslant \xi_1 < \xi_2 \leqslant 1\}$, was investigated in [20, 21]. The behaviour of Spectrum Curves and CPs trajectories in Phase Space $S_{\boldsymbol{\xi}}$ was analysed. We note interesting previous results, where special cases were presented: $\boldsymbol{\xi} = (0,1)$ and $\boldsymbol{\xi} = (1/4, 3/4)$ in [4], $\boldsymbol{\xi} = (0,\xi)$ and $\boldsymbol{\xi} = (\xi, 1)$, $\xi \in [0,1]$ in [16], $\boldsymbol{\xi} = (\xi, 1-\xi)$, $\xi \in [0, 1/2])$ in [22]. In the case of full integral ($\boldsymbol{\xi} = (0,1)$) in the NBC, there exist only real eigenvalues [4, 15]. The cases with complex eigenvalues are less investigated. Some new results about complex eigenvalues are published in [20, 25].

Discrete SLP

$$-\frac{U_{i-1}-2U_i+U_{i+1}}{h^2}=\lambda U_i,\quad i=1,\dots,n-1,$$
$$\frac{U_0}{2}+\frac{U_n}{2}+\sum_{i=1}^{n-1}U_i=0,\qquad \frac{U_n}{2}+\sum_{i=1}^{n-1}U_i ih=0$$

was analysed by Jachimavičienė, Jesevičiūtė and Sapagovas [11]. Jachimavičienė gathered some new results [10] about the spectrum for NBCs

$$U_0=\gamma_0 h\left(\frac{U_0}{2}+\frac{U_n}{2}+\sum_{i=1}^{n-1}U_i\right),\qquad U_n=\gamma_1 h\left(\frac{U_0}{2}+\frac{U_n}{2}+\sum_{i=1}^{n-1}U_i\right).$$

In this paper, we approximate the differential SLP (1)–(2) by FDS (trapezoidal formula is used for integral approximation) and investigate the Spectrum Curves for discrete SLP. In Section 2, we describe the discrete problem and prove some auxiliary formulae. In Section 3, we analyse CE points, Poles and Zeros of CF. The structure of the spectrum (Spectrum Curves and at the points $q=0$, $q=n$, $q=\infty$) of discrete SLP is investigated in Section 4. Section 5 contains some brief conclusions and comments.

2 Discrete Sturm–Liouville problem

We introduce a uniform grid and we use the notation $\overline{\omega}^h=\{t_j=jh,\ j=0,\dots,n;\ nh=1\}$ for $2\leqslant n\in\mathbb{N}:=\{1,2,3,\dots\}$. $\mathbb{N}_\mathrm{o}$ and $\mathbb{N}_\mathrm{e}$ are sets for odd and even numbers and $\mathbb{N}^h:=(0,n)\cap\mathbb{N}$, $\overline{\mathbb{N}}^h:=\mathbb{N}^h\cup\{0,n\}$. Also, we make an assumption that ξ_1 and ξ_2 are located on the grid, i.e., $\xi_1=m_1h=m_1/n$, $\xi_2=m_2h=m_2/n$, $\boldsymbol{m}\in S^h_{\boldsymbol{\xi}}:=\{(m_1,m_2)\colon\ 0\leqslant m_1<m_2\leqslant n,\ m_1,m_2\in\overline{\mathbb{N}}^h\}$. So, $\boldsymbol{\xi}=\boldsymbol{m}/n=(m_1/n,\ m_2/n)$, $\xi=\xi_1/\xi_2=m_1/m_2$, $\xi_+=\xi_1+\xi_2=m_+/n$, $\xi_-=\xi_2-\xi_1=m_-/n$, where $m_+:=m_1+m_2$, $m_-:=m_2-m_1$. Let us denote $\gcd(n_1,n_2)$ the greatest common divisor of $n_1\in\mathbb{N}$ and $n_2\in\mathbb{N}$.

Let us introduce a space H of real grid function on $\overline{\omega}^h$. We will use the notation

$$[U,V]:=\sum_{j=0}^{n}U_jV_j \tag{3}$$

to approximate the integral in NBC

$$\int_a^b u\,\mathrm{d}t\approx u_{m_1}\frac{h}{2}+\sum_{i=m_1+1}^{m_2-1}u_ih+u_{m_2}\frac{h}{2}=[\chi_{[a,b]},u],$$

where

$$\chi_{[a,b],j}=\chi_{[a,b]}(t_j)=\begin{cases}0 & \text{for } t_j<a \text{ or } t_j>b,\\ \frac{h}{2} & \text{for } t_j=a \text{ or } t_j=b,\\ h & \text{for } a<t_j<b,\end{cases}\qquad t_j\in\overline{\omega}^h,$$

$a, b \in \overline{\omega}^h$ and $a < b$, i.e., $a = t_\alpha = \alpha h$, $b = t_\beta = \beta h$, $\alpha, \beta \in \overline{\mathbb{N}}^h$. For real functions, notation (3) corresponds to the inner product in the space H. We will use this definition for complex function, too.

We approximate differential SLP (1)–(2) by the Finite-Difference Scheme (FDS) and get a discrete Sturm–Liouville Problem (dSLP)

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j \in \mathbb{N}^h, \tag{4}$$

$$U_0 = 0, \qquad U_n = \gamma[\chi_{[\xi_1,\xi_2]}, U]. \tag{5}$$

Lemma 1. *Let $t \in \overline{\omega}^h$, $z \in \mathbb{C}$, then the following equalities hold*:

$$\left[\chi_{[a,b]}(t), \mathrm{e}^{\imath zt}\right] = h\mathrm{e}^{\imath z(a+b)/2} \sin\frac{z(b-a)}{2} \tan^{-1}\frac{zh}{2}, \tag{6}$$

$$\left[\chi_{[a,b]}(t), \cos(zt)\right] = h\cos\frac{z(b+a)}{2} \sin\frac{z(b-a)}{2} \tan^{-1}\frac{zh}{2}, \tag{7}$$

$$\left[\chi_{[a,b]}(t), \sin(zt)\right] = h\sin\frac{z(b+a)}{2} \sin\frac{z(b-a)}{2} \tan^{-1}\frac{zh}{2}. \tag{8}$$

Proof. We can consider the grid function $Y = Y_j = y^j$ for $y \in \mathbb{C}$, i.e., $Y_0 = 1$, $Y_1 = y, \dots, Y_n = y^n$. If $y \neq 1$, then we have

$$\begin{aligned}
[\chi_{[a,b]}, Y] &= h\left(\frac{y^\alpha + y^\beta}{2} + \sum_{j=\alpha+1}^{\beta-1} y^j\right) \\
&= h\left(\frac{y^\alpha + y^\beta}{2} + \sum_{j=0}^{\beta-1} y^j - \sum_{j=0}^{\alpha} y^j\right) = h\,\frac{(y^\beta - y^\alpha)(y+1)}{2(y-1)} \\
&= \imath h\,\frac{y^{\frac{\beta+\alpha}{2}}(y^{(\beta-\alpha)/2} - y^{-(\beta-\alpha)/2})}{2\imath} \cdot \frac{y^{1/2} + y^{-1/2}}{y^{1/2} - y^{-1/2}}.
\end{aligned}$$

In the case $y = \mathrm{e}^{\imath zh}$, $z \neq 0$, we have

$$\left[\chi_{[a,b]}(t), \mathrm{e}^{\imath zt}\right] = h\mathrm{e}^{(\imath z(a+b))/2} \sin\frac{z(b-a)}{2} \tan^{-1}\frac{zh}{2}.$$

Using the formula above, we get (7)–(8). If $z = 0$, then the trapezoidal formula is exact. So, $[\chi_{[a,b]}, 1] = b - a$. □

Remark 1. In the case $z = 0$, we understand equalities (6)–(7) as $[\chi_{[a,b]}, 1] = b - a$ and equality (8) as $[\chi_{[a,b]}, 0] = 0$. The trapezoidal formula is exact for linear functions. So,

$$\left[\chi_{[a,b]}(t), t\right] = \frac{b^2 - a^2}{2}.$$

Lemma 2. *Let $t \in \overline{\omega}^h$, then the following equalities hold*:

$$\left[\chi_{[a,b]}(t_j), (-1)^j\right] = 0, \qquad \left[\chi_{[a,b]}(t_j), (-1)^j t_j\right] = \frac{h^2((-1)^{b/h} - (-1)^{a/h})}{4}.$$

Proof. The first equality follows from (7) if we take $z = n\pi$. The simple equalities

$$\left[\chi_{[b-h,b]}(t_j), (-1)^j t_j\right] = \frac{(-1)^\beta h^2}{2}, \quad 0 < \beta \leqslant n,$$
$$\left[\chi_{[b-h,b+h]}(t_j), (-1)^j t_j\right] = 0, \qquad 0 < \beta < n,$$

are valid, where $b = \beta h$. If $\beta - \alpha \in \mathbb{N}_{\mathrm{e}}$, then $[\chi_{[a,b]}(t_j), (-1)^j t_j] = 0$, too. If $\beta - \alpha \in \mathbb{N}_{\mathrm{o}}$, then $[\chi_{[a,b]}(t_j), (-1)^j t_j] = (-1)^\beta h^2/2$. We rewrite both cases as

$$\left[\chi_{[a,b]}(t_j), (-1)^j t_j\right] = \frac{((-1)^\beta - (-1)^\alpha)h^2}{4}. \qquad \square$$

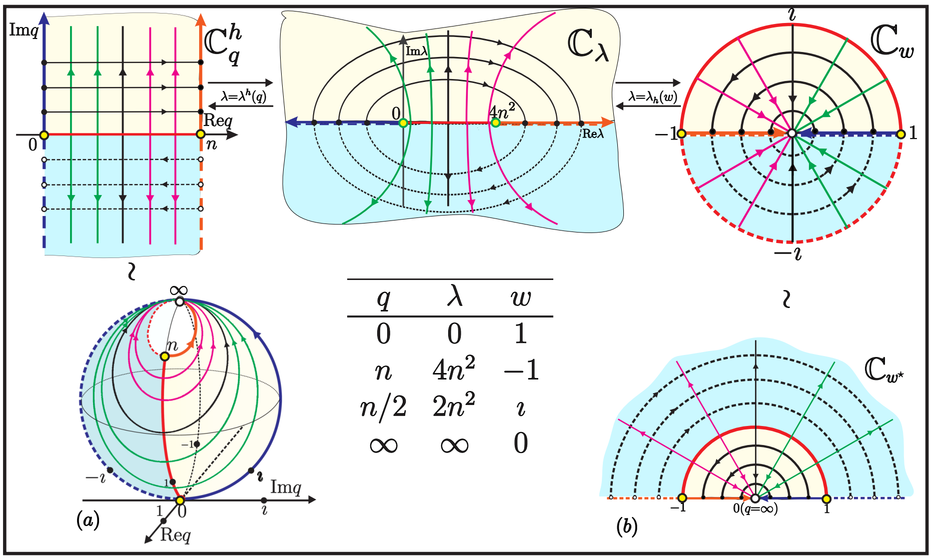
The function λ^h: $\mathbb{C} \to \mathbb{C}$,

$$\lambda^h(z) := \frac{4}{h^2}\sin^2\frac{\pi z h}{2} = \frac{2}{h^2}\bigl(1 - \cos(\pi z h)\bigr)$$

is holomorphic (entire) function. Inverse function is multivalued. Function $\lambda^h(z)$ has an infinite set of Branch Points (BP) of the second order at $\lambda = 0$ and $\lambda = 4n^2$ and a logarithmic BP at $\lambda = \infty$. Points $q = 0$, $q = n$ are Ramification Points (RP) of this function (see Fig. 1). We make inverse function single-valued if we make branch cuts along the intervals $(-\infty, 0)$ and $(4n^2, +\infty)$. Now we can consider a bijection

$$\lambda = \lambda^h(q) := \frac{4}{h^2}\sin^2\frac{\pi q h}{2} \tag{9}$$

between $\mathbb{C}_\lambda := \mathbb{C}$ and $\mathbb{C}_q^h$ (see Fig. 1), where $\mathbb{C}_q^h := \mathbb{R}_y^- \cup \{0\} \cup \mathbb{R}_x^h \cup \{n\} \cup \mathbb{R}_q^{h+} \cup \mathbb{C}_y^{h+} \cup \mathbb{C}_q^{h-}$, $\mathbb{R}_x^h := \{q = x\colon 0 < x < n\}$, $\mathbb{R}_y^- := \{q = \imath y\colon y > 0\}$, $\mathbb{R}_y^{h+} := \{q = n + \imath y\colon y > 0\}$, $\mathbb{C}_q^{h+} := \{q = x + \imath y\colon 0 < x < n,\ y > 0\}$, $\mathbb{C}_q^{h-} := \{q = x + \imath y\colon 0 < x < n,\ y < 0\}$. We use the notation $\overline{\mathbb{C}}_q^h = \mathbb{C}_q^h \cup \{\infty\}$ for domain on *Riemann sphere* (see Fig. 1). Then for any eigenvalue $\lambda \in \mathbb{C}_\lambda$, there exists the Eigenvalue Point (EP) $q \in \mathbb{C}_q^h$. Let us denote $\mathring{\mathbb{C}}_q^h := \mathbb{C}_q^h \setminus \{0, n\}$ the relative complement of RP points in $\mathbb{C}_q^h$ and $\mathring{\mathbb{R}}_q^h := \mathbb{R}_y^- \cup \mathbb{R}_x^h \cup \mathbb{R}_y^{h+}$. It follows that $\lambda < 0$ for $q \in \mathbb{R}_y^{h-}$, $0 < \lambda < 4/h^2$ for $q \in \mathbb{R}_x^h$, $\lambda > 4/h^2$ for $q \in \mathbb{R}_y^{h+}$. So, if $q \in \mathbb{R}_q^h := \mathring{\mathbb{R}}_q^h \cup \{0, n\}$, then corresponding eigenvalue is real. If EP is RP, then we call this point Branch Eigenvalue Point (BEP). For differential problem (1)–(2), eigenvalues are defined by the formula $\lambda = (\pi q)^2$, $q \in \mathbb{C}_q := \mathbb{R}_y^- \cup \{0\} \cup \{q = x + \imath y\colon x > 0,\ y \in \mathbb{R}\}$ [25]. We also use bijection $\lambda = \lambda_h(w) := (2/h^2)(1 - (w + w^{-1})/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w := \{w \in \mathbb{C}\colon |w| \leqslant 1,\ w \neq 0\}$ (see Fig. 1). The domain $\mathbb{C}_w$ will be used for investigation of eigenvalues in the neighborhood of $\lambda = \infty$ ($w = 0$). This bijection maps $\lambda < 0$ to the interval $w \in (0, 1)$, $0 < \lambda < 4/h^2$ to the upper unit semicircle, $\lambda > 4/h^2$ to



q	λ	w
0	0	1
n	$4n^2$	-1
$n/2$	$2n^2$	$\imath$
∞	∞	0

Figure 1. Bijective mappings: $\lambda = (4/h^2)\sin^2(\pi qh/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q^h$; $\lambda = (2/h^2)(1-(w+w^{-1})/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w$, ○ – Branch Point, ○ – Ramification Point; (a) $\overline{\mathbb{C}}_q^h$ on Riemann sphere; (b) domain $\mathbb{C}_{w^\star}$ on the upper half-plane.

the interval $(-1, 0)$, respectively. Complex λ points correspond to the points w, $\operatorname{Im} w \neq 0$, inside the unit circle (see Fig. 1). The points $w = \pm 1$ are RPs in $\mathbb{C}_w$ of the function $\lambda_h(w)$ and correspond to two BP of the second order at $\lambda = 0$ and $\lambda = 4n^2$. The function $w = \mathrm{e}^{\imath\pi hq}$ maps $\mathbb{C}_q^{h+}$ to the upper unit semidisk in $\mathbb{C}_w$ or $\mathbb{C}_{w^\star}$ and $\mathbb{C}_q^{h-}$ to outer part of the unit semidisk in $\mathbb{C}_{w^\star}$. The corresponding points in the different domains are shown in the table (see Fig. 1). Using formula (9), equation (4) can be rewritten in form

$$U_{j+1} - 2\cos(\pi qh)U_j + U_{j-1} = 0, \quad j \in \mathbb{N}^h, \tag{10}$$

where $q = x + \imath y \in \mathbb{C}_q^h$. The general solution of the difference equation (10) is

$$U = \begin{cases} C\sin(\pi q t_j) + C_1\cos(\pi q t_j) & \text{for } q \neq 0, n; \\ C t_j + C_1 & \text{for } q = 0; \\ C(-1)^j t_j + C_1(-1)^j & \text{for } q = n. \end{cases}$$

From classical BC $U_0 = 0$ we get $C_1 = 0$. So, we are looking for solutions which have the following form:

$$U = \begin{cases} C t_j & \text{for } q = 0; \\ C(-1)^j t_j & \text{for } q = n; \\ C\sin(\pi q t_j) & \text{for } q \neq 0, n. \end{cases} \tag{11}$$

Let us substitute expressions (11) into NBC (5). We have three cases.

Case 1: $q = 0$ ($\lambda = 0$). In this case, we have equality (see Lemma 1)

$$C = \gamma\big[\chi_{[\xi_1,\xi_2]}(t), Ct\big] = \frac{C\gamma(\xi_2^2 - \xi_1^2)}{2}.$$

Nontrivial ($C \neq 0$) solution (eigenfunction) for $\lambda = 0$ exists if

$$1 = \frac{\gamma(\xi_2^2 - \xi_1^2)}{2}.$$

So, $\lambda = 0$ exists if and only if $\gamma = 2n^2/(m_2^2 - m_1^2)$, and $q = 0$ is BEP for all ξ_1, ξ_2 values (see Fig. 2(d)–(f)). Note that for the differential case, $\lambda = 0$ if $\gamma = 2/(\xi_2^2 - \xi_1^2)$ [21].

Case 2: $q = n$ ($\lambda = 4/h^2$). In this case, we have equality (see Lemma 2)

$$C(-1)^n = \gamma\big[\chi_{[\xi_1,\xi_2]}(t_j), C(-1)^j t_j\big] = \frac{C\gamma h^2((-1)^{\xi_2/h} - (-1)^{\xi_1/h})}{4}.$$

Nontrivial ($C \neq 0$) solution (eigenfunction) for $q = n$ exists if

$$1 = \frac{\gamma h^2(-1)^n((-1)^{m_2} - (-1)^{m_1})}{4}.$$

So, the eigenvalue $\lambda = 4/h^2$ ($q = n$) exists if and only if

$$m_2 - m_1 \in \mathbb{N}_{\mathrm{o}} \quad \text{and} \quad \gamma = \frac{2(-1)^{n-m_2}}{h^2},$$

and $q = n$ is BEP for $m_2 - m_1 \in \mathbb{N}_{\mathrm{o}}$ (see Fig. 2(e), (f)). If $m_2 - m_1 \in \mathbb{N}_{\mathrm{e}}$, the eigenvalue $\lambda = 4/h^2$ does not exist ($\gamma = \infty$). In this case, we have a pole at RP (see Fig. 2(d)).

Case 3: $q \in \mathring{\mathbb{C}}_q^h$. If we substitute $V = \sin(\pi q t_j)$ into the second BC (5), then by Lemma 1 we get equation for $q \in \mathbb{C}_q^h$:

$$\sin(\pi q) = \frac{1}{2}\gamma h\big(\cos(\pi q\xi_1) - \cos(\pi q\xi_2)\big)\tan^{-1}\frac{\pi q h}{2}. \tag{12}$$

Equation (12) can be rewritten in a more convenient form:

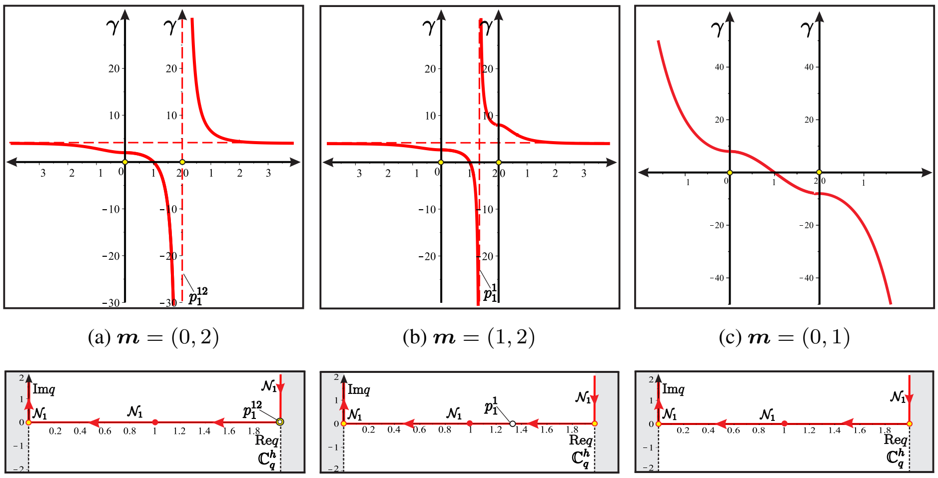
$$\frac{\sin(\pi q)}{\pi q} \cdot \frac{\sin(\pi q h/2)}{\pi q h\cos(\pi q h/2)} = \gamma\frac{\sin\frac{\pi q(\xi_2-\xi_1)}{2}\sin\frac{\pi q(\xi_2+\xi_1)}{2}}{\pi^2 q^2}. \tag{13}$$

This equation is valid (as limit cases) for $q = 0, n$, too. Roots of this equation are EPs for dSLP (4)–(5). Bijection (9) allows to find corresponding eigenvalues.

Remark 2. If hq is sufficiently small, then $\tan\pi qh/2 \approx \pi qh/2$. So, in the limiting case, equation (13) is the same as for differential problem [21].

If $\gamma = 0$, we have the classical BCs, and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters ξ_1 and ξ_2:

$$\lambda_k(0) = \lambda^h\big(q_k(0)\big), \qquad U_j^k(0) = \sin\big(\pi q_k(0)t_j\big), \qquad q_k(0) = k \in \mathbb{N}^h. \tag{14}$$



(a) $\boldsymbol{m} = (0,2)$ (b) $\boldsymbol{m} = (1,2)$ (c) $\boldsymbol{m} = (0,1)$

(d) $\boldsymbol{m} = (0,2)$ (e) $\boldsymbol{m} = (1,2)$ (f) $\boldsymbol{m} = (0,1)$

Figure 2. (a)–(c) Real CF; (d)–(f) Spectrum Curves for $n = 2$, ● – Zero Point, ○ – Pole Point of the first order, ○ – Branch Eigenvalue Point, ◎ – Pole at Ramification Point.

3 Constant eigenvalues points, characteristic function

We introduce entire functions:

$$Z^h(z) := Z(z) \cdot \frac{\sin(\pi z h/2)}{\pi z h \cos(\pi z h/2)}, \qquad Z(z) := \frac{\sin(\pi z)}{\pi z};$$

$$P_{\boldsymbol{\xi}}(z) := 2P^1_{\boldsymbol{\xi}}(z) P^2_{\boldsymbol{\xi}}(z),$$

$$P^1_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_1 + \xi_2)/2)}{\pi z}, \qquad P^2_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_2 - \xi_1)/2)}{\pi z}; \quad z \in \mathbb{C}.$$

Zeroes of these functions in the domain $\mathbb{C}^h_q$ are simple and positive. Zeroes of the function $Z^h(q)$, $q \in \mathbb{C}^h_q$, coincide with EPs in the classical case $\gamma = 0$ (see (14)), i.e., a set of zeroes for this function is $\mathcal{Z} := \mathbb{N}^h = \{1, \dots, n-1\}$.

Sets of zeroes in $\mathbb{C}^h_q$ for the functions $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ are

$$\overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} = \left\{ p^1_k = \frac{2nk}{m_+},\ k = 1, \dots, \left\lfloor \frac{n}{p^1_1} \right\rfloor \right\}, \qquad \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} = \left\{ p^2_l = \frac{2nl}{m_-},\ l = 1, \dots, \left\lfloor \frac{n}{p^2_1} \right\rfloor \right\}.$$

Remark 3. If $\boldsymbol{m} = (0,1)$, then there are no zeroes of the functions $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ in $\mathbb{C}^h_q$, i.e., $\overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} = \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} = \varnothing$. If $m_2 - 1 = m_1 > 0$, then exists $p^1_1 \in \mathbb{C}^h_q$ and $\overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} = \varnothing$. If $m_2 - m_1 > 1$, then the both sets are nonempty.

Lemma 3. *A set $\mathcal{Z}^P_{\boldsymbol{\xi}} = \mathcal{Z}^1_{\boldsymbol{\xi}} + \mathcal{Z}^2_{\boldsymbol{\xi}} + \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ describes all zeroes of $P_{\boldsymbol{\xi}}$, where $\mathcal{Z}^1_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} \setminus \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ and $\mathcal{Z}^2_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} \setminus \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ are two families of the first-order zeroes of $P^1_{\boldsymbol{\xi}}(z)$, $P^2_{\boldsymbol{\xi}}(z)$,* *respectively, and* $\mathcal{Z}_{\boldsymbol{\xi}}^{12} := \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^{1} \cap \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^{2}$ *is family of the second-order zeroes*

$$\mathcal{Z}_{\boldsymbol{\xi}}^{12} = \left\{ p_s^{12} = \frac{2ns}{\gcd(m_+; m_-)},\ s = 1, \dots, \left\lfloor \frac{n}{p_1^{12}} \right\rfloor \right\}.$$

Remark 4. If $m_1 = 0$, then all zeroes of $P_{\boldsymbol{\xi}}$ are of the second order and $\mathcal{Z}_{\boldsymbol{\xi}} = \mathcal{Z}_{\boldsymbol{\xi}}^{12}$. The point $q = n$ is zero of $P_{\boldsymbol{\xi}}^1$ iff $m_2 + m_1$ is even and $q = n$ is zero of $P_{\boldsymbol{\xi}}^2$ iff $m_2 - m_1$ is even. So, if $q = n$ is zero of $P_{\boldsymbol{\xi}}$, then it is of the second order and belongs to $\mathcal{Z}_{\boldsymbol{\xi}}^{12}$.

We can rewrite equality (13) in the form:

$$Z^h(q) = \gamma P_{\boldsymbol{\xi}}(q), \quad q \in \mathbb{C}_q^h. \tag{15}$$

We define the Constant Eigenvalue (CE) as the eigenvalue that does not depend on parameter γ. For any CE $\lambda_c \in \mathbb{C}$, there exists a Constant Eigenvalue Point (CEP) $q_c \in \mathbb{C}_q^h$ [25] and $\lambda_c = \lambda^h(q_c)$. For dSLP (4)–(5), all CEPs are real, and we can find them as solutions of the following system:

$$Z(q) = 0, \quad P_{\boldsymbol{\xi}}(q) = 0, \quad q \in (0, n).$$

CEPs belongs to two sets ($Z(q) = 0$, $P_{\boldsymbol{\xi}}^1(q) = 0$ and $Z(q) = 0$, $P_{\boldsymbol{\xi}}^2(q) = 0$):

$$\begin{aligned}
\overline{\mathcal{C}}_{\boldsymbol{\xi}}^1 &= \mathcal{Z} \cap \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^1 = \left\{ c_k^1 = \frac{2nk}{\gcd(2n; m_+)},\ k = 1, \dots, \left\lfloor \frac{n-1}{c_1^1} \right\rfloor \right\}, \\
\overline{\mathcal{C}}_{\boldsymbol{\xi}}^2 &= \mathcal{Z} \cap \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^2 = \left\{ c_l^2 = \frac{2nl}{\gcd(2n; m_-)},\ l = 1, \dots, \left\lfloor \frac{n-1}{c_1^2} \right\rfloor \right\}.
\end{aligned}$$

Lemma 4. *A set* $\mathcal{C}_{\boldsymbol{\xi}} = \mathcal{C}_{\boldsymbol{\xi}}^1 + \mathcal{C}_{\boldsymbol{\xi}}^2 + \mathcal{C}_{\boldsymbol{\xi}}^{12}$ *describes all Constant Eigenvalue Points, where* $\mathcal{C}_{\boldsymbol{\xi}}^1 := \overline{\mathcal{C}}_{\boldsymbol{\xi}}^1 \setminus \mathcal{C}_{\boldsymbol{\xi}}^{12} = \mathcal{Z} \cap \mathcal{Z}_{\boldsymbol{\xi}}^1$, $\mathcal{C}_{\boldsymbol{\xi}}^2 := \overline{\mathcal{C}}_{\boldsymbol{\xi}}^2 \setminus \mathcal{C}_{\boldsymbol{\xi}}^{12} = \mathcal{Z} \cap \mathcal{Z}_{\boldsymbol{\xi}}^2$ *and*

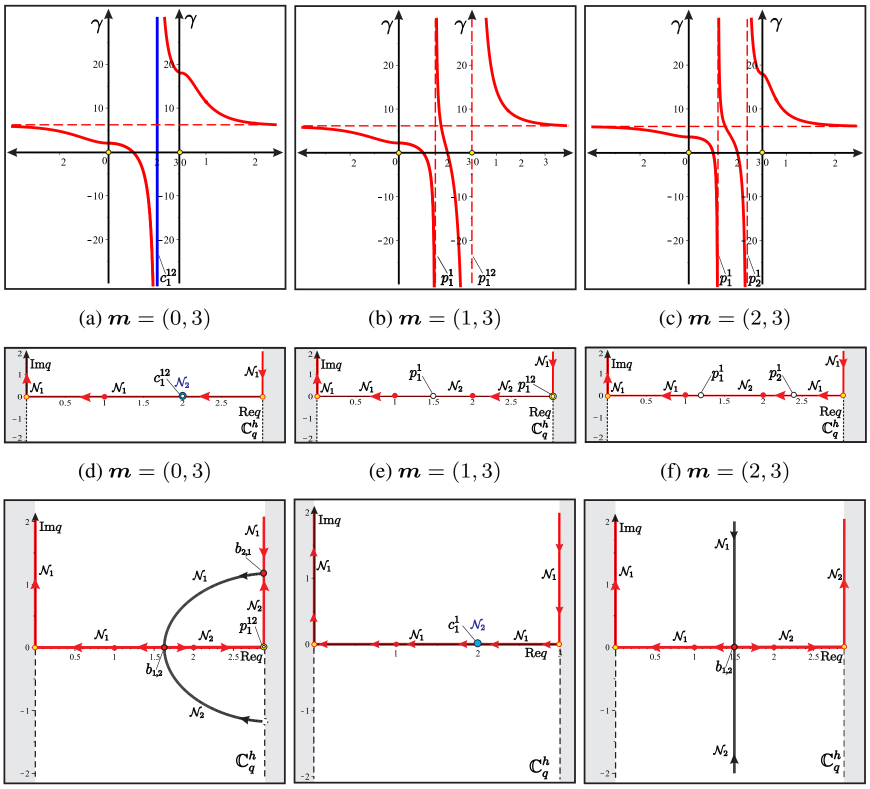
$$\mathcal{C}_{\boldsymbol{\xi}}^{12} = \overline{\mathcal{C}}_{\boldsymbol{\xi}}^1 \cap \overline{\mathcal{C}}_{\boldsymbol{\xi}}^2 = \mathcal{Z} \cap \mathcal{Z}_{\boldsymbol{\xi}}^{12} = \left\{ c_s^{12} = \frac{2ns}{\gcd(2n; m_+; m_-)},\ s = 1, \dots, \left\lfloor \frac{n-1}{c_1^{12}} \right\rfloor \right\}.$$

Remark 5. If $m_2 + m_1 = n$, then $c_1^1 = 2$. So, CEPs exist for all $n \geqslant 3$ (see Fig. 3(b)).

If $q \notin \mathbb{N}^h$, i.e., $Z^h(q) \neq 0$, and q satisfies equation $P_{\boldsymbol{\xi}}(q) = 0$, then equality (15) is not valid for all γ and such point q is a Pole Point (PP) (see Fig. 2(e)). The notation of PP is related to meromorphic function $\gamma_c(z) = Z^h(z)/P_{\boldsymbol{\xi}}(z)$, $z \in \mathbb{C}$. This function is obtained by expressing γ from equation (15). We call the restriction of meromorphic function γ_c on $\mathbb{C}_q^h$ as Complex Characteristic Function (Complex CF) [25] and denote this function as $\gamma(q)$, $q \in \mathbb{C}_q^h$:

$$\gamma(q) = \frac{Z^h(q)}{P_{\boldsymbol{\xi}}(q)} = \frac{\sin(\pi q)}{\sin(\pi q(\xi_2 + \xi_1)/2)\sin(\pi q(\xi_2 - \xi_1)/2)} \cdot \frac{\tan(\pi q h/2)}{h}. \tag{16}$$

Zeroes of Complex CF form a set $Z_{\boldsymbol{\xi}} = \mathcal{Z} \setminus \mathcal{C}_{\boldsymbol{\xi}}$ (see Fig. 2(d)–(f)).



(a) $\boldsymbol{m} = (0, 3)$ (b) $\boldsymbol{m} = (1, 3)$ (c) $\boldsymbol{m} = (2, 3)$

(d) $\boldsymbol{m} = (0, 3)$ (e) $\boldsymbol{m} = (1, 3)$ (f) $\boldsymbol{m} = (2, 3)$

(g) $\boldsymbol{m} = (0, 2)$ (h) $\boldsymbol{m} = (1, 2)$ (i) $\boldsymbol{m} = (0, 1)$

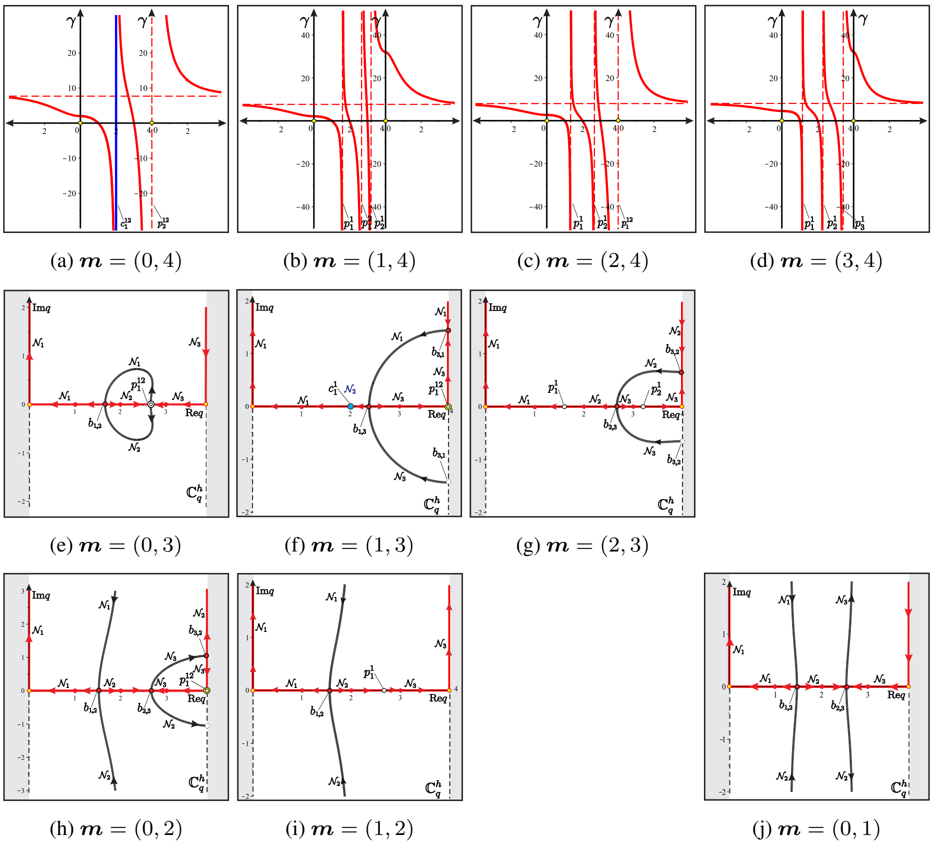
Figure 3. (a)–(c) Real CF; (d)–(i) Spectrum Curves for $n = 3$, ● – Constant Eigenvalue Point, ◎ – Constant Eigenvalue Point and Pole, ● – Critical Point.

Remark 6. Zeroes and poles of Complex CF γ_c are the same as in differential case [21] for rational ξ_1 and ξ_2, but in the discrete case, we use $\mathbb{C}_q^h$ instead $\mathbb{C}_q$.

Lemma 5. *A set $\mathcal{P}_{\boldsymbol{\xi}} = \mathcal{P}_{\boldsymbol{\xi}}^1 + \mathcal{P}_{\boldsymbol{\xi}}^2 + \mathcal{P}_{\boldsymbol{\xi}}^{12} + \mathcal{C}_{\boldsymbol{\xi}}^{12}$ describes all Pole Points of Complex CF, where $\mathcal{P}_{\boldsymbol{\xi}}^1 := \mathcal{Z}_{\boldsymbol{\xi}}^1 \setminus \mathcal{Z} = \mathcal{Z}_{\boldsymbol{\xi}}^1 \setminus \mathcal{C}_{\boldsymbol{\xi}}^1$, $\mathcal{P}_{\boldsymbol{\xi}}^2 := \mathcal{Z}_{\boldsymbol{\xi}}^2 \setminus \mathcal{Z} = \mathcal{Z}_{\boldsymbol{\xi}}^2 \setminus \mathcal{C}_{\boldsymbol{\xi}}^2$, $\mathcal{P}_{\boldsymbol{\xi}}^{12} := \mathcal{Z}_{\boldsymbol{\xi}}^{12} \setminus \mathcal{Z} = \mathcal{Z}_{\boldsymbol{\xi}}^{12} \setminus \mathcal{C}_{\boldsymbol{\xi}}^{12}$.*

Proof. From Lemma 3 we have that all zeroes of $P_{\boldsymbol{\xi}}$ are in the set $\mathcal{Z}_{\boldsymbol{\xi}}^1 + \mathcal{Z}_{\boldsymbol{\xi}}^2 + \mathcal{Z}_{\boldsymbol{\xi}}^{12}$. If $q \in \mathcal{Z}_{\boldsymbol{\xi}}^i$, $i = 1, 2$, $q \notin \mathcal{Z}$, then Complex CF has the first-order pole at the point q. If $q \in \mathcal{Z}_{\boldsymbol{\xi}}^{12}$, $q \notin \mathcal{Z}$, then Complex CF has the second-order pole at the point q. If $q \in \mathcal{Z}_{\boldsymbol{\xi}}^i$, $i = 1, 2$, $q \in \mathcal{Z}$, i.e., $q \in \mathcal{C}_{\boldsymbol{\xi}}^{12}$, then Complex CF has Removable Singularity Point and $0 \neq \gamma(q) \neq \infty$. If $q \in \mathcal{Z}_{\boldsymbol{\xi}}^{12}$, $q \in \mathcal{Z}$, i.e., $q \in \mathcal{C}_{\boldsymbol{\xi}}^{12}$, then Complex CF has the first-order pole at the point q (Z^h has the first-order zero, $P_{\boldsymbol{\xi}}$ has the second-order zero). □

Remark 7. If $q \in \mathcal{C}_{\boldsymbol{\xi}}^{12}$, then we have PP of Complex CF and CEP at this point (see Fig. 3(d)). If $q \in \mathcal{P}_{\boldsymbol{\xi}}^{12}$, then we have PP of the second order (see Fig. 4(e)).



(a) $\boldsymbol{m} = (0,4)$ (b) $\boldsymbol{m} = (1,4)$ (c) $\boldsymbol{m} = (2,4)$ (d) $\boldsymbol{m} = (3,4)$

(e) $\boldsymbol{m} = (0,3)$ (f) $\boldsymbol{m} = (1,3)$ (g) $\boldsymbol{m} = (2,3)$

(h) $\boldsymbol{m} = (0,2)$ (i) $\boldsymbol{m} = (1,2)$ (j) $\boldsymbol{m} = (0,1)$

Figure 4. (a)–(d) Real CF; (e)–(j) Spectrum Curves for $n = 4$, ◎ – Pole Point of the second order.

Remark 8. If the point $q \in \mathcal{C}^1_{\boldsymbol{\xi}}$ or $\in \mathcal{C}^2_{\boldsymbol{\xi}}$, then it is Removable Singularity Point of Complex CF.

The point $q = \infty \notin \mathbb{C}^h_q$. In the domain $\mathbb{C}_w$, this point corresponds to $w = 0$. Expression for Complex CF (16) (see (12), too) can be rewritten in the following form:

$$\gamma(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi_1} + e^{-\imath\pi q\xi_1} - e^{\imath\pi q\xi_2} - e^{-\imath\pi q\xi_2}} \cdot \frac{1 - e^{\imath\pi qh}}{1 + e^{\imath\pi qh}} \cdot \frac{2}{h}.$$

So, we can investigate Complex CF in the neighborhood $w = 0$ in the domain $\mathbb{C}_w$:

$$\begin{aligned}
\gamma(w) &= \frac{w^n - w^{-n}}{w^{m_1} + w^{-m_1} - w^{m_2} - w^{-m_2}} \cdot \frac{1 - w}{1 + w} \cdot \frac{2}{h} \\
&= \frac{w^{-n}}{w^{-m_2}} \cdot \frac{(1 - w^{2n})(1 - w)}{(1 - w^{m_2+m_1} - w^{m_2-m_1} + w^{2m_2})(1 + w)} \cdot \frac{2}{h} \\
&= \frac{1}{w^{n-m_2}} \cdot \frac{2}{h} \cdot \bigl(1 + \mathcal{O}(w)\bigr), \quad w = e^{\imath\pi qh}. \qquad (17)
\end{aligned}$$

So, the function $1/(w^{n-m_2})$ describes the properties Complex CF in the neighborhood of the isolated singularity point $w = 0 \notin \mathbb{C}_w^h$. If $m_2 = n$, then $\lim_{w\to 0} \gamma(w) = 2/h$, and we have Removable Singularity Point. If $m_2 \leqslant n-1$, then point $w = 0$ is $(n-m_2)$-order PP. We have the same classification for $q = \infty$ (see Figs. 2(a)–(c) and 3(a)–(c), a horizontal dashed red line).

All *nonconstant eigenvalues* (which depend on the parameter γ) are γ-points of Complex-Real Characteristic Function (CF) [25]. CF $\gamma(q)$ is the restriction of Complex CF $\gamma_c(q)$ on a set $\mathcal{D}_{\boldsymbol{\xi}} := \{q \in \mathbb{C}_q^h\colon \mathrm{Im}\gamma_c(q) = 0\}$. *Real CF* $\gamma(q)$ is defined on the domain $\{q \in \mathbb{R}_q^h\}$ and Real CF describes only real eigenvalues. We plot the graph of Real CF for eigenvalue points $0 < x < n$ in the middle graph; $x = 0$, $y > 0$ in the left half-plane and $x = n$, $y > 0$ in the right half-plane. Two γ-axes correspond to RPs $q = 0, n$ (see Figs. 2(a)–(c), 3(a)–(c) and 4(a)–(d)).

One can see the Real CF graphs in Figs. 2(a)–(c), 3(a)–(c) and 4(a)–(d) for $n = 2, 3, 4$. Vertical blue solid and red dashes lines are added at the CEPs and PPs. We note that there is real eigenvalues only in the case $n = m_2$ for any m_1 value (see Figs. 2–4) and for $\boldsymbol{m} = (0, 1)$, $n = 2$ (see Fig. 2(c)), $\boldsymbol{m} = (1, 2)$, $n = 3$ (see Fig. 3(h)). The second-order pole exists for $n = 4$, $\boldsymbol{m} = (0, 3)$ at the point $p_1^{12} = 2.(6)$ (see Fig. 4(e)); $n = 5$, $\boldsymbol{m} = (0, 4)$ at the point $p_1^{12} = 2.5$ (see Fig. 7(a)); $n = 5$, $\boldsymbol{m} = (0, 3)$ at the point $p_1^{12} = 3.(3)$ (see Fig. 7(e)); $n = 6$, $\boldsymbol{m} = (0, 5)$ at the points $p_1^{12} = 2.4$, $p_2^{12} = 4.8$ (see Fig. 8(a)).

More results about Poles and CE Points are presented in [21] (the cases when ξ_1 and ξ_2 are rational).

3.1 Ramification Points

Taylor series for $\gamma(q)$ at RP $q = 0$ is

$$\gamma(q) = \frac{2n^2}{m_2^2 - m_1^2} + \frac{\pi^2}{6(m_2^2 - m_1^2)}\big(1 - 2n^2 + m_2^2 + m_1^2\big)q^2 + \mathcal{O}\big(q^4\big).$$

We estimate $m_1^2 + m_2^2 \leqslant (n-1)^2 + n^2 \leqslant 2n^2 - 1$. So, a coefficient of the second term is negative.

In the case $m_- = m_2 - m_1 \in \mathbb{N}_o$, Taylor series for $\gamma(q)$ at RP $q = n$ is

$$\begin{aligned}\gamma(q) = {} & 2(-1)^{n-m_2}n^2 \\ & + \frac{\pi^2(-1)^{n-m_2}}{2}\left(\left(m_2^2 + m_1^2\right) - \frac{2n^2+1}{3}\right)(q-n)^2 \\ & + \frac{\pi^4(-1)^{n-m_2}}{8n^2}\left(\frac{6n^4 + 10n^2 - 1}{45} - \frac{m_2^4 + m_1^4}{3}\right. \\ & \left. + \left(\left(m_2^2 + m_1^2\right) - \frac{2n^2+1}{3}\right)\left(m_2^2 + m_1^2\right)\right)(q-n)^4 \\ & + \mathcal{O}\big((q-n)^6\big).\end{aligned} \tag{18}$$

If conditions

$$m_1^2 + m_2^2 = \frac{2n^2+1}{3}, \tag{19}$$

$$m_1^4 + m_2^4 = \frac{6n^2+10n^2-1}{15} \tag{20}$$

are valid, then the second term and the third term in (18) are vanished. Equation (20) can be replaced with

$$m_1^2 m_2^2 = \frac{(n^2-1)(n^2-4)}{45} \tag{21}$$

or

$$K(K+3) = 3^2 \cdot 5 \cdot M^2, \quad M = m_1 m_2,\ K = n^2 - 4. \tag{22}$$

Then $K = 3 \cdot 5 \cdot k$, $k \in \mathbb{N}$, or $K = 3 \cdot 5 \cdot k - 3$, $k \in \mathbb{N}$. We can rewrite (22) as $k(5k \pm 1) = M^2$. We see that M has divisor k, i.e., $M = kM_1$. Finally, $5k \pm 1 = kM_1^2$. This equation (and equation (21), too) has no solution. So, we prove that system (19)–(20) has no solution and both terms (the second and the third) in (18) can not vanish simultaneously for all values n, m_1, m_2.

The second term in (18) vanishes if the equation (19) has solution, where $n, m_1, m_2 \in \mathbb{N}$. From (19) we have that $n^2 = 1 + 3t$, $t \in \mathbb{N}$, and $m_2^2 + m_1^2 = 1 + 2t$. In the case $n \in \mathbb{N}_\mathrm{e}$, we get that t must be odd number and $m_1^2 + m_2^2 = 4t - 1 \equiv 3 \pmod 4$. Then from *Sum of two squares theorem* (An integer greater than one can be written as a sum of two squares if and only if its prime decomposition contains no prime congruent to 3 (mod 4) raised to an odd power, see [6]) follows that in the case $n \in \mathbb{N}_\mathrm{e}$ equation (19) has no solutions. If $n = 2k+1$, $k \in \mathbb{N}$, then t must be even, i.e., $t = 2p$, $p \in \mathbb{N}$, and we have equation $2k(k+1) = 3p$. Now, we see $3|m$ or $3|(m+1)$. So, $p = 2\tau(3\tau \pm 1)$, $\tau \in \mathbb{N}$, and $n = 6\tau \pm 1$, $m_1^2 + m_2^2 = 24\tau^2 \pm 8\tau + 1$.

Remark 9. In the case $n = 6\tau \pm 1$, we must additionally look for prime decomposition of $24\tau^2 \pm 8\tau + 1$. For example, if $n = 5$, then (19) has solution $m_1^2 + m_2^2 = 17 = 1^2 + 4^2$. In the case $n = 7$, $m_1^2 + m_2^2 = 33 = 3 \cdot 11$, there is no solution (sum of two squares theorem). For prime $24\tau^2 \pm 8\tau + 1$, such solution exists.

In the case $m_- = m_2 - m_1 \in \mathbb{N}_\mathrm{e}$, Laurent series for $\gamma(q)$ at BP $q = n$ is

$$\gamma(q) = \frac{8(-1)^{n-m_2+1} n^4}{\pi^2(m_2^2 - m_1^2)} \cdot \frac{1}{(q-n)^2} + \mathcal{O}(1). \tag{23}$$

Remark 10. If at RP $q = n$, we have pole, then this pole is of the second order. Indeed, if $p_l^2 = n$, then $m_2 - m_1 = 2l \in \mathbb{N}_\mathrm{e}$ and $m_2 + m_1 = 2s \in \mathbb{N}_\mathrm{e}$. So, $p_s^1 = n$. But at $\lambda = 4n^2 \in \mathbb{C}_\lambda$, we have pole of the first order.

3.2 Critical points

If $\gamma_c'(b) = 0$, $b \in \mathbb{C}_q$, then we have CP b of the Complex CF, and value $\gamma_c(b)$ is a *critical value* of the Complex CF [25]. Critical points of the Complex CF are saddle points of this function (see Fig. 3(g), (i)). For Real CF, Critical points can be maximum, minimum points or inflection (saddle) points. If the function γ_c at the Critical Point $b \in \mathbb{C}_q$ satisfies $\gamma_c'(b) = 0, \ldots, \gamma_c^k(b) = 0, \gamma_c^{(k+1)}(b) \neq 0$, then b is called Critical Point of the kth order (kCP).

Remark 11. A point $q = 0$ is 1CP in the domain $\mathbb{C}_q$, but $\lambda = 0$ is not Critical point in $\mathbb{C}_\lambda$ because q is a BP of $\lambda = \lambda(q)$. The order of a CP at BP is not invariant. Therefore, we investigate these points separately.

Critical points of the CF are important for investigation of multiple eigenvalues.

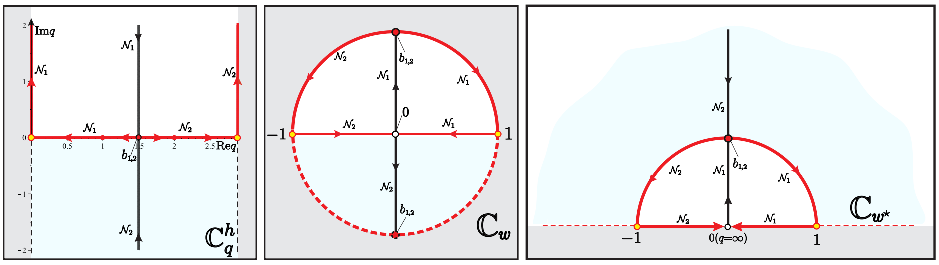
4 Spectrum curves

Spectrum Domain is the set $\mathcal{N}_{\boldsymbol{\xi}} = \mathcal{D}_{\boldsymbol{\xi}} \cup \mathcal{C}_{\boldsymbol{\xi}}$. Function γ_c has real values on $\mathcal{D}_{\boldsymbol{\xi}}$ except pole points. A set $\mathcal{E}_{\boldsymbol{\xi}}(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all nonconstant eigenvalue points for $\gamma_0 \in \mathbb{R}$. So, $\mathcal{D}_{\boldsymbol{\xi}} = \cup_{\gamma\in\mathbb{R}}\mathcal{E}_{\boldsymbol{\xi}}(\gamma)$. If $q \in \mathcal{D}_{\boldsymbol{\xi}}$ and $\gamma_c'(q) \neq 0$ (q is not a critical point of CF), then $\mathcal{E}_{\boldsymbol{\xi}}(\gamma)$ is smooth parametric curve $\mathcal{N}$: $\mathbb{R} \to \mathbb{C}_q^h$ locally and we can add arrow on this curve (arrows show the direction in which $\gamma \in \mathbb{R}$ is increasing). We call such curves *regular Spectrum Curves*. We can enumerate those Spectrum Curves for our problem by classical case ($\gamma = 0$): if $z_k = k \in \mathbb{N}$ belongs to Spectrum Curve, then the index of this Spectrum Curve is k. So, $\mathcal{N}_k(0) = z_k = k$. Few Spectrum Curves may intersect at the Critical Point. At this point, Curves change direction and the angle between the old and the new direction is $\pi/(k+1)$ for the Critical Point of the kth order (see Fig. 3(g), (i)). We use the "right-hand rule". So, the Spectrum Curve turns to the right. For the $\gamma \to \pm\infty$, Spectrum Curve $\mathcal{N}_k(\gamma)$ approaches a pole point or the point ∞. The index of a Critical Point is formed of the indices of the Spectrum Curves which intersect at this Critical point. If Critical Point of the first order is real, then the left index coincides with the index of Spectral Curve which is defined by the smaller real λ values, and the right index is defined by greater λ values.

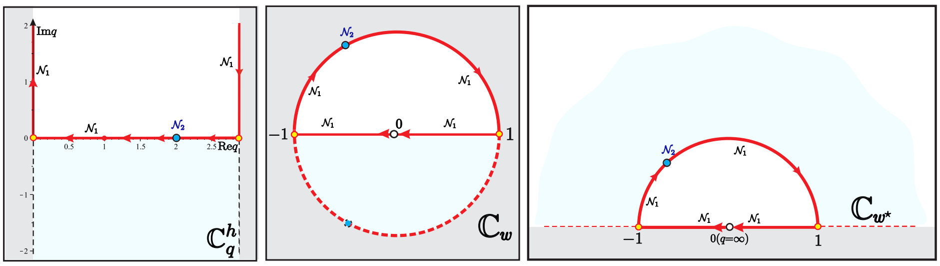
For every Constant Eigenvalue Point $c_j = j$, we define *nonregular Spectrum Curve* $\mathcal{N}_j = \{c_j\}$. We note that nonregular Spectrum Curves can overlap with a point of a regular Spectrum Curve. Finally, we have that $\mathcal{N}_{\boldsymbol{\xi}}$ is a finite union of Spectrum Curves $\mathcal{N}_l$, where $l = \overline{1, n-1}$.

One can see the Spectrum Curves in Figs. 2(d)–(f), 3(d)–(i) and 4(e)–(j) for $n = 2, 3, 4$. We have pole at RP $q = n$ ($n = 2$, $\boldsymbol{m} = (0, 2)$; $n = 3$, $\boldsymbol{m} = (1, 3)$, $\boldsymbol{m} = (0, 2)$; $n = 4$, $\boldsymbol{m} = (0, 4)$, $\boldsymbol{m} = (2, 4)$, $\boldsymbol{m} = (1, 3)$, $\boldsymbol{m} = (0, 2)$). In other cases, we have BEP at RPs $q = 0, n$. In the case $m_2 = n$, all Spectrum Curves belong to $\mathbb{R}_q^h$ (see Figs. 2(d), (e), 3(d)–(f)) and all eigenvalues are real. So, we present Real CF (see Figs. 2(a), (b), 3(a)–(c) and 4(a)–(d)). We have all real eigenvalues in the case $n = 2$, $\boldsymbol{m} = (0, 1)$ (see Fig. 2(c), (f)); $n = 3$, $\boldsymbol{m} = (1, 2)$ (see Fig. 3(h)), too. CEPs (Spectrum Curve $\mathcal{N}_2$) exist only for $n \geqslant 3$: $n = 3$, $\boldsymbol{m} = (0, 3)$, $\boldsymbol{m} = (1, 2)$; $n = 4$, $\boldsymbol{m} = (0, 4)$, $\boldsymbol{m} = (1, 3)$. We note that in the first case CEP is in PP of CF. In the case $n = 4$, $\boldsymbol{m} = (0, 3)$, we have the PP of the second order.

Some Spectrum Curves in $\mathbb{C}_q^h$, $\mathbb{C}_w$ and $\mathbb{C}_{w^\star}$ for $n = 3$ are presented Fig. 5. The domain $\mathbb{C}_w$ is useful for investigation of Spectrum Curves near the point $q = \infty$ ($w = 0$).

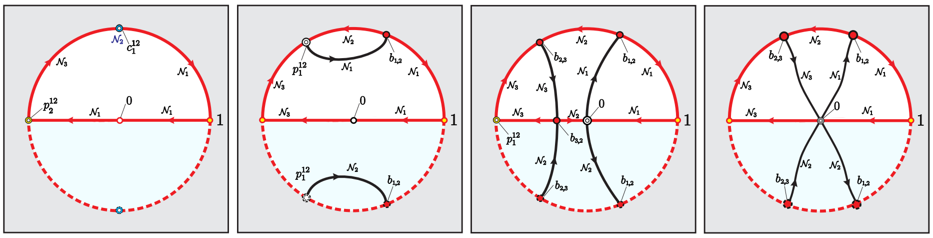


(a) $\boldsymbol{m} = (0, 1)$



(b) $\boldsymbol{m} = (1, 2)$

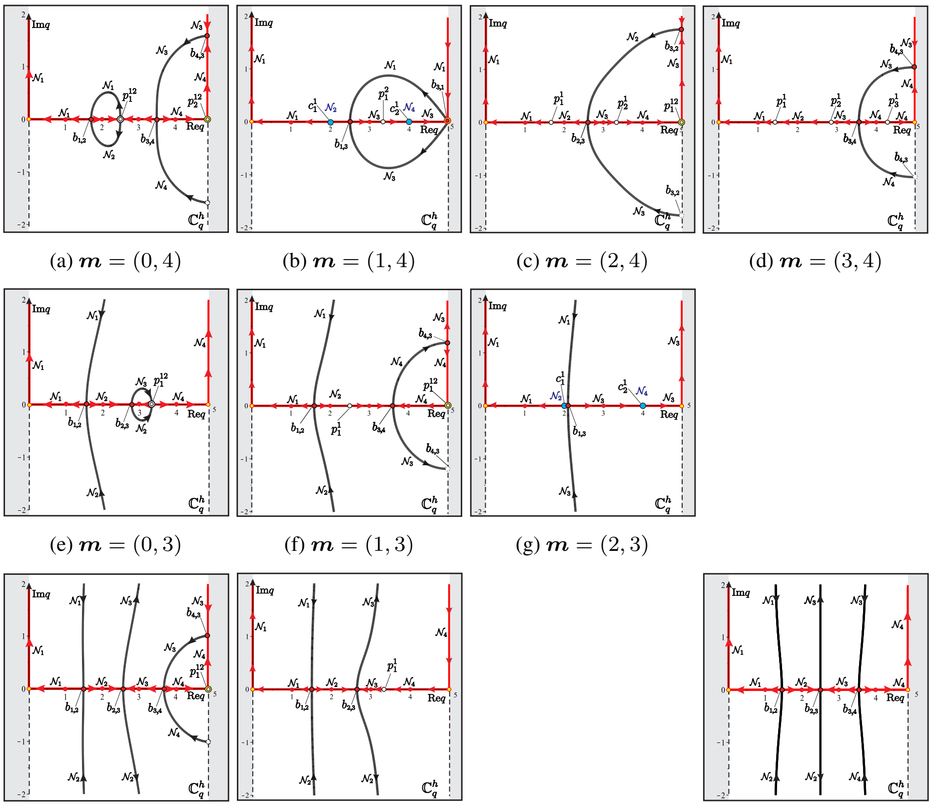
Figure 5. Spectrum Curves in $\mathbb{C}^h_q$, $\mathbb{C}_w$ and $\mathbb{C}_{w^\star}$, $n = 3$.



(a) $\boldsymbol{m} = (0, 4)$ (b) $\boldsymbol{m} = (0, 3)$ (c) $\boldsymbol{m} = (0, 2)$ (d) $\boldsymbol{m} = (0, 1)$

Figure 6. Spectrum Curves in $\mathbb{C}_w$ for different $\boldsymbol{m}$ values ($n = 4$), o – Removable Singularity Point at $q = \infty$.

We see Spectrum Curves in $\mathbb{C}_w$ for $n = 4$, $\boldsymbol{m} = (0, m)$, $m = 4, 3, 2, 1$. Formula (17) shows that this point is PP for $m_2 > 0$ and Removable Singularity Point for $m_2 = n$. In the case of removable singularity, we have the same Spectrum Curve $\mathcal{N}_1$ for $w \in (-1, 0)$ and $w \in (0, 1)$ (see Fig. 6(a)). So, the same Spectrum Curve enters and leaves the point $w = 0$ (or $q = \infty$). Additionally, there are no complex Spectrum Curves and all eigenvalues are real. The difference $n - m_2$ shows the order of a pole (see equation (17) and Fig. 6). In the case $n - m_2 = 1$, complex eigenvalues exist, but there are no complex Spectrum Curves (except in the case $n = 3$, $\boldsymbol{m} = (1, 2)$ when CEP exist instead of complex Spectrum Curve) that enter or leave PP $w = 0$ (see Fig. 6(b)). We have Spectrum Curve $\mathcal{N}_1$ to the right of PP of the first order $w = 0$ and Spectrum Curve $\mathcal{N}_3$ to the left of this point. If $n - m_2 = 2$, then the point $w = 0$ is PP of the second order in $\mathbb{C}^h_w$ and



(a) $\boldsymbol{m} = (0, 4)$ (b) $\boldsymbol{m} = (1, 4)$ (c) $\boldsymbol{m} = (2, 4)$ (d) $\boldsymbol{m} = (3, 4)$

(e) $\boldsymbol{m} = (0, 3)$ (f) $\boldsymbol{m} = (1, 3)$ (g) $\boldsymbol{m} = (2, 3)$

(h) $\boldsymbol{m} = (0, 2)$ (i) $\boldsymbol{m} = (1, 2)$ (j) $\boldsymbol{m} = (0, 1)$

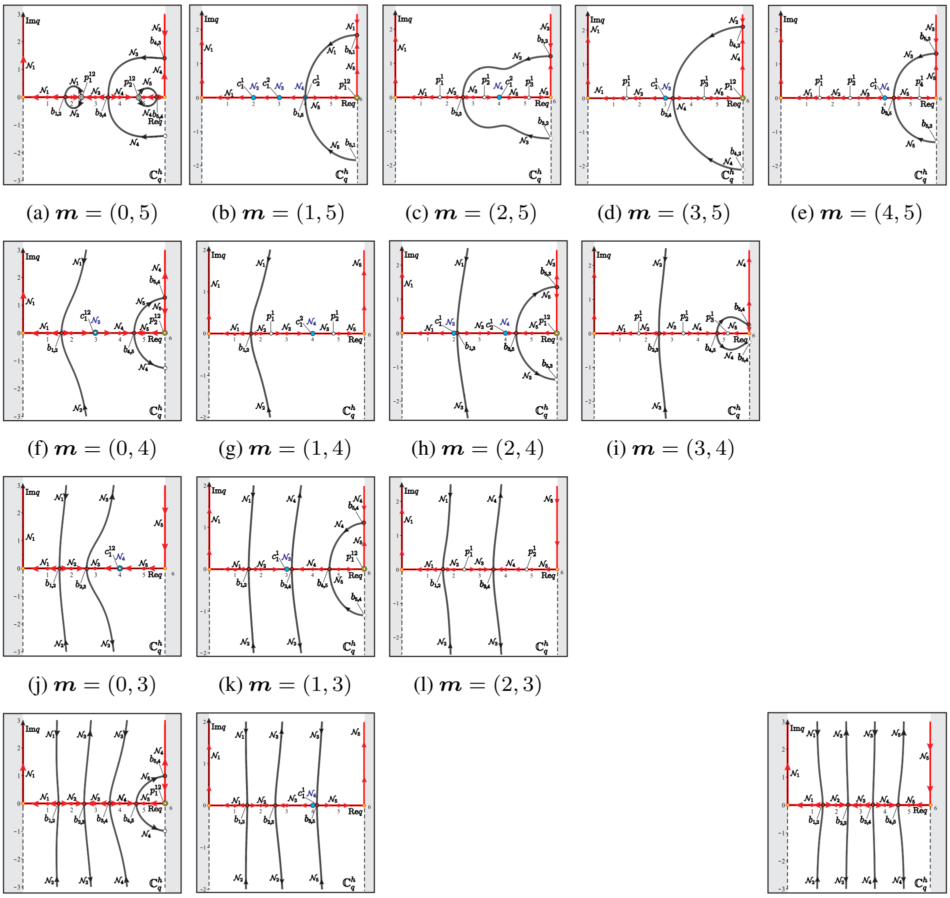
Figure 7. Spectrum Curves for $n = 5$, ◉ – Critical Point at Branch Eigenvalue Point.

one real and one complex Spectrum Curve goes to $w = 0$ and one real and one complex Spectrum Curves leaves the point. So, if $n - m_2 \geqslant 2$, then there exist complex Spectrum Curves that enter and leave the point $w = 0$.

The point $q = 0$ is BEP. At this point, Spectrum Curve $\mathcal{N}_1$ turns orthogonally to the right, i.e., the first positive eigenvalue point reaches $q = 0$ ($\gamma = 2n^2/(m_2^2 - m_1^2)$) and then this point moves along imaginary axis (describes negative eigenvalues). An image of this Spectrum Curve in $\mathbb{C}_\lambda$ at BP $\lambda = 0$ belongs to real axis and is straight.

At RP $q = n$, situation is more complicated. If $m_2 - m_1 \in \mathbb{N}_{\mathrm{e}}$, then this point is PP of the second order (see formula (23) and Remark 10). At corresponding BP $\lambda = 4n^2 \in \mathbb{C}_\lambda$, the pole is of the first order. In Remark 9, we formulate conditions for existence of critical point of the second order at $q = n$ (or the first order in $\mathbb{C}_\lambda$). For other cases, we have BEP at $q = n$ which has properties of CP of the first order, but really is not a CP in $\mathbb{C}_\lambda$. So, we do not use "right-hand rule" and Spectrum Curves may turn to the left at this point.

Spectrum Curves for $n = 5, 6$ are presented in Figs. 7 and 8, respectively. For example, we have CP at BEP $q = n = 5$ for $\boldsymbol{m} = (1, 4)$ and CP at CEP for $n = 6$ for



(a) $\boldsymbol{m} = (0,5)$ (b) $\boldsymbol{m} = (1,5)$ (c) $\boldsymbol{m} = (2,5)$ (d) $\boldsymbol{m} = (3,5)$ (e) $\boldsymbol{m} = (4,5)$

(f) $\boldsymbol{m} = (0,4)$ (g) $\boldsymbol{m} = (1,4)$ (h) $\boldsymbol{m} = (2,4)$ (i) $\boldsymbol{m} = (3,4)$

(j) $\boldsymbol{m} = (0,3)$ (k) $\boldsymbol{m} = (1,3)$ (l) $\boldsymbol{m} = (2,3)$

(m) $\boldsymbol{m} = (0,2)$ (n) $\boldsymbol{m} = (1,2)$ (o) $\boldsymbol{m} = (0,1)$

Figure 8. Spectrum Curves for $n = 6$, ● – Critical Point at Constant Eigenvalue Point.

$\boldsymbol{m} = (1,5)$. For some cases, there exists one CP which belongs to $\mathbb{R}_y^{h+}$. In this case, complex Spectrum Curve comes from another CP which belongs to $\mathbb{R}_x^h$. In the case $m_2 = n$, we have only real eigenvalues. Then Spectrum Curves belong to $\mathbb{R}_q^h$, and we do not present graphs of Real CFs (see [22] for some Real CFs graphs).

5 Some remarks and conclusions

If $m_2 = n$ (independent of m_1), complex eigenvalues do not exist. In this case, the point $q = \infty$ is removable singularity point (the same Spectrum Curve goes to this point and leaves it). Also, there exists a horizontal asymptote $\gamma(\infty) = \lim_{q\to\infty} \gamma(q) = 2/h$.

We have $n-1$ Spectrum Curve for every $n \in \mathbb{N}$, $n \geqslant 2$. Nonregular Spectrum Curves are CEPs and belong to $\mathbb{R}_x^h = (0,n)$. The number of such Spectrum Curves is equal to

$$n_{ce} = \left\lfloor \frac{n-1}{2n} \gcd(2n; m_+) \right\rfloor + \left\lfloor \frac{n-1}{2n} \gcd(2n; m_-) \right\rfloor - \left\lfloor \frac{n-1}{2n} \gcd(2n; m_+; m_-) \right\rfloor.$$

Then the number of regular Spectrum Curves is equal to $n_{nce} = n - 1 - n_{ce}$.

The poles of CF belong to $\mathbb{R}_x^h \cup \{n\} \cup \{\infty\}$, and the pole at RP $q = n$ (at BP $\lambda = 4n^2$) is of the second (the first) order. The pole at $q = \infty$ is of $n_\infty = n - m_2$ order. Then n_∞ Spectrum Curves go to this point and the same number of Spectrum Curves leave this point. Note that incoming Spectral Curves alternate with outgoing (see figures). If we denote the number of poles at $\mathbb{R}_x^h \cup \{n\}$ corresponding to function $\gamma(\lambda)$ by n_p (including the order, and at BP $\lambda = 4n^2$ poles are of the first order), then $n_p + n_\infty = n_{nce}$. So, we have formula for poles

$$n_p = m_2 - 1 - n_{ce}.$$

Particulary, $n_{ce}, n_p \leqslant m_2 - 1$.

The number poles of the second order in $\mathbb{R}_x^h$ is equal to

$$n_{2p} = \left\lfloor \frac{n-1}{2n} \gcd(m_+; m_-) \right\rfloor - \left\lfloor \frac{n-1}{2n} \gcd(2n; m_+; m_-) \right\rfloor.$$

If n_c is the number of parts Spectrum Curves in the complex part of C_q^h between two CP and n_{cr} is number of CP for real λ (including the order), then the following relation is valid:

$$n_{cr} = n_\infty + n_c + n_{2p} - 1, \quad n > m_2,$$

and there are no CPs for $n = m_2$. One CP can belong to R_y^{h+}.

When the value n is increasing, then the Spectrum Curves of dSLP (4)–(5) become more similar to the Spectrum Curves of the differential problem [21]. Every zero point, pole point of CF in $\mathbb{R}_x^h$ and CEP for dSLP is zero point, pole point and CEP for SLP, respectively.

References

1. J.R. Cannon, The solution of the heat equation subject to specification of energy, *Quart. Appl. Math.*, **21**(2):155–160, 1963.
2. R. Čiegis, A. Štikonas, O. Štikonienė, O. Subočc, Stationary problems with nonlocal boundary conditions, *Math. Model. Anal.*, **6**(2):178–191, 2001, `https://doi.org/10.1080/13926292.2001.9637157`.
3. R. Čiegis, A. Štikonas, O. Štikonienė, O. Suboč, A monotonic finite-difference scheme for a parabolic problem with nonlocal conditions, *Differ. Equ.*, **38**(7):1027–1037, 2002, `https://doi.org/10.1023/A:1021167932414`.

4. R. Čiupaila, Ž. Jesevičiūtė, M. Sapagovas, On the eigenvalue problem for one-dimensional differential operator with nonlocal integral condition, *Nonlinear Anal. Model. Control*, **9**(2):109–116, 2004, `http://www.mii.lt/na/issues/NA_0902/NA09201.pdf`.
5. W. Day, *Heat Conduction within Linear Thermoelasticity*, Springer-Verlag, 1985.
6. U. Dudley (1978). Elementary Number Theory (2 ed.), *Elementary Number Theory*, W.H. Freeman and Company, 2 edition, 1978.
7. A.V. Gulin, A.Yu. Mokin, Stability of a family of weighted finite-difference schemes, *Comput. Math. Model.*, **20**(2):152–172, 2009, `https://doi.org/10.1007/s10598-009-9026-1`.
8. N.I. Ionkin, E.I. Moiseev, Problem for the heat-transfer equation with two point boundary conditions, *Differ. Equ.*, **15**(7):915–924, 1980. (Translated from Russian, Differ. Uravn., **15**(7):1284–1295,1979)
9. N.I. Ionkin, E.A. Valikova, On eigenvalues and eigenfunctions of a non-classical boundary value problem, *Matem. Mod.*, **8**(1):53–63, 1996. (in Russian)
10. J. Jachimavičienė, The finite-difference method for a third-order pseudoparabolic equation with integral conditions, in V. Kleiza, S. Rutkauskas, A. Štikonas (Eds.), Proceedings of International Conference *Differential Equations and their Applications* (DETA-2009), *September 10-12, 2009, Panevėžys, Lithuania*, Kaunas University of Technology, 2009, pp. 49–58.
11. J. Jachimavičienė, Ž. Jesevičiūtė, M. Sapagovas, The stability of finite-difference schemes for a pseudoparabolic equation with nonlocal conditions, *Numer. Funct. Anal. Optim.*, **30**(9):988–1001, 2009, `https://doi.org/10.1080/01630560903405412`.
12. Ž. Jesevičiūtė, M. Sapagovas, On the stability of the finite-difference schemes for parabolic equations subject to integral conditions with applications for thermoelasticity, *Comp. Meth. Appl. Math.*, **8**(4):360–373, 2008, `https://doi.org/10.2478/cmam-2008-0026`.
13. L.I. Kamynin, A boundary value problem in the theory of the heat conduction with nonclassical boundary condition, *USSR Computational Mathematics and Mathematical Physics*, **4**(6):33–59, 1964, `https://doi.org/10.1016/0041-5553(64)90080-1`.
14. A.M. Nakhushev, An approximation method for solving boundary value problems for differential equations with applications to the dynamics of soil moisture and groundwater, *Differ. Uravn.*, **18**(1):72–81, 1982. (in Russian)
15. J. Novickij, A. Štikonas, On the stability of a weighted finite difference scheme for wave equation with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):460–475, 2014, `https://doi.org/10.15388/NA.2014.3.10`.
16. S. Pečiulytė, O. Štikonienė, A. Štikonas, Sturm–Liouville problem for stationary differential operator with nonlocal integral boundary condition, *Math. Model. Anal.*, **10**(4):377–392, 2005, `https://doi.org/10.1080/13926292.2005.9637295`.
17. M.P. Sapagovas, The solution of nonlinear ordinary differential equation with integral condition, *Lith. Math. J.*, **24**(1):60–68, 1984, `https://doi.org/10.1007/BF00970337`.
18. M.P. Sapagovas, The eigenvalues of some problem with a nonlocal condition, *Differ. Equ.*, **38**(7):1020–1026, 2002, `https://doi.org/10.1023/A:1021115915575`.

19. M.P. Sapagovas, A.D. Štikonas, On the structure of the spectrum of a differential operator with a nonlocal condition, *Differ. Equ.*, **41**(7):1010–1018, 2005, https://doi.org/10.1007/s10625-005-0242-y.

20. A. Skučaitė, K. Skučaitė-Bingelė, S. Pečiulytė, A. Štikonas, Investigation of the spectrum for the Sturm–Liouville problem with one integral boundary condition, *Nonlinear Anal. Model. Control*, **15**(4):501–512, 2010, http://www.mii.lt/na/issues/NA_1504/NA15412.pdf.

21. A. Skučaitė, A. Štikonas, Spectrum curves for Sturm–Liouville Problem with Integral Boundary Condition, *Math. Model. Anal.*, **20**(6):802–818, 2015, https://doi.org/10.3846/13926292.2015.1116470.

22. A. Skučaitė, *Investigation of the spectrum for Sturm–Liouville problem with a nonlocal integral boundary condition*, PhD thesis, Vilnius University, 2016.

23. A. Štikonas, Investigation of characteristic curve for Sturm–Liouville problem with nonlocal boundary conditions on torus, *Math. Model. Anal.*, **16**(1):1–22, 2011, https://doi.org/10.3846/13926292.2011.552260.

24. A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):301–334, 2014, https://doi.org/10.15388/NA.2014.3.1.

25. A. Štikonas, O. Štikonienė, Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions, *Math. Model. Anal.*, **14**(2):229–246, 2009, https://doi.org/10.3846/1392-6292.2009.14.229-246.