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Global attracting solutions to Hilfer fractional differential inclusions of Sobolev type with noninstantaneous impulses and nonlocal conditions*

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Abstract. In this paper, we establish the existence of decay mild solutions on an unbounded interval of nonlocal fractional semilinear differential inclusions with noninstantaneous impulses and involving the Hilfer derivative. Our argument uses fixed point theorems, semigroup theory, multi-functions and a measure of noncompactness on the space of piecewise weighted continuous functions defined on an unbounded interval. An example is provided to illustrate our results.

Keywords: Hilfer fractional derivatives, differential inclusions, noninstantaneous impulses, nonlocal conditions, global attracting solutions.

1 Introduction

Fractional differential equations and inclusions arise in various fields of physics, mechanics and engineering [3,12,17], and there are many papers on the existence of solutions and controls for fractional differential equations and inclusions; see [16,19–21,23,25,29–31] and the references therein. The action of instantaneous impulsive effect does not describe certain dynamics of evolution processes in therapy using pharmaceutical drugs. Take into

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consideration the hemodynamic equilibrium of a person: in the case of a decompensation (for example, high or low levels of glucose), one can prescribe some intravenous drugs (insulin), and the introduction of drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. This situation falls into a new case of impulsive action, which starts at any arbitrary fixed point and stays active on a finite time interval (noninstantaneous impulsive differential equation was introduced by Hernández and O'Regan [11]). A strong motivation for investigating nonlocal Cauchy problems (which is a generalization of classical Cauchy problems with an initial condition) comes from physical problems; for example, it is used to determine the unknown physical parameters in some inverse heat condition problems. Abstract nonlocal semilinear initial-value problems was initiated by Byszewski [6], where the existence and uniqueness of mild solutions for nonlocal differential equations without impulsive was considered.

Hilfer [12] introduced the Hilfer fractional derivative, which have two fractional parameters α and β, and this fractional derivative is used to extend Riemann–Liouville or Caputo-type Nutting's law to Hilfer-type Nutting's law, which can be used in the stress–strain relationship for more complex elastic solids. For other contributions on Hilfer-type equations, we refer the reader to [8, 10, 14, 27, 28].

The study of Sobolev-type equations can be traced back to the work of Barenblat et al. [4], in which the author initiated a model of flow liquid in fissured rocks, i.e., $\partial_t(u(t,x) - \partial_x^2 u(t,x)) - \partial_x^2 u(t,x) = 0$. This model was developed and studied in [5, 22] when the authors considered the abstract nonlinear evolution equation $(\mathrm{d}/\mathrm{d}t)B(u(t)) - Au(t) = f(t,u(t))$ in Banach spaces, where A and B are unbounded operators. Fečkan et al. [7] used two new characteristic solution operators and studied the controllability of fractional functional evolution equations of Sobolev type in Banach spaces.

In this paper, we study the global attracting of mild solutions to the following Hilfer fractional noninstantaneous impulsive differential inclusions of Sobolev type with nonlocal conditions on the unbounded interval $[0,\infty)$:

$$\begin{aligned} &D_{s_i^+}^{\alpha,\beta} Bx(t) \in Ax(t) + F\bigl(t, x(t)\bigr), \quad \text{a.e. } t \in (s_i, t_{i+1}],\ i \in \{0\} \cup \mathbb{N}, \\ &x(t_i^+) = g_i\bigl(t_i, x(t_i^-)\bigr), \quad x(t) = g_i\bigl(t, x(t_i^-)\bigr), \quad t \in (t_i, s_i],\ i \in \mathbb{N}, \\ &I_{0^+}^{1-\gamma} x(0) = g(x), \quad I_{s_i^+}^{1-\gamma} x(s_i^+) = g_i\bigl(s_i, x(t_i^-)\bigr), \quad i \in \mathbb{N}; \end{aligned} \tag{1}$$

we find a mild solution $x : [0,\infty) \to E$ for (1) satisfying $\lim_{t\to\infty} x(t) = 0$, where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, $D_{s_i+}^{\alpha,\beta} x(t)$ is the left-sided Hilfer derivative with lower limit at s_i of order α and type β (for definitions concerning the left-sided Hilfer fractional integral and derivative, see [12]), E is a real Banach space, A, B are linear closed operators on E such that $D(B) \subseteq D(A) \subseteq E$ and $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} < \cdots$. The symbol $x(t_i^+)$, $x(t_i^-)$ are the right and left limits of x at the point t_i, respectively, $I_{s_i^+}^{1-\gamma}$ is the left-sided Riemann–Liouville integral of order $1-\gamma$ with lower limit at s_i and $I_{s_i^+}^{1-\gamma} x(s_i^+) = \lim_{t\to s_i^+} I_{s_i^+}^{1-\gamma}(t)$. Moreover, $F : [0,\infty) \times E \to 2^E - \{\emptyset\}$ is a multifunction, $g : PC_{1-\gamma}^0([0,\infty), E) \to D(B)$, the domain of B, and $g_i : [t_i, s_i] \times E \to D(B)$, $i = 1, 2, \dots, m$, are functions. The space $PC_{1-\gamma}^0([0,\infty), E)$ will be specified in Section 4.

Associated with (1), we address the large time behavior of its solution. Anh et al. [1] found decay integral solutions for a class of neutral fractional differential equations with infinite delay, and in [9, 15], the authors studied some models of semilinear fractional differential equations in Banach spaces involving nonlocal conditions and impulsive effects, in which the existence of attracting solutions was established by employing the contraction mapping principle. Wang et al. [24] studied the controllability of Sobolev-type fractional evolution systems, and Le et al. [18] established the global attracting solutions to impulsive fractional differential inclusions of Sobolev-type involving the Caputo derivative. Associated with (1), we recall that in [26], the authors study nonlocal problems for impulsive fractional differential inclusions of Caputo-type, and existence and compactness results of PC-mild solutions are established.

To the best of our knowledge, no work has reported on attracting solutions to the Hilfer fractional noninstantaneous impulsive differential inclusion with nonlocal conditions and on an unbounded interval. We now consider the results in [18,26] and the differences with this paper: (i) The impulse effect in our paper is noninstantaneous (while in [18, 26] it is instantaneous). (ii) In [18, 26], the authors considered the Caputo fractional derivative, while in our paper we consider the Hilfer fractional derivative, which includes the Caputo and Riemann–Liouville fractional derivative. Note if we put $\beta = 0$ in the formula of the Hilfer fractional derivative, we obtain the Caputo fractional derivative, and if we put $\beta = 1$ in the formula of the Hilfer fractional derivative, we obtain the Riemann–Liouville fractional. (iii) The problem considered in [26] is not of Sobolev type. Moreover, the fixed point theory for multifunctions is different from the theory we use (and so the arguments are different). Note for Hilfer fractional evolution equations, the initial value includes singular kernels (so more complex than the Riemann–Liouville case since the initial condition does not include singular kernels), and we introduce new weighted piecewise continuous functions space to deal with such problems. In [18, 26], the lower limit of the Caputo-type fractional derivative is fixed and keeps it at the initial value. However, in our paper, the lower limit of the Hilfer-type fractional derivative is varying and changes at impulsive points. In fact, (1) can be used to characterize some possible control problems, where the impulsive equations can be considered as impulsive control conditions.

The paper is organized as follows. In Section 2, we collect some background material about multifunctions and fractional calculus to be used later. We introduce a measure of noncompactness on the space of piecewise weighted continuous functions. In Section 3, we establish an existence result for (1) on a compact interval. In Section 4, we introduce a regular measure of noncompactness on the space of piecewise weighted continuous functions defined on $[0, \infty)$, and then we prove the existence of solutions for (1). At the end of Section 4, an example is provided to illustrate our results.

2 Preliminaries and notation

Let $P_{\mathrm{b}}(E) = \{Z \subseteq E\colon\ Z$ is nonempty and bounded$\}$, $P_{\mathrm{cl}}(E) = \{Z \subseteq E\colon\ Z$ is nonempty, convex and closed$\}$, $P_{\mathrm{ck}}(E) = \{Z \subseteq E\colon$ is nonempty, convex and compact$\}$, $\mathrm{co}(Z)$ (respectively, $\overline{\mathrm{co}}(Z)$) be the convex hull (respectively, convex closed hull in E)

of a subset B, and $C(\Omega, E)$ be the Banach space of all E valued continuous functions from Ω to E with the norm $\|x\|_{C(\Omega,E)} = \sup_{t\in\Omega} \|x(t)\|$. For $a \in [0,b)$ and $0 \leqslant \gamma \leqslant 1$, consider the weighted space of continuous functions $C_\gamma([a,b],E) = \{x \in C((a,b],E)\colon (t-a)^\gamma x(t) \in C([a,b],E)\}$. Obviously $C_\gamma([a,b],E)$ is a Banach spaces with the norm $\|x\|_{C_\gamma([a,b],E)} = \sup_{t\in[a,b]}(t-a)^\gamma\|x(t)\|$.

Definition 1. Let E be a Banach space and $(\mathcal{A}, \geqslant)$ a partially ordered set. A function $\chi_E : \mathcal{P}_{\mathrm{b}}(E) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in E if $\chi_E(\overline{\mathrm{co}}\Omega) = \chi_E(\Omega)$ for every $\Omega \in \mathcal{P}_{\mathrm{b}}(E)$, where $\mathcal{P}_{\mathrm{b}}(E)$ is the family of bounded subsets of X.

The well-known Hausdorff measure of noncompactness defined by $\chi_E(\Omega) = \inf\{\epsilon > 0\colon\ \Omega \text{ has a finite } \epsilon\text{-net}\}$ is monotone, semiadditive, subadditive, nonsingular and regular.

Definition 2. Let E and Y be two Banach spaces. A multifunction $G : E \to 2^Y - \{\phi\}$ is said to be χ-condensing, where χ is a measure of noncompactness defined on bounded subset of E if for every bounded subset $\mathcal{D}$ of E that is not relatively compact, $\chi_E(F(\mathcal{D})) < \chi_E(\mathcal{D})$.

We need the following lemmas:

Lemma 1. (*See* [22].) *Let $\mathcal{C} \subset L^1([a,b],E)$ be a countable set such that there is $h \in L^1([0,b],E)$ with $f(t) \leqslant h(t)$ for a.e. $t \in [a,b]$ and every $f \in \mathcal{C}$. Then the function $t \to \chi\{f(t)\colon f \in \mathcal{C}\}$ belongs to $L^1([a,b],E)$ and satisfies $\chi_E\{\int_a^b f(s)\,\mathrm{d}s\colon f \in \mathcal{C}\} \leqslant 2\int_a^b \chi_E\{f(s)\colon f \in \mathcal{C}\}\,\mathrm{d}s$.*

Lemma 2. (*See* [9].) *Let $\chi_{C([a,b],E)}$ be the Hausdorff measure of noncompactness on $C([a,b],E)$. If $W \subseteq C([a,b],E)$ is bounded, then, for every $t \in [a,b]$, $\chi_E(W(t)) \leqslant \chi_{C([a,b],E)}(W)$, where $W(t) = \{x(t)\colon x \in W\}$. Furthermore, if W is equicontinuous on $[a,b]$, then the map $t \to \chi_E\{x(t)\colon x \in W\}$ is continuous on $[a,b]$ and $\chi_{C([a,b],E)}(W) = \sup_{t\in[a,b]} \chi\{x(t)\colon x \in W\}$.*

Lemma 3. (*See* [13, Cor. 3.3.1].) *If W is a convex closed subset of a Banach space X and $R\colon W \to P_{\mathrm{ck}}(X)$ is a closed γ-condensing multifunction, where γ is a monotone nonsingular measure of noncompactness defined on bounded subsets of X. Then the set of fixed points for R is nonempty.*

3 Existence of solutions for (1) on compact intervals

[Existence of solutions for (1) on compact intervals] In this section, we consider (1) on a compact interval $J = [0, t_{m+1}]$ and set $b = t_{m+1}$. That is, we are interested in the problem

$$\begin{aligned}
&D^{\alpha,\beta}_{s_i^+} Bx(t) \in Ax(t) + F\big(t, x(t)\big), \quad \text{a.e. } t \in (s_i, t_{i+1}],\ i = 0,1,\dots,m,\\
&x(t_i^+) = g_i\big(t_i, x(t_i^-)\big), \quad x(t) = g_i\big(t, x(t_i^-)\big), \quad t \in (t_i, s_i],\ i = 1,\dots,m, \qquad (2)\\
&I^{1-\gamma}_{0^+} x(0) = g(x), \quad I^{1-\gamma}_{s_i^+} x(s_i^+) = g_i\big(s_i, x(t_i^-)\big), \quad i = 1,\dots,m,
\end{aligned}$$

where $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = b$, $J_k = (s_k, t_{k+1}]$, $\overline{J}_k = [s_k, t_{k+1}]$, $k = 0, 1, \ldots, m$, $T_i = (t_i, s_i]$ and $\overline{T}_i = [t_i, s_i]$, $i = 1, 2, \ldots, m$.

Consider $PC_{1-\gamma}(J, E) = \{x\colon (t - s_k)^{1-\gamma} x \in C(J_k, E),\ \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} x(t),$ $k = 0, 1, \ldots, m,\ x \in C(T_i, E),\ \lim_{t \to t_i^+} x(t)$ exist, $i = 1, \ldots, m\}$ with $\|x\|_{PC_{1-\gamma}(J,E)} = \max\{\max_{k=0,1,\ldots,m} \sup_{t \in J_k} (t - s_k)^{1-\gamma} \|x(t)\|_E,\ \max_{i=1,2,\ldots,m} \sup_{t \in T_i} \|x(t)\|_E\}$.

Remark 1. Similar to [27, Remark 1], if $x \in PC_{1-\gamma}(J, E)$, then, for any $k = 0, 1, \ldots, m$,

(i) x is not necessarily defined at s_k, but $\lim_{t \to s_k+} (t - s_k) x(t)$ and $x(s_{i+1}^-)$ exist.

(ii) $x(t_{k+1}) = x(t_{k+1}^-)$ and $x(t_{k+1}^+)$ exists. Moreover, $(t_{k+1} - s_k)^{1-\gamma} \|x(t_{k+1}^-)\| \leqslant \|x\|_{PC_{1-\gamma}(J,E)}$.

(iii) If $x_n \to x$ in $PC_{1-\gamma}(J, E)$, then $x_n(t) \to x(t),\ t \in (t_i, s_i], i = 1, \ldots, m$, and $(t - s_k)^{1-\gamma} x_n(t) \to (t - s_k)^{1-\gamma} x(t),\ t \in (s_k, t_{k+1}]$. Consequently, $x_n(t) \to x(t),\ t \in (s_i, t_{i+1}]$, and hence $x_n(t_{i+1}) = x_n(t_{i+1}^-) \to x(t_{i+1}) = x(t_{i+1}^-)$, $i = 0, 1, \ldots, m$. It follows that $x_n(t) \to x(t)$ a.e. for $t \in J$.

Next, $\chi_{PC_{1-\gamma}(J,E)} : P_{\mathrm{b}}(PC_{1-\gamma}(J, E)) \to [0, \infty)$ defined by $\chi_{PC_{1-\gamma}(J,E)}(Z) = \max\{\max_{k=0,1,\ldots,m} \chi_{C(\overline{J}_k,E)}(Z_{|\overline{J}_k}), \max_{i=1,\ldots,m} \chi_{C(\overline{T}_i,E)}(Z_{|\overline{T}_i})\}$ is a monotone, nonsingular, semiadditive and regular measure of noncompactness on $PC_{1-\gamma}(J, E)$, where $\chi_{C(\overline{J}_k,E)}(Z_{|\overline{J}_k})$ is the Hausdorff measure of noncompactness on $C(\overline{J}_k, E)$, $Z_{|\overline{J}_k} = \{y^* \in C(\overline{J}_k, E)\colon y^*(t) = (t - s_k)^{1-\gamma} y(t),\ t \in J_k,\ y^*(s_k) = \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} y(t),$ $y \in Z\}$, and $Z_{|\overline{T}_i} = \{y^* \in C(\overline{T}_i, E)\colon y^*(t) = y(t),\ t \in T_i,\ y^*(t_i) = y(t_i^+),\ y \in Z\}$.

In the proof of the results in this paper, we do not need $\chi_{PC_{1-\gamma}(J,E)}$ to be semiadditive. In fact, we need only monotone and nonsingular to apply Lemma 2, and regular to conclude that $R(x)$ is relatively compact in our results (see the third last line in the proof of Theorem 1).

Lemma 4. *The measure of noncompactness $\chi_{PC_{1-\gamma}(J,E)}$ is monotone, nonsingular, and regular.*

Proof. (i) Let Z and W be two bounded subsets in $PC_{1-\gamma}(J, E)$ such that $Z \subseteq W$ and $k \in \{0, 1, \ldots, m\}$ be fixed. From the definition of the Hausdorff measure of noncompactness on $C(\overline{J}_k, E)$ (see [13]) and $Z_{|\overline{J}_k} \subseteq W_{|\overline{J}_k}$, we get $\chi_{C(\overline{J}_k,E)}(Z_{|\overline{J}_k}) \leqslant \chi_{C(\overline{J}_k,E)}(W_{|\overline{J}_k})$. Similarly, $\chi_{C(\overline{T}_i,E)}(Z_{|\overline{T}_i}) \leqslant \chi_{C(\overline{T}_i,E)}(W_{|\overline{T}_i})$, for any $i = 1, \ldots, m$, and hence $\chi_{PC_{1-\gamma}(J,E)}(Z) \leqslant \chi_{PC_{1-\gamma}(J,E)}(W)$. Thus $\chi_{PC_{1-\gamma}(J,E)}$ is monotone.

(ii) Let Z be a bounded subset in $PC_{1-\gamma}(J, E)$ and $w \in PC_{1-\gamma}(J, E)$. Notice that $Z_{|\overline{J}_k} \cup \{w_{|\overline{J}_k}\} = (Z \cup \{w\})_{|\overline{J}_k}$. Since $\chi_{C(\overline{J}_k,E)}$ is the Hausdorff measure of noncompactness on $C(\overline{J}_k, E)$, we have that $\chi_{PC_{1-\gamma}(J,E)}(Z_{|\overline{J}_k} \cup \{w_{|\overline{J}_k}\}) = \chi_{PC_{1-\gamma}(J,E)}(Z \cup \{w\})_{|\overline{J}_k} = \chi_{PC_{1-\gamma}(J,E)}(Z)_{|\overline{J}_k}$ for any $k = 0, 1, \ldots, m$. Similarly, $\chi_{PC_{1-\gamma}(J,E)}(Z_{|\overline{T_k}} \cup \{w_{|\overline{J}_k}\}) = \chi_{PC_{1-\gamma}(J,E)}(Z)_{|\overline{T_k}}$, for any $i = 1, \ldots, m$, and hence $\chi_{PC_{1-\gamma}(J,E)}(Z \cup \{w\}) \leqslant \chi_{PC_{1-\gamma}(J,E)}(Z)$. Thus $\chi_{PC_{1-\gamma}(J,E)}$ is nonsingular.

(iii) In order to show that $\chi_{PC_{1-\gamma}(J,E)}$ is regular, let $\chi_{PC_{1-\gamma}(J,E)}(Z) = 0$. Then $\chi_{C(\overline{J}_k,E)}(Z_{|\overline{J}_k}) = 0$ for any $k = 0, 1, \ldots, m$, and $\chi_{C(\overline{T}_i,E)}(Z_{|\overline{T}_i}) = 0$ for any $i = 1, \ldots, m$. From the fact that $\chi_{C(\overline{J}_k,E)}$ and $\chi_{C(\overline{T}_i,E)}$ are the Hausdorff measure of noncompactness on $C(\overline{J}_k, E)$ and $C(\overline{T}_i, E)$, respectively, we conclude that $Z_{|\overline{J}_k}$ and $Z_{|\overline{T}_i}$ are relatively compact for any $k = 0, 1, \dots, m$ and any $i = 1, \dots, m$. Now let (z_n) be a sequence in Z, $k \in \{0, 1, \dots, m\}$ and $i \in \{1, \dots, m\}$. We define $z^*_{n,k}(t) = (t - s_k)^{1-\gamma} z_n(t)$, $t \in J_k$, $z^*_{n,k}(s_k) = \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} z_n(t)$ and $z^*_{n,i}(t) = z_n(t)$, $t \in T_k$, $z^*_{n,i}(t_i) = z_n(t_i^+)$. It follows from the relative compactness of $Z_{|\overline{J}_k}$ and $Z_{|\overline{T}_i}$ that there are two subsequences of $(z^*_{n,k})$ and $(z^*_{n,i})$, denoted again by $(z^*_{n,k})$ and $(z^*_{n,i})$ such that $z^*_{n,k} \to z^*_k$ in $C(\overline{J}_k, E)$ and $z^*_{n,i} \to z^*_k$ in $C(\overline{T}_i, E)$. Next, we define $z^* : J \to E$ as follows: $z^*(t) = z^*_k(t)$, $t \in J_k$, $k = 0, 1, \dots, m$ and $z^*(t) = z^*_i(t)$, $t \in J_i$, $i = 1, \dots, m$. From the definition of the norm in $PC_{1-\gamma}(J, E)$ we have that (z_n) has a subsequence that converging to z^* in $PC_{1-\gamma}(J, E)$.

Now assume that Z is relatively compact. If we show that $Z_{|\overline{J}_k}$ and $Z_{|\overline{T}_i}$ are relatively compact for any $k = 0, 1, \dots, m$ and any $i = 1, \dots, m$, then from the fact that $\chi_{C(\overline{J}_k, E)}$ and $\chi_{C(\overline{T}_i, E)}$ are the Hausdorff measure of noncompactness on $C(\overline{J}_k, E)$ and $C(\overline{T}_i, E)$ it follows that $\chi_{PC_{1-\gamma}(J,E)}(Z) = 0$. Now, let $z^*_n \in Z_{|\overline{J}_k}$ for some $k = 0, 1, \dots, m$. Then $z^*_n(t) = (t - s_k)^{1-\gamma} z_n(t)$, $t \in J_k$, $z^*_n(s_k) = \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} z_n(t)$, $z_n \in Z$. From the relative compactness of Z, without loss generality, we can assume that $z_n \to z$ in Z. From the definition of the norm in $PC_{1-\gamma}(J, E)$ we get $\lim_{n\to\infty} \sup_{t \in J_k} (t - s_k)^{1-\gamma} \times \|z_n(t) - z(t)\| = 0$.

Let $z^* : \overline{J}_k \to E$ be such that $z^*(t) = (t - s_k)^{1-\gamma} z(t)$, $t \in J_k$, $z^*(s_k) = \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} z(t)$. Then $\lim_{n\to\infty} \sup_{t \in J_k} \|z^*_n(t) - z^*(t)\| = \lim_{n\to\infty} \sup_{t \in T_k} (t - s_k)^{1-\gamma} \times \|z_n(t) - z(t)\| = 0$. Next, $\lim_{n\to\infty} z^*_n(s_k) = \lim_{n\to\infty} \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} z_n(t) = z^*(s_k)$. Thus there is a subsequence in (z^*_n) that converges to z^* in $C(\overline{J}_k, E)$. This show that $Z_{|\overline{J}_k}$ is relatively compact. Similarly, we can show that $Z_{|\overline{J}_k}$ and $Z_{|\overline{T}_i}$ are relatively compact, for $k = 0, 1, \dots, m$ and $i = 1, \dots, m$. □

Definition 3. (See [10, Def. 2.13].) Let $f : [0, b] \times E \to E$ be a function, A, B be linear operators on a Banach space E such that $D(B) \subseteq D(A) = E$, B is bijective, has a bounded inverse B^{-1} and AB^{-1} is the infinitesimal generator of a C_0-semigroup $\{T(t)\colon t \geqslant 0\}$. By a mild solution of $D^{\alpha,\beta}_{a+} x(t) = AB^{-1}x(t) + f(t, x(t))$, $t \in (0, b]$, with $I^{1-\gamma}_{0+} x(0^+) = x_0$. We mean a function $x \in C((0, b], E)$ satisfying $x(t) = S_{\alpha,\beta}(t)x_0 + \int_0^t K_\alpha(t - s) f(s, x(s))\, \mathrm{d}s$, $t \in (0, b]$, where $K_\alpha(t) = t^{\alpha-1} P_\alpha(t)$, $P_\alpha(t) = \int_0^\infty \alpha\theta \times M_\alpha(\theta) T(t^\alpha \theta)\, \mathrm{d}\theta$, $t \geqslant 0$, $M_\mu(\theta) = \sum_{n=1}^\infty (-\theta)^{n-1}/((n - 1)\Gamma(1 - \mu n))$, $\mu \in (0, 1)$, $\theta \in \mathbb{C}$, and $S_{\alpha,\beta}(t) = I^{\beta(1-\alpha)}_{0+} K_\alpha(t)$. Note that the weight function $M_\mu(\theta)$ satisfies the equality $\int_0^\infty \theta^\tau M_\mu(\theta)\, \mathrm{d}\theta = \Gamma(1 + \tau)/\Gamma(1 + \tau\mu)$ for $\theta \geqslant 0$.

Remark 2. (See [10, Remark 2.14].) $D^{\beta(1-\alpha)}_{0+} S_{\alpha,\beta}(t) = K_\alpha(t)$, $t \in (0, b]$. When $\beta = 0$, the fractional equation (2) reduces to the classical Riemann–Liouville fractional equation, which was studied by Zhou and Jiao [30]. Note $S_{\alpha,0}(t) = K_\alpha(t) = t^{\alpha-1} P_\alpha(t)$. When $\beta = 1$, the fractional equation (2) reduces to the classical Caputo fractional equation, which was studied by Zhou et al. [31]. Note $S_{\alpha,1} = S_\alpha(t)$, where $S_\alpha(t) = \int_0^\infty M_\alpha(\theta) T(t^\alpha \theta)\, \mathrm{d}\theta$.

In the following, we present some properties for the operators $S_{\alpha,\beta}(\cdot)$ and $K_\alpha(\cdot)$ [10, Props. 2.15, 2.16].

Lemma 5. *Suppose $\{T(t), t \geqslant 0\}$ is continuous for the uniform operator topology for $t > 0$, and there is a $M > 1$ such that $\sup_{t \geqslant 0} \|T(t)\| \leqslant M$. Then we have the following results*:

(i) *$P_\alpha(t)$ is continuous for the uniform operator topology for $t > 0$.*

(ii) *For any fixed $t > 0$, $S_{\alpha,\beta}(t)$ and $K_\alpha(t)$ are linear bounded operators, and for any fixed $x \in E$,*

$$\left\|S_{\alpha,\beta}(t)x\right\| \leqslant \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|x\|, \quad \gamma = \alpha + \beta - \alpha\beta, \tag{3}$$

and

$$\left\|P_\alpha(t)\right\| \leqslant \frac{M}{\Gamma(\alpha)}. \tag{4}$$

(iii) *$\{K_\alpha(t)\colon t > 0\}$ and $\{S_{\alpha,\beta}(t)\colon t > 0\}$ are strongly continuous, which means that for any $x \in E$ and $0 < t_1 < t_2 \leqslant b$, we have $\|K_\alpha(t_1)x - K_\alpha(t_2)x\| \to 0$ and $\|S_{\alpha,\beta}(t_1)x - S_{\alpha,\beta}(t_2)x\| \to 0$ as $t_1 \to t_2$.*

To prove the existence of mild solutions, we need to the following conditions:

(F) $F : J \times E \to P_{\mathrm{ck}}(E)$ is a multifunction satisfying:

1. For every $x \in E$, the multifunction $t \to F(t,x)$ has a strong measurable selection, and for almost every $t \in J$, $z \to F(t,z)$ is upper semicontinuous.

2. There exist a function $\varphi \in L^p(J,\mathbb{R}^+)$, $p > 1/\alpha$, and a continuous nondecreasing function $\Omega : [0,\infty) \to (0,\infty)$ such that for any $x \in PC_{1-\gamma}(J,E)$ and any $i = 0,1,\dots,m$, $\|F(t,x(t))\| \leqslant \varphi(t)\Omega((t-s_i)^{1-\gamma}\|x(t)\|)$ for $t \in (s_i, t_{i+1}]$.

3. There exists a function $\varsigma \in L^p(J,\mathbb{R}^+)$ such that for any bounded subset $D \subseteq E$ and any $k = 0,1,\dots,m$, $\chi_E(F(t,D)) \leqslant (t-s_k)^{1-\gamma}\varsigma(t)\chi_E(D)$ for a.e. $t \in J_k$, where χ is the Hausdorff measure of noncompactness on E.

(H_1) The operator B is bijective, has a bounded inverse B^{-1}, and AB^{-1} is the infinitesimal generator of a C_0-semigroup $\{T(t)\colon t \geqslant 0\}$, which is continuous for the uniform operator topology for $t > 0$, and there is a $M > 1$ such that $\sup_{t \geqslant 0} \|T(t)\| \leqslant M$.

(H_g) The function $g : PC_{1-\gamma}(J,E) \to D(B)$ obeys to the following conditions:

1. $Bg : PC_{1-\gamma}(J,E) \to E$ is continuous, and there is a nondecreasing function $\Psi : [0,\infty) \to [0,\infty)$ such that $\|Bg(x)\| \leqslant \Psi(\|x\|_{PC_{1-\gamma}(J,E)})$ and

$$\liminf_{n\to\infty} \frac{\Psi(n)}{n} = w < \infty. \tag{5}$$

2. There is $\kappa_1 > 0$ such that for any bounded subset D of $PC_{1-\gamma}(J,E)$,

$$\chi_E\bigl(Bg(D)\bigr) \leqslant \kappa_1\chi_{PC_{1-\gamma}(J,E)}(D). \tag{6}$$

(H) The function $g_i : [t_i, s_i] \times E \to D(B)$, $i = 1, 2, \dots, m$, is uniformly continuous on bounded sets and satisfies:

1. There exists a positive constant h_i such that for any $x \in E$ $\|Bg_i(t, x))\| \leqslant h_i(t_i - s_{i-1})^{1-\gamma}\|x\|$, $t \in [t_i, s_i]$, $x \in E$.
2. There is a $\kappa_2 > 0$ such that for any bounded subset D of $PC_{1-\gamma}(J, E)$

$$\chi_E\bigl(Bg_i\bigl(t, \{h(t_i)\colon h \in D\}\bigr)\bigr) \leqslant \kappa_2 \chi_{PC_{1-\gamma}(J,E)}(D), \quad t \in [t_i, s_i]. \tag{7}$$

Let $h \in L^1(J, X)$. Consider the impulsive problem

$$\begin{aligned} &D^{\alpha,\beta}_{s_i^+} Bx(t) = Ax(t) + h(t), \quad \text{a.e. } t \in (s_i,\ t_{i+1}],\ i = 0, 1, \dots, m,\\ &x(t_i^+) = g_i\bigl(t_i, x(t_i^-)\bigr), \quad x(t) = g_i\bigl(t, x(t_i^-)\bigr), \quad t \in (t_i, s_i],\ i = 1, \dots, m,\\ &I^{1-\gamma}_{0^+} x(0) = g(x), \quad I^{1-\gamma}_{s_i^+} x(s_i^+) = g_i\bigl(s_i, x(t_i^-)\bigr), \quad i = 1, \dots, m. \end{aligned} \tag{8}$$

In order to formulate the solution function of (8), let $v(\cdot) = Bx(\cdot)$, i.e., $x(\cdot) = B^{-1}v(\cdot)$. Then (8) can be rewritten as

$$\begin{aligned} &D^{\alpha,\beta}_{s_i^+} v(t) = AB^{-1}v(t) + h(t), \quad \text{a.e. } t \in (s_i, t_{i+1}],\ i = 0, 1, \dots, m,\\ &v(t_i^+) = Bg_i\bigl(t_i, B^{-1}v(t_i^-)\bigr), \quad v(t) = Bg_i\bigl(t, B^{-1}v(t_i^-)\bigr),\\ &\quad t \in (t_i, s_i],\ i = 1, \dots, m,\\ &I^{1-\gamma}_{0^+} v(0) = Bg\bigl(B^{-1}v(0)\bigr), \quad I^{1-\gamma}_{s_i^+} v(s_i^+) = Bg_i\bigl(s_i, B^{-1}v(t_i^-)\bigr), \quad i = 1, \dots, m. \end{aligned} \tag{9}$$

From Definition 3, the mild solution of (9) is a function $v \in PC_{1-\gamma}([0, b], X)$ such that

$$v(t) = \begin{cases} S_{\alpha,\beta}(t)(Bg(B^{-1}v)) + \frac{1}{\Gamma(\alpha)} \int_0^t K_\alpha(t-s)h(s)\,\mathrm{d}s, & t \in [0, t_1],\\ Bg_i(t, B^{-1}v(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, &\\ S_{\alpha,\beta}(t - s_i)Bg_i(s_i, B^{-1}v(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)h(s), &\\ \quad t \in (s_i, t_{i+1}], i = 1, \dots, m. & \end{cases} \tag{10}$$

By substituting in (10) $v = Bx$, we get

$$Bx(t) = \begin{cases} S_{\alpha,\beta}(t)(Bg(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t K_\alpha(t-s)h(s)\,\mathrm{d}s, & t \in (0, t_1],\\ Bg_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, &\\ S_{\alpha,\beta}(t - s_i)Bg_i(t, x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)h(s), &\\ \quad t \in (s_i, t_{i+1}], i = 1, \dots, m. & \end{cases}$$

Then

$$x(t) = \begin{cases} B^{-1}[S_{\alpha,\beta}(t)(Bg(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t K_\alpha(t-s)h(s)\,\mathrm{d}s], & t \in [0, t_1],\\ g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i \in \mathbb{N}, &\\ B^{-1}[S_{\alpha,\beta}(t - s_i)Bg_i(s_i, \quad x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)h(s)\,\mathrm{d}s], &\\ \quad t \in (s_i, t_{i+1}], i = 1, \dots, m. & \end{cases}$$

Based on the above discussion, we give the concept of mild solutions of (2).

Definition 4. A function $x \in PC_{1-\gamma}(J,E)$ is called a mild solution of problem (2) if there is $f \in L^p(J,E)$ such that $f(t) \in F(t,x(t))$ for a.e. $t \in J$ and

$$x(t) = \begin{cases} B^{-1}[S_{\alpha,\beta}(t)(Bg(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t K_\alpha(t-s)f(s)\,\mathrm{d}s], \quad t \in [0,t_1], \\ g_i(t,x(t_i^-)),\ t \in (t_i,s_i], \quad i = 1,\dots,m, \\ B^{-1}[S_{\alpha,\beta}(t-s_i)Bg_i(s_i,x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)f(s)\,\mathrm{d}s], \\ \quad t \in (s_i,t_{i+1}],\ i = 1,\dots,m. \end{cases}$$

Now we prove the following lemma.

Lemma 6. *Let $F : J \times E \to P_{\mathrm{ck}}(E)$ be a multifunction satisfying conditions* $(\mathrm{F})_1$ *and* $(\mathrm{F})_2$. *Then, for every $x \in PC_{1-\gamma}(J,E)$, there is a $f \in L^p(J,E)$ such that $f(t) \in F(t,x(t))$ for a.e. $t \in J$.*

Proof. Let $x \in PC_{1-\gamma}(J,E)$. Then one can find a sequence of step functions (v_n) that converges uniformly to x. Hence, see $(\mathrm{F})_1$ for any $n \geqslant 1$, there a strongly measurable function h_n satisfying $h_n(t) \in F(t,v_n(t))$ for a.e. $t \in J$. Moreover, since (v_n) converges uniformly to x, the set $\overline{\{v_n(t)\colon n \geqslant 1\}}$, $t \in J$, is compact. It follows from the upper semicontinuity of $F(t,\cdot)$ that the set $C(t) = \cup\{F(t,v_n(t))\colon n \geqslant 1\}$ is relatively compact for a.e. $t \in J$. Note that $h_n(t) \in C(t)$ for a.e. $t \in J$. Furthermore, from $(\mathrm{F})_2$ the sequence (h_n) is integrably bounded by an $L^p(J,\mathbb{R}^+)$-integrable function. Therefore, (h_n) is weakly compact in $L^p(J,E)$. Let $h_n \rightharpoonup f$. From Mazur's lemma, for every natural number j, there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = k_0(j),\dots,j$, such that $\sum_{k=j}^{k_0} \lambda_{j,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0} \lambda_{j,k} h_k$, $j \geqslant 1$, converges strongly to f in $L^p(J,E)$ as $j \to \infty$, and then $z_j(t) \to f(t)$ for a.e. $t \in J$ up to a subsequence. Since F has compact values, the upper semicontinuity of $F(t,\cdot)$ for a.e. $t \in J$ implies $F(t,v_n(t)) \subseteq F(t,x(t)) + B_\epsilon$ for a.e. $t \in J$ and for large n, here $\epsilon > 0$ is given, and $B_\epsilon = \{y \in E\colon \|y\| < \epsilon\}$. Thus, $h_n(t) \in F(t,x(t)) + B_\epsilon$ for a.e. $t \in J$ and for large n. From the convexity of the values of F, $z_n(t) \in F(t,x(t)) + B_\epsilon$ for a.e. $t \in J$ and for large n. Then $f(t) \in F(t,x(t))$ for a.e. $t \in J$. $\square$

Theorem 1. *Let $F : J \times E \to P_{\mathrm{ck}}(E)$ be a multifunction, $A, B : E \to E$ the linear closed operators, $g_i : [t_i,s_i] \times E \to E$, $i = 1,2,\dots,m$, and p be a real number such that $p > 1/\alpha$. If conditions* (F), (H_1), (H_g) *and* (H) *are satisfied, then problem* (2) *admits a solution on J, provided that*

$$\liminf_{n\to\infty} \frac{\|\Omega(n)\|}{n} = \upsilon < \infty, \tag{11}$$

$$\|B^{-1}\|\left[\frac{Mw}{\Gamma(\gamma)} + \frac{M\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} + h + \frac{hM}{\Gamma(\gamma)}\right] < 1 \tag{12}$$

and

$$\frac{(\kappa_1+\kappa_2)\|B^{-1}\|}{\Gamma(\gamma)}+2\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\eta\|\varsigma\|_{L^p(J,\mathbb{R}^+)}+\kappa_2\|B^{-1}\|<1, \tag{13}$$

where $h=\sum_{i=1}^{i=m}h_i$ *and* $\eta=b^{\alpha-1/p}((p-1)/(p\alpha-1))^{(p-1)/p}$.

Proof. From Lemma 6 we can define a multifunction $R: PC_{1-\gamma}(J,E)\to 2^{PC_{1-\gamma}(J,E)}$ as follows: let $x\in PC_{1-\gamma}(J,E)$ and a function $y\in R(x)$ if and only if

$$y(t)=\begin{cases}B^{-1}[S_{\alpha,\beta}(t)(Bg(x))+\frac{1}{\Gamma(\alpha)}\int_0^t K_\alpha(t-s)f(s)\,\mathrm{d}s], \quad t\in(0,t_1],\\ g_i(t,x(t_i^-)), \quad t\in(t_i,s_i],\ i=1,\dots,m,\\ B^{-1}[S_{\alpha,\beta}(t-s_i)Bg_i(s_i,x(t_i^-))+\int_{s_i}^t K_\alpha(t-s)f(s)\,\mathrm{d}s],\\ \quad t\in(s_i,t_{i+1}], i=1,\dots,m,\end{cases}$$

where $f\in S^p_{F(\cdot,x(\cdot))}=\{f\in L^p(J,E){:}\ f(t)\in F(t,x(t)),\ \text{a.e. } t\in J\}$.

We prove, using Lemma 3, that R has a fixed point. The proof will be given in several steps. First, note that the values of R are convex.

Step 1. In this step, we claim that there is a natural number n such that $R(B_n)\subseteq B_n$, where $B_n=\{x\in PC_{1-\gamma}(J,E){:}\ \|x\|_{PC_{1-\gamma}(J,E)}\leqslant n\}$. Suppose the contrary. Then, for any natural number n, there are $x_n,y_n\in PC_{1-\gamma}(J,E)$ with $y_n\in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)}\leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)}>n$. Then, for any $n\in\mathbb{N}$, there is a $f_n\in S^p_{F(\cdot,x_n(\cdot))}$ such that

$$y_n(t)=\begin{cases}B^{-1}[S_{\alpha,\beta}(t)(Bg(x_n))+\frac{1}{\Gamma(\alpha)}\int_0^t K_\alpha(t-s)f_n(s)\,\mathrm{d}s], \quad t\in(0,t_1],\\ g_i(t,x_n(t_i^-)), \quad t\in(t_i,s_i], i=1,\dots,m,\\ B^{-1}[S_{\alpha,\beta}(t-s_i)Bg_i(s_i,x_n(t_i^-))+\int_{s_i}^t K_\alpha(t-s)f_n(s)\,\mathrm{d}s],\\ \quad t\in(s_i,t_{i+1}], i=1,\dots,m.\end{cases} \tag{14}$$

It follows from (3), (4), (14), $(\mathrm{H}_g)_1$, $(\mathrm{F})_2$ and Hölder's inequality that

$$\begin{aligned}&\sup_{t\in[0,t_1]}t^{1-\gamma}\|y_n(t)\|\\ &\quad\leqslant\|B^{-1}\|\sup_{t\in[0,t_1]}t^{1-\gamma}\|S_{\alpha,\beta}(t)\big(Bg(x_n)\big)\|\\ &\qquad+\sup_{t\in[0,t_1]}\frac{\|B^{-1}\|Mt^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)\varOmega\big(s^{1-\gamma}\|x_n(s)\|\big)\,\mathrm{d}s\\ &\quad\leqslant\frac{M\|B^{-1}\|}{\Gamma(\gamma)}\varPsi(n)+\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\varOmega\big(\|x_n\|_{PC_{1-\gamma}(J,E)}\big)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta\\ &\quad\leqslant\frac{M\|B^{-1}\|}{\Gamma(\gamma)}\varPsi(n)+\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\varOmega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta.\end{aligned} \tag{15}$$

Let $i \in \{1, 2, \dots, m\}$. From (14), $(\mathrm{H})_1$ and Remark 1(ii) (buvo (19)) we have

$$\begin{aligned}
&\sup_{t\in[t_i,s_i]} \|y_n(t)\| \\
&\quad = \sup_{t\in[t_i s_i]} \|g_i(t, x_n(t_i^-))\| \leqslant \|B^{-1}\| \sup_{t\in[t_i,s_i]} \|B^{-1}Bg_i(t, x_n(t_i^-))\| \\
&\quad \leqslant h\|B^{-1}\|(t_i - s_{i-1})^{1-\gamma}\|x_n(t_i^-)\| \\
&\quad \leqslant \|B^{-1}\|h\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant \|B^{-1}\|hn. \qquad (16)
\end{aligned}$$

Also, from (14), $(\mathrm{H})_1$, $(\mathrm{H}_g)_1$, $(\mathrm{F})_2$ and Hölder inequality, for $t \in (s_i, t_{i+1}]$, we get

$$\begin{aligned}
&\sup_{t\in[s_i,t_{i+1}]} (t - s_i)^{1-\gamma}\|y_n(t)\| \\
&\quad \leqslant \sup_{t\in[s_i,t_{i+1}]} \frac{\|B^{-1}\|M\|Bg_i(s_i, x_n(t_i^-))\|}{\Gamma(\gamma)} + \frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\quad \leqslant \frac{\|B^{-1}\|Mhn}{\Gamma(\gamma)} + \frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta. \qquad (17)
\end{aligned}$$

From (15), (16) and (17) it follows that

$$\begin{aligned}
n &< \|y_n\|_{PC_{1-\gamma}(J,E)} \\
&\leqslant \frac{M\|B^{-1}\|}{\Gamma(\gamma)}\Psi(n) + \frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\quad + \|B^{-1}\|hn + \frac{\|B^{-1}\|Mhn}{\Gamma(\gamma)}.
\end{aligned}$$

By dividing both sides by n, taking into account (11) and passing to the limit as $n \to \infty$, we obtain

$$1 \leqslant \|B^{-1}\|\left[\frac{Mw}{\Gamma(\gamma)} + \frac{M\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} + h + \frac{hM}{\Gamma(\gamma)}\right],$$

which contradicts (12).

Thus we deduce that there is a natural number n_0 such that $R(B_{n_0}) \subseteq B_{n_0}$.

Step 2. Let $K = \{z \in PC_{1-\gamma}(J,E)\colon z \in R(B_{n_0})\}$. We claim that the subsets $K_{|\overline{J}_k}$ $(k = 0, 1, \dots, m)$ and $K_{\overline{T}_i}$ $(i = 1, 2, \dots, m)$ are equicontinuous, where $K_{|\overline{J}_k} = \{z : \overline{J}_k \to E, z(t) = (t - s_k)^{1-\gamma}y(t),\ t \in J_k,\ z(s_k) = \lim_{t\to s_k}(t - s_k)^{1-\gamma}z(t),\ y \in R(x),\ x \in B_{n_0}\}$, and $K_{|\overline{T}_i} = \{y^* \in C(\overline{T}_i, E)\colon y^*(t) = y(t),\ t \in (t_i, s_i],\ y^*(t_i) = y(t_i^+),\ y \in R(x),\ x \in B_{n_0}\}$.

Case 1. Let $z \in K_{|\overline{J}_0}$. Then there is a $x \in B_{n_0}$ and a $f \in S^p_{F(\cdot,x(\cdot))}$ such that for $t \in (0, t_1]$, $z(t) = t^{1-\gamma}[B^{-1}S_{\alpha,\beta}(t)(Bg(x)) + B^{-1}\int_0^t K_\alpha(t - s)f(s)\,\mathrm{d}s]$ and $z(0) = \lim_{t\to0+} t^{1-\gamma}y(t)$. It follows for $t = 0$, $\delta \in (0, t_1]$ that $\lim_{\delta\to0+} z(\delta) = \lim_{\delta\to0+} \delta^{1-\gamma}y(\delta) = \lim_{t\to0+} t^{1-\gamma}y(t) = z(0)$.

Let t, $t+\delta$ be two points in $(0, t_1]$. Then

$$\begin{aligned}
&\big\|z(t+\delta) - z(t)\big\| \\
&\quad \leqslant \big\|B^{-1}\big\|\big\|(t+\delta)^{1-\gamma} S_{\alpha,\beta}(t+\delta)\big(Bg(x)\big) - t^{1-\gamma} S_{\alpha,\beta}(t)\big(Bg(x)\big)\big\| \\
&\qquad + \big\|B^{-1}\big\|\left\|(t+\delta)^{1-\gamma}\int_0^{t+\delta} K_\alpha(t+\delta-s)f(s)\,\mathrm{d}s - t^{1-\gamma}\int_0^t K_\alpha(t-s)f(s)\,\mathrm{d}s\right\| \\
&\quad \leqslant \sum_{i=1}^{i=5} I_i,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \big\|B^{-1}\big\|(t+\delta)^{1-\gamma}\big\|S_{\alpha,\beta}(t+\delta)\big(Bg(x)\big) - S_{\alpha,\beta}(t)\big(Bg(x)\big)\big\|, \\
I_2 &:= \big|(t+\delta)^{1-\gamma} - t^{1-\gamma}\big|\big\|S_{\alpha,\beta}(t)\big\|\Psi(n_0), \\
I_3 &:= \big\|B^{-1}\big\|\left\|(t+\delta)^{1-\gamma}\int_t^{t+\delta} K_\alpha(t+\delta-s)f(s)\,\mathrm{d}s\right\|, \\
I_4 &:= \big\|B^{-1}\big\|\left\|\int_0^t \big[(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s)f(s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)f(s)\big]\,\mathrm{d}s\right\|, \\
I_5 &:= \big\|B^{-1}\big\|\left\|\int_0^t \big[t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}K_\alpha(t-s)\big]f(s)\,\mathrm{d}s\right\|.
\end{aligned}$$

In view of Lemma 5, it follows that

$$\lim_{\delta\to 0} I_1 = \lim_{\delta\to 0}(t+\delta)^{1-\gamma}\big\|S_{\alpha,\beta}(t+\delta)\big(Bg(x)\big) - S_{\alpha,\beta}(t)\big(Bg(x)\big)\big\| = 0,$$

and

$$\begin{aligned}
\lim_{\delta\to 0} I_2 &= \lim_{\delta\to 0}\big|(t+\delta)^{1-\gamma} - t^{1-\gamma}\big|\big\|S_{\alpha,\beta}(t)\big\|\Psi(n_0) \\
&\leqslant \frac{M\,t^{\gamma-1}}{\Gamma(\gamma)}\Psi(n_0)\lim_{\delta\to 0}\big|(t+\delta)^{1-\gamma} - t^{1-\gamma}\big| = 0.
\end{aligned}$$

For I_3, from (4), Lemma 5 and $(\mathrm{F})_2$ we get

$$\begin{aligned}
\lim_{\delta\to 0} I_3 &= \big\|B^{-1}\big\|\left\|\lim_{\delta\to 0}\|(t+\delta)^{1-\gamma}\int_t^{t+\delta} K_\alpha(t+\delta-s)f(s)\,\mathrm{d}s\right\| \\
&\leqslant \frac{\|B^{-1}\|M\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}(t+\delta)^{1-\gamma}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s = 0.
\end{aligned}$$

Similarly,

$$\lim_{\delta\to 0} I_4 \leqslant \|B^{-1}\| \lim_{\delta\to 0} \left\| \int_0^t \big[(t+\delta)^{1-\gamma} K_\alpha(t+\delta-s) f(s) - t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s)\big] \,\mathrm{d}s \right\|$$

$$= \|B^{-1}\| \lim_{\delta\to 0} \left\| \int_0^t \big[(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s) - t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s)\big] \,\mathrm{d}s \right\|$$

$$\leqslant \frac{\|B^{-1}\| M \varOmega(n_0)}{\Gamma(\alpha)} \lim_{\delta\to 0} \int_0^t \big|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}\big| \varphi(s) \,\mathrm{d}s.$$

Since $\varphi \in L^p(J, \mathbb{R}^+)$, $\int_0^t [(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}]\varphi(s)\,\mathrm{d}s$ exists, and from the Lebesgue dominated convergence theorem we see that $\lim_{\delta\to 0} I_4 = 0$.

For I_5, note

$$\lim_{\delta\to 0} I_5 = \|B^{-1}\| \lim_{\delta\to 0} \left\| \int_0^t t^{1-\gamma} \big[(t-s)^{\alpha-1} P_\alpha(t+\delta-s) - K_\alpha(t-s)\big] f(s) \,\mathrm{d}s \right\|$$

$$= \|B^{-1}\| \lim_{\delta\to 0} \left\| \int_0^t t^{1-\gamma}(t-s)^{\alpha-1} \big[P_\alpha(t+\delta-s) - P_\alpha(t-s)\big] f(s) \,\mathrm{d}s \right\|.$$

To find this limit, let $\epsilon > 0$ be enough small. We have

$$\begin{aligned}\lim_{\delta\to 0} I_5 \leqslant{}& \|B^{-1}\| \varOmega(n_0) t^{1-\gamma} \\ &\times \lim_{\delta\to 0} \int_0^{t-\epsilon} (t-s)^{\alpha-1} \varphi(s) \sup_{s\in[0,\, t-\epsilon]} \big\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\big\| \,\mathrm{d}s \\ &+ \frac{\|B^{-1}\| 2M \varOmega(n_0)}{\Gamma(\alpha)} \lim_{\delta\to 0} \Bigg[\int_0^t t^{1-\gamma}(t-s)^{\alpha-1} \varphi(s) \,\mathrm{d}s \\ &- \int_0^{t-\epsilon} (t-\epsilon)^{1-\gamma}(t-\epsilon-s)^{\alpha-1} \varphi(s) \,\mathrm{d}s \Bigg]\end{aligned}$$

$$+\frac{\|B^{-1}\|2M\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\Bigg[\int_0^{t-\epsilon}(t-\epsilon)^{1-\gamma}(t-\epsilon-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s$$
$$-\int_0^{t-\epsilon}t^{1-\gamma}(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s\Bigg].$$

From Lemma 5, $\lim_{\delta\to 0}\sup_{s\in[0,t-\epsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\|=0$, and since $\varphi\in L^p(J,\mathbb{R}^+)$, then from the Lebesgue dominated convergence theorem we see that $I_5\to 0$ as $\delta\to 0$ and $\epsilon\to 0$.

Case 2. Let $y\in K_{|T_i}$, $i\in\{1,2,\dots,m\}$ be fixed. Then $y(t)=g_i(t,x(t_i^-))$, $x\in B_{n_0}$. Since $\|x\|_{PC_{1-\gamma}}(J,E)\leqslant n_0$, it follows from the uniform continuity of g_i on bounded sets that for t, $t+\delta\in(t_i,s_i]$, $\lim_{\delta\to 0}\|y(t+\delta)-y(t)\|=\lim_{\delta\to 0}\|g_i(t+\delta,x(t_i^-))-g_i(t,x(t_i^-))\|=0$, independent of x.

When $t=t_i$, let $\delta>0$ and $\lambda>0$ be such that $t_i<\lambda<t_i+\delta\leqslant s_i$, and we have $\|y^*(t_i+\delta)-y^*(t_i)\|=\lim_{\lambda\to t_i^+}\|y(t_i+\delta)-y(\lambda)\|=0$.

Case 3. Let $z\in K_{|J_k}$, $k\in\{1,2,\dots,m\}$ be fixed. Then there is a $x\in B_{n_0}$ and a $f\in S^p_{F(\cdot,x(\cdot))}$ such that for $t\in(s_k,t_{k+1}]$,

$$z(t)=(t-s_k)^{1-\gamma}B^{-1}\Bigg[S_{\alpha,\beta}(t-s_k)Bg_k\big(s_k,x(t_k^-)\big)+\int_{s_k}^{t}K_\alpha(t-s)f_n(s)\,\mathrm{d}s\Bigg].$$

If $t=s_k$ and $\delta>0$, then

$$\begin{aligned}\lim_{\delta\to 0+}z(s_k+\delta)&=\lim_{\delta\to 0+}(s_k+\delta-s_k)^{1-\gamma}y(s_k+\delta)\\&=\lim_{t\to s_k+}(t-s_k)^{1-\gamma}y(t)=z(s_k).\end{aligned}$$

Next, let t, $t+\delta\in(s_i,t_{i+1}]$, $\delta>0$. Then we have

$$\begin{aligned}&\big\|z(t+\delta)-z(t)\big\|\\&=\big\|B^{-1}\big\|\big\|(t+\delta-s_k)^{1-\gamma}B^{-1}S_{\alpha,\beta}(t+\delta-s_k)Bg_k\big(s_k,x(t_k^-)\big)\\&\quad-(t-s_k)^{1-\gamma}S_{\alpha,\beta}(t-s_k)Bg_k\big(s_k,x(t_k^-)\big)\big\|\\&\quad+\big\|B^{-1}\big\|\Bigg\|(t+\delta-s_k)^{1-\gamma}\int_{s_k}^{t}K_\alpha(t+\delta-s)f_n(s)\,\mathrm{d}s\\&\quad-(t-s_k)^{1-\gamma}\int_{s_k}^{t+\delta}K_\alpha(t-s)f_n(s)\,\mathrm{d}s\Bigg\|.\end{aligned}$$

By arguing as in Case 1, we conclude that $\lim_{\delta\to 0}\|z(t+\delta)-z(t)\|=0$.

Step 3. The graph of the multivalued function $R_{|B_{n_0}}: B_{n_0} \to 2^{B_{n_0}}$ is closed. Consider a sequence $\{x_n\}_{n\geqslant 1}$ in B_{n_0} with $x_n \to x$ in B_{n_0} and let $y_n \in R(x_n)$ with $y_n \to y$ in $PC_{1-\gamma}(J, E)$. We need to show that $y \in R(x)$. Recalling the definition of R, for any $n \geqslant 1$, there is a $f_n \in S^p_{F(\cdot,x_n(\cdot))}$ such that (14) holds. In view of (F)$_2$, $\|f_n(t)\| \leqslant \varphi(t)\Omega(n_0)$ for every $n \geqslant 1$ and for a.e. $t \in J$. Then $\{f_n\colon n \geqslant 1\}$ is bounded in $L^p(J, E)$. Because $p > 1$, $L^p(J, E)$ is reflexive, without loss of generality, we can assume that (f_n) converges weakly to a function $f \in L^p(J, E)$. From Mazur's lemma, for every natural number j, there are a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = k_0(j), \dots, j$, such that $\sum_{k=j}^{k_0} \lambda_{j,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0} \lambda_{j,k} f_k$, $j \geqslant 1$, converges strongly to f in $L^1(J, E)$ as $j \to \infty$.

Take $\overline{y}_n(t) = \sum_{k=n}^{k_0(n)} \lambda_{n,k} y_k(t)$, Then

$$\overline{y}_n(t) = \begin{cases} B^{-1}[S_{\alpha,\beta}(t)\big(Bg(x_n)\big) + \int_0^t K_\alpha(t-s) z_n(s)\,\mathrm{d}s], \quad t \in (0, t_1], \\ g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ B^{-1} S_{\alpha,\beta}(t - s_i) B g_i(s_i, x_n(t_i^-)) + B^{-1} \int_{s_i}^t K_\alpha(t-s) z_n(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases}$$

From the continuity of Bg and the uniform continuity of g_i on bounded sets it follows from the Lebesgue dominated convergence theorem that $\overline{y}_n(t) \to v(t)$, where

$$v(t) = \begin{cases} B^{-1}[S_{\alpha,\beta}(t)(Bg(x)) + \int_0^t K_\alpha(t-s) f(s)\,\mathrm{d}s], \quad t \in (0, t_1], \\ g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ B^{-1} S_{\alpha,\beta}(t - s_i) B g_i(s_i, x(t_i^-)) + B^{-1} \int_{s_i}^t K_\alpha(t-s) f(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases}$$

Since $y_n \to y$, then $y = v$. Almost everywhere $F(t, \cdot)$ is upper semicontinuous with closed convex values, so from [2, Chap. 1, Sect. 4, Thm. 1] it follows that $f(t) \in F(t, x(t))$ for a.e. $t \in J$, and hence R is closed.

Step 4. R is $\chi_{PC_{1-\gamma}(J,E)}$-condensing. Suppose that D is a bounded subset of $PC_{1-\gamma}(J, E)$ and $Z = R(D)$. Let $k = 0$. Since $Z_{|\overline{J}_0}$ is equicontinuous, then by Lemma 2 we get $\chi_{C(\overline{J}_k,E)}(Z_{|\overline{J}_0}) = \max_{t\in\overline{J}_0} \chi_E\{y^*(t)\colon y \in Z\}$.

Let $t \in (0, t_1]$. Then, from Lemma 1 we get

$$\begin{aligned} &\chi_E\big\{y^*(t)\colon y \in Z\big\} \\ &\quad = \chi_E\big\{t^{1-\gamma} y(t)\colon y \in Z\big\} \\ &\quad \leqslant \chi_E\big\{t^{1-\gamma} B^{-1} S_{\alpha,\beta}(t)\big(Bg(x)\big)\colon x \in D\big\} \\ &\qquad + \chi_E\left\{t^{1-\gamma} B^{-1} \int_0^t K_\alpha(t-s) f(s)\,\mathrm{d}s\colon f \in S^p_{F(\cdot,x(\cdot))}, x \in D\right\} \end{aligned}$$

$$\begin{aligned}&\leqslant \frac{\|B^{-1}\|}{\Gamma(\gamma)}\chi_E\big\{\big(Bg(x)\big)\colon\ x\in D\big\}\\&\quad+2\frac{\|B^{-1}\|}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\chi_E\big\{f(s)\,\mathrm{d}s\colon\ f\in S^p_{F(\cdot,x(\cdot))},x\in D\big\}\,\mathrm{d}s.\end{aligned}\tag{18}$$

From $(\mathrm{F})_3$, for a.e. $s\in J_0$, we have

$$\begin{aligned}&\chi_E\big\{f(s)\colon\ f\in S^p_{F(\cdot,x(\cdot))},x\in D\big\}\\&\quad\leqslant \chi\big\{F\big(s,x(s)\big)\colon\ x\in D\big\}\leqslant \varsigma(s)s^{1-\gamma}\chi_E\big\{x(s)\colon\ x\in D\big\}\\&\quad=\varsigma(s)\chi_E\big\{s^{1-\gamma}x(s)\colon\ x\in D\big\}\leqslant \varsigma(s)\chi_{PC_{1-\gamma}(J,E)}(D).\end{aligned}\tag{19}$$

It follows from (6), (18) and (19) that

$$\begin{aligned}&\chi_E\big\{y^*(t)\colon\ y\in Z\big\}\\&\quad\leqslant \chi_{PC_{1-\gamma}(J,E)}(D)\left[\frac{\kappa_1\|B^{-1}\|}{\Gamma(\gamma)}+2\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\varsigma(s)\,\mathrm{d}s\right]\\&\quad\leqslant \chi_{PC_{1-\gamma}(J,E)}(D)\left[\frac{\kappa_1\|B^{-1}\|}{\Gamma(\gamma)}+2\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\eta\|\varsigma\|_{L^p(J,\mathbb{R}^+)}\right].\end{aligned}\tag{20}$$

If $t=0$, then

$$\begin{aligned}&\chi_E\big\{y^*(0)\colon\ y\in Z\big\}\\&\quad=\chi_E\Big\{\lim_{t\to 0^+}t^{1-\gamma}y(t)\colon\ y\in Z\Big\}=\big\|B^{-1}\big\|\chi_E\big\{\big(Bg(x)\big)\colon\ x\in D\big\}\\&\quad\leqslant \kappa_1\big\|B^{-1}\big\|\chi_{PC_{1-\gamma}(J,E)}(D).\end{aligned}$$

Now, let $i\in\{1,\dots,m\}$ and $t\in(t_i,s_i]$. From (7) we have

$$\begin{aligned}&\chi_E\big\{g_i\big(t,x(t_i^-)\big)\colon\ x\in D\big\}\\&\quad=\chi_E\big\{B^{-1}Bg_i\big(t,x(t_i^-)\big)\colon\ x\in D\big\}\leqslant \big\|B^{-1}\big\|\chi_E\big\{Bg_i\big(t,x(t_i^-)\big)\colon\ x\in D\big\}\\&\quad\leqslant \kappa_2\big\|B^{-1}\big\|\chi_{PC_{1-\gamma}(J,E)}(D).\end{aligned}\tag{21}$$

Arguing as above, if $i\in\{1,\dots,m\}$ and $t\in(s_i,t_{i+1}]$, then

$$\begin{aligned}&\chi_E\big\{y^*(t)\colon\ y\in Z\big\}\\&\quad=\chi_E\big\{(t-s_i)^{1-\gamma}y(t)\colon\ y\in Z\big\}\\&\quad\leqslant \chi_E\big\{(t-s_i)^{1-\gamma}B^{-1}S_{\alpha,\beta}(t-s_i)Bg_i\big(s_i,x(t_i^-)\big)\colon\ x\in D\big\}\\&\qquad+\chi_E\left\{(t-s_i)^{1-\gamma}B^{-1}\int\limits_0^t K_\alpha(t-s)f(s)\,\mathrm{d}s\colon\ f\in S^p_{F(\cdot,x(\cdot))},\ x\in D\right\}\\&\quad\leqslant \frac{\|B^{-1}\|}{\Gamma(\gamma)}\kappa_2\chi_{PC_{1-\gamma}(J,E)}(D)+2\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\eta\|\varsigma\|_{L^p(J,\mathbb{R}^+)}.\end{aligned}\tag{22}$$

It follows from (18)–(21) and (22) that

$$\begin{aligned}&\chi_{PC_{1-\gamma}(J,E)}\big(R(D)\big)\\&\quad\leqslant \chi_{PC_{1-\gamma}(J,E)}(D)\left[\frac{(\kappa_1+\kappa_2)\|B^{-1}\|}{\Gamma(\gamma)}+\frac{2\eta\|\varsigma\|_{L^p(J,\mathbb{R}^+)}\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}+\kappa_2\|B^{-1}\|\right].\end{aligned}$$

This inequality with (13) establishes that R is $\chi_{PC_{1-\gamma}(J,E)}$-condensing. Note that since R is $\chi_{PC_{1-\gamma}(J,E)}$-condensing, then $\chi_{PC_{1-\gamma}(J,E)}(R(x)) < \chi_{PC_{1-\gamma}(J,E)}\{x\} = 0$, and hence $R(x)$ is relatively compact. Furthermore, by arguing as in Step 3, one can see that the values of R is closed. Therefore, the values of R is compact. From Lemma 3 we conclude that (2) has a mild solution. □

Theorem 2. *Assume that the hypotheses of Theorem* 1 *with* (F)$_2$ *replaced by the following condition:* (F)$_4$ *for any natural number n there is a function $\varphi_n \in L^p(J,\mathbb{R}^+)$ such that* $\sup_{\|x\|\leqslant n}\|F(t,x)\| \leqslant \varphi_n(t)$ *for a.e.* $t\in J$ *and* $\liminf_{n\to\infty}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}/n = 0$. *Then problem* (2) *has a mild solution, provided* (13) *and*

$$\|B^{-1}\|\left[\frac{Mw}{\Gamma(\gamma)}+h+\frac{hM}{\Gamma(\gamma)}\right] < 1. \tag{23}$$

Proof. We only have to prove that there is a natural number n such that $R(B_n)\subseteq B_n$, where $B_n = \{x\in PC_{1-\gamma}(J,E)\colon \|x\|_{PC_{1-\gamma}(J,E)}\leqslant n\}$. Suppose the contrary. Then, for any natural n, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n\in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)}\leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$. Then there is a $f_n\in S^p_{F(\cdot,x_n(\cdot))}$, $n\geqslant 1$, such that (14) holds. Let $t\in[0,t_1]$. As in Step 1, it follows from (3), (4), (14), (H$_g$)$_1$, (F)$_4$ and Hölder's inequality that

$$\begin{aligned}\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| &\leqslant \|B^{-1}\|\sup_{t\in[0,t_1]} t^{1-\gamma}\big\|S_{\alpha,\beta}(t)\big(Bg(x_n)\big)\big\|\\&\quad+\sup_{t\in[0,t_1]}\frac{\|B^{-1}\|Mt^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_n(s)\,\mathrm{d}s\\&\leqslant \frac{M\|B^{-1}\|}{\Gamma(\gamma)}\Psi(n)+\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta.\end{aligned}$$

Let $i\in\{1,2,\dots,m\}$ and, as in (16), we get $\sup_{t\in[t_i,s_i]}\|y_n(t)\|\leqslant\|B^{-1}\|hn$. Also, by (14), (H)$_1$, (H$_g$)$_1$, (F)$_2$ and Hölder inequality, we get for $t\in(s_i,t_{i+1}]$

$$\begin{aligned}&\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^{1-\gamma}\|y_n(t)\|\\&\quad\leqslant \sup_{t\in[s_i,t_{i+1}]}\frac{\|B^{-1}\|M\|Bg_i(s_i,x_n(t_i^-))\|}{\Gamma(\gamma)}+\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta\\&\quad\leqslant \frac{\|B^{-1}\|Mhn}{\Gamma(\gamma)}+\frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta.\end{aligned}$$

It follows that $n < \|y_n\|_{PC_{1-\gamma}(J,E)} \leqslant (M\|B^{-1}\|/\Gamma(\gamma))\Psi(n) + (\|B^{-1}\|Mb^{1-\gamma}/\Gamma(\alpha))\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta + \|B^{-1}\|hn + \|B^{-1}\|Mhn/\Gamma(\gamma)$. By dividing both sides by n and passing to the limit as $n \to \infty$, we obtain $1 \leqslant \|B^{-1}[\|Mw/\Gamma(\gamma) + h + hM/\Gamma(\gamma)]$, which contradicts (23). Thus we deduce that there is a natural number n_0 such that $R(B_{n_0}) \subseteq B_{n_0}$. □

Theorem 3. *Assume the hypotheses of Theorem* 2 *with* (H_g) *replaced by the condition*

(H_g^*) *$g : PC_{1-\gamma}(J,E) \to D(B)$ is such that Bg is Lipschitz with Lipschitz constant k.*

Then problem (2) *has a solution, provided that*

$$\|B^{-1}\|\left[\frac{Mk}{\Gamma(\gamma)} + h + \frac{Mh}{\Gamma(\gamma)}\right] < 1, \tag{24}$$

and (13) *with $\kappa_1 = k$ are satisfied.*

Proof. Since Bg is Lipschitz with constant k, it follows that Bg is continuous and for any bounded subset D of $PC_{1-\gamma}(J,E)$, $\chi_E(Bg(D)) \leqslant \kappa\chi_{PC_{1-\gamma}(J,E)}(D)$. We only have to prove that there is a natural number n such that $R(B_n) \subseteq B_n$, where $B_n = \{x \in PC_{1-\gamma}(J,E)\colon \|x\|_{PC_{1-\gamma}(J,E)} \leqslant n\}$. Suppose the contrary. Then, for any natural number n, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$. Then there is a $f_n \in S^p_{F(\cdot,x_n(\cdot))}$, $n \geqslant 1$, such that (14) holds. We have

$$\begin{aligned}
&\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| \\
&\quad\leqslant \|B^{-1}\| \sup_{t\in[0,t_1]} t^{1-\gamma}\|S_{\alpha,\beta}(t)(Bg(x_n))\| \\
&\qquad + \sup_{t\in[0,t_1]} \frac{\|B^{-1}\|Mt^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi_n(s)\,\mathrm{d}s \\
&\quad\leqslant \frac{M\|B^{-1}\|}{\Gamma(\gamma)}\left(\|Bg(x_n) - Bg(0)\| + \|Bg(0)\|\right) + \frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\quad\leqslant \frac{M\|B^{-1}\|}{\Gamma(\gamma)}\left(kn + \|Bg(0)\|\right) + \frac{\|B^{-1}\|Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta.
\end{aligned}$$

Thus, $n < \|y_n\|_{PC_{1-\gamma}(J,E)} \leqslant \|B^{-1}\|[(M/\Gamma(\gamma))(kn + \|Bg(0)\|) + (Mb^{1-\gamma}/\Gamma(\alpha)) \times \|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta + hn + Mhn/\Gamma(\gamma)]$. By dividing both sides by n and passing to the limit as $n \to \infty$, we get $1 \leqslant \|B^{-1}\|[Mk/\Gamma(\gamma) + h + Mh/\Gamma(\gamma)]$, which contradicts (24). □

Remark 3. According to [7, Lemma 3.2], if B^{-1} is compact, then the operator $B^{-1}P_\alpha(t)$, $t > 0$, is compact, and hence we do not need (F)$_3$ in Theorems 1–3.

4 Globally attracting solutions

In this section, we establish the existence of globally attracting solutions for (1). Consider the Banach space $PC^0_{1-\gamma}([0,\infty),E) = \{x \in PC_{1-\gamma}([0,\infty),E)\colon\ \lim_{t\to\infty} x(t) = 0\}$ with $\|x\|_\infty = \max\{\max_{k=\{0\}\cup\mathbb{N}} \sup_{t\in J_k}(t-s_k)^{1-\gamma}\|x(t)\|_E, \max_{i\in\mathbb{N}}\sup_{t\in T_i}\|x(t)\|_E\}$.

Moreover, we define a measure of noncompactness on $PC^0_{1-\gamma}([0,\infty),E)$ as $\chi^*\colon P_{\mathrm{b}}(PC^0_{1-\gamma}([0,\infty),E)) \to [0,\infty)$, $\chi^*(Z) = \chi_\infty(Z) + d_\infty(Z)$, where $\chi_\infty(Z) = \sup_{i\in\mathbb{N}} \chi_{PC_{1-\gamma}([0,t_i],E)}\{x_{|[0,t_i]}\colon\ x\in Z\}$ and $d_\infty(Z) = \lim_{m\to\infty}\sup_{x\in Z} d_m(x)$,

$$d_m(x) = \max\Big\{\max_{k\geqslant m}\sup_{t\in J_k}(t-s_k)^{1-\gamma}\big\|x(t)\big\|_E,\ \max_{i\geqslant m}\sup_{t\in T_i}\big\|x(t)\big\|_E\Big\}.$$

We show, in the following proposition, that χ^* is regular.

Proposition 1. *χ^* is regular on $PC^0_{1-\gamma}([0,\infty),E)$.*

Proof. Let Z be a bounded subset of $PC^0_{1-\gamma}([0,\infty),E)$ satisfying $\chi^*(Z) = 0$ and $\epsilon > 0$. Because $d_\infty(Z) = 0$, there is a natural number m_0 such that

$$d_{m_0}(x) < \frac{\epsilon}{2} \quad \forall x \in Z. \tag{25}$$

Since $\chi_\infty(Z) = 0$, $\chi_{PC([0,t_{m_0}],E)}\{x_{|[0,t_{m_0}]}\colon\ x\in Z\} = 0$.

It follows that $\{x_{|T_{m_0}}\colon\ x \in Z\}$ is relatively compact, and hence there are $u_r \in PC([0,t_{m_0}],E)$, $r = 1,2,\dots,N$, such that

$$\{x_{|[0,t_{m_0}]}\colon\ x\in Z\} \subseteq \bigcup_{r=1}^{r=N} B\left(u_r, \frac{\epsilon}{2}\right), \tag{26}$$

where $B(u_r,\epsilon/2)$ denotes a ball in $PC_{1-\gamma}([0,t_{m_0}],E)$ centered at u_r with radius $\epsilon/2$. For any $r = 1,2,\dots,N$, define

$$\widehat{u}_r(t) = \begin{cases} u_r(t), & t\in[0,t_{m_0}],\\ 0, & t\notin[0,t_{m_0}].\end{cases}$$

We now show that $Z \subseteq \cup_{r=1}^{r=N} B_\infty(\widehat{u}_r,\epsilon)$, where $B_\infty(\widehat{u}_r,\epsilon)$ denotes a ball in $PC^0_{1-\gamma}([0,\infty),E)$ centered at $\widehat{u}_r$ with radius $\epsilon/2$. Let $x\in Z$. From (26) there is a $r_0 \in \{1,2,\dots,N\}$ such that

$$\|x_{|[0,t_{m_0}]} - u_{r_0}\|_{PC_{1-\gamma}([0,t_{m_0}],E)} \leqslant \frac{\epsilon}{2}. \tag{27}$$

From (25) and (27) we have

$$\|x - \widehat{u}_{r_0}\|_\infty \leqslant \|x_{|[0,t_{m_0}]} - u_{r_0}\|_{PC_{1-\gamma}([0,t_{m_0}],E)} + d_{m_0}(x) \leqslant \epsilon,$$

which completes the proof. □

Definition 5. A function $x \in PC^0_{1-\gamma}([0,\infty),E)$ is called a globally attracting mild solution of problem (1) if there is a $f \in L^p_{\mathrm{loc}}([0,\infty),E)$ such that $f(t) \in F(t,x(t))$ for a.e. $t \in [0,\infty)$ and

$$x(t) = \begin{cases} B^{-1}[S_{\alpha,\beta}(t)(Bg(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t K_\alpha(t-s)f(s)\,\mathrm{d}s], & t \in [0,t_1] \\ g_i(t,x(t_i^-)), \quad t \in (t_i,s_i],\ i \in \mathbb{N}, \\ B^{-1}[S_{\alpha,\beta}(t-s_i)Bg_i(s_i,x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)f(s)\,\mathrm{d}s], \\ \quad t \in (s_i,t_{i+1}], i \in \mathbb{N}, \end{cases}$$

and $\lim_{t\to\infty} x(t) = 0$.

Consider the following assumptions:

(F^*) $F : [0,\infty) \times E \to P_{\mathrm{ck}}(E)$ is a multifunction satisfying:

1. For every $x \in PC^0([0,\infty),E)$, there is a $f \in L^p_{\mathrm{loc}}([0,\infty),E)$ such that $f(t) \in F(t,x(t))$ for a.e. $t \in [0,\infty)$, and for a.e. $t \in [0,\infty)$, $z \to F(t,z)$ is upper semicontinuous.
2. There exist $\varphi \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R}^+)$ satisfying $\sup_{i\in N}(\int_{s_i}^{t_i+1}|\varphi(s)|^p\,\mathrm{d}s)^{1/p} = \sigma < \infty$, and for any $i = \{0\} \cup \mathbb{N}$ and any $x \in PC^0_{1-\gamma}([0,\infty),E)$,

$$\big\|F\big(t,x(t)\big)\big\| \leqslant \varphi(t)(t-s_i)^{1-\gamma}\|x(t)\| \quad \text{for } t \in (s_i,t_{i+1}]. \tag{28}$$

3. There exists $\varsigma \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R}^+)$ satisfying $\sup_{i\in N}(\int_{s_i}^{t_i+1}|\varsigma(s)|^p\,\mathrm{d}s)^{1/p} := \xi < \infty$ and such that for any bounded subset $D \subseteq E$ and any $k \in \{0\} \cup \mathbb{N}$, $\chi_E(F(t,D)) \leqslant (t-s_k)^{1-\gamma}\varsigma(t)\chi_E(D)$ for a.e. $t \in J_k$, where χ is the Hausdorff measure of noncompactness on E.

(H_g^{**}) The function $g : PC^0_{1-\gamma}([0,\infty),E) \to D(B)$ satisfies the following conditions:

1. $Bg : PC^0([0,\infty),E) \to E$ is continuous, and there is a continuous nondecreasing function $\Psi : [0,\infty) \to [0,\infty)$ such that $\|Bg(x)\| \leqslant \Psi(\|x\|_\infty)$ and (5) is satisfied.
2. There is a $\kappa_1 > 0$ such that for any bounded subset D of $PC^0_{1-\gamma}([0,\infty),E)$, $\chi_E(Bg(D)) \leqslant \kappa_1\chi^*(D)$.

(H^*) The function $g_i : [t_i,s_i] \times E \to D(B)$, $i \in \mathbb{N}$, is uniformly continuous on bounded sets and satisfies:

1. For any $i \in \mathbb{N}$, there exists a positive constant h_i such that $\sum_{i=1}^\infty h_i = h < \infty$, and for any $x \in E$, $\|Bg_i(t,x))\| \leqslant h_i(t_i-s_{i-1})^{1-\gamma}\|x\|$, $t \in [t_i,s_i]$, $x \in E$.
2. There is a $\kappa_2 > 0$ such that for any bounded subset D of $PC_{1-\gamma}([0,\infty),E)$, $\chi_E(Bg_i(t,\{z(t_i)\colon z \in D\})) \leqslant \kappa_2\chi^*(D)$, $t \in [t_i,s_i]$, $i \in \mathbb{N}$.

Theorem 4. *Let $A : D(A) \subseteq E \to E$, $B\colon D(B) \subseteq E \to E$ be closed linear operators such that $D(B) \subseteq D(A)$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} < \cdots$, $g_i : [t_i,s_i] \times E \to E (i = 1,2,\ldots,m)$, and p be a real number such that $p > 1/\alpha$.*

Suppose (F^*), (H_1), (H_g^{**}) *and* (H^*) *are satisfied. Then problem* (1) *admits a solution* $x : [0,\infty) \to E$ *satisfying* $\lim_{t\to\infty} x(t) = 0$*, provided that*

$$\sup_{i\in\{0\}\cup\mathbb{N}} |t_{i+1} - s_i| = l < \infty, \tag{29}$$

$$\|B^{-1}\|\left[\frac{M(h+w)}{\Gamma(\gamma)} + \frac{\|B^{-1}\|Ml^{1-\gamma+\alpha-1/P}}{\Gamma(\alpha)}\sigma\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} + h\right] < 1 \tag{30}$$

and

$$\begin{aligned}\|B^{-1}\|\Bigg[&l^{1-\gamma+\alpha-1/p}\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}\sigma + \frac{\kappa_1+\kappa_2}{\Gamma(\gamma)}\\ &+ \frac{2l^{\alpha-1/p}(\frac{p-1}{p\alpha-1})^{(p-1)/p}\xi}{\Gamma(\alpha)} + \kappa_2\Bigg] < 1.\end{aligned} \tag{31}$$

Proof. Let $x \in PC^0([0,\infty),E)$. From $(F^*)_1$, there is a $f \in L^p_{\mathrm{loc}}([0,\infty),E)$ such that $f(t) \in F(t,x(t))$ for a.e. $t \in [0,\infty)$. Then we can define a multifunction R^* on $PC^0([0,\infty),E)$ as follows: a function $y \in R^*(x)$ if and only if

$$y(t) = \begin{cases} B^{-1}[S_{\alpha,\beta}(t)(Bg(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t K_\alpha(t-s)f(s)\,\mathrm{d}s], \quad t\in[0,t_1]\\ g_i(t,x(t_i^-)), \quad t\in(t_i,s_i],\ i\in\mathbb{N},\\ B^{-1}[S_{\alpha,\beta}(t-s_i)Bg_i(s_i,x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)f(s)\,\mathrm{d}s],\\ \quad t\in(s_i,t_{i+1}],\ i\in\mathbb{N}.\end{cases}$$

Step 1. We show that $R^*(PC^0([0,\infty),E)) \subseteq PC^0([0,\infty),E)$. Let $y \in R^*(x), x \in PC^0([0,\infty),E)$. We prove that $\lim_{t\to\infty} y(t) = 0$. Let $\epsilon > 0$. Since $\sum_{k=1}^\infty h_k < \infty$, there is a natural number $N_1 > 1$ such that

$$\sum_{k=N_1}^\infty h_k < \frac{\epsilon}{2\|B^{-1}\|\frac{M}{\Gamma(\gamma)}\|x\|_\infty}. \tag{32}$$

Now $x \in PC^0([0,\infty),E)$ implies that $\lim_{t\to\infty}\|x(t)\| = 0$, and so there is a natural number N_2 such that

$$\|x(t)\| < \frac{\epsilon}{2\|B^{-1}\|l^{2-2\gamma+\alpha-\frac{1}{p}}\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{(p-1)/p}\sigma} \quad \forall t > N_2. \tag{33}$$

Now let i be such that i and t_i are greater than $\max\{N_1,N_2\}$. If $t \in (t_i,s_i]$, then from $(H^*)_1$ and (32) we have

$$\begin{aligned}\|y(t)\| &\leqslant \|g_i(t,x(t_i^-))\| \leqslant \|B^{-1}\|\|Bg_i(t,x(t_i^-))\|\\ &\leqslant \|B^{-1}\|(t_i-s_{i-1})^{1-\gamma}\|x(t_i^-)\|h_i \leqslant \|B^{-1}\|\|x\|_\infty\sum_{k=i}^\infty h_k < \frac{\epsilon}{2}.\end{aligned} \tag{34}$$

If $t \in (s_i, t_{i+1}]$, then from (3), (4), (F*)$_2$, (28), (32), (33) and Hölder's inequality we get

$$
\begin{aligned}
&(t-s_i)^{1-\gamma}\|y(t)\| \\
&\quad\leqslant (t-s_i)^{1-\gamma}\|B^{-1}\|\|S_{\alpha_\beta}(t-s_i)\|\|Bg_i(s_i, x(t_i^-))\| \\
&\qquad + \|B^{-1}\|(t-s_i)^{1-\gamma}\int\limits_{s_i}^{t}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\varphi(s)\|(s-s_i)^{1-\gamma}x(s)\|\,\mathrm{d}s \\
&\quad\leqslant (t-s_i)^{1-\gamma}\|B^{-1}\|\frac{M(t-s_i)^{\gamma-1}}{\Gamma(\gamma)}\|x\|_\infty\sum_{k=i}^{\infty}h_k \\
&\qquad + \|B^{-1}\|\frac{M}{\Gamma(\alpha)}(t_{i+1}-s_i)^{2(1-\gamma)}\sup_{s\in[s_i,t_{i+1}]}\|x(s)\|\int\limits_{s_i}^{t}(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\
&\quad\leqslant \|B^{-1}\|\frac{M}{\Gamma(\gamma)}\|x\|_\infty\sum_{k=i}^{\infty}h_k \\
&\qquad + \|B^{-1}\|l^{2(1-\gamma)}\frac{M}{\Gamma(\alpha)}(t_{i+1}-s_i)^{\alpha-1/p}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}\sigma\sup_{s\in[s_i,t_{i+1}]}\|x(s)\| \\
&\quad\leqslant \frac{\epsilon}{2} + \|B^{-1}\|l^{2-2\gamma+\alpha-1/p}\frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}\sigma\sup_{s\in[s_i,t_{i+1}]}\|x(s)\| \\
&\quad\leqslant \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

From this inequality and (34) we conclude that $y \in PC^0_{1-\gamma}([0,\infty),E)$, and hence $R^*(PC^0_{1-\gamma}([0,\infty),E)) \subseteq PC^0_{1-\gamma}([0,\infty),E)$.

Step 2. In this step, we claim that there is a natural number n such that $R^*(B_n) \subseteq B_n$, where $B_n = \{x \in PC^0_{1-\gamma}([0,\infty),E)\colon \|x\|_\infty \leqslant n\}$. Suppose the contrary. Then, for any natural number n, there are $x_n, y_n \in PC^0_{1-\gamma}([0,\infty),E)$ with $y_n \in R^*(x_n)$, $\|x_n\|_\infty \leqslant n$ and $\|y_n\|_\infty > n$. Then there is a $f_n \in L^p_{\mathrm{loc}}([0,\infty),E)$ such that $f_n(t) \in F(t, x_n(t))$ for a.e. $t \in [0,\infty)$ and

$$
y_n(t) = \begin{cases}
B^{-1}[S_{\alpha,\beta}(t)(Bg(x_n)) + \frac{1}{\Gamma(\alpha)}\int_0^t K_\alpha(t-s)f_n(s)\,\mathrm{d}s], & t \in (0,t_1], \\
g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i \in \mathbb{N}, \\
B^{-1}[S_{\alpha,\beta}(t-s_i)Bg_i(t, x_n(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)f_n(s)\,\mathrm{d}s], \\
\quad t \in (s_i, t_{i+1}],\ i \in \mathbb{N}.
\end{cases}
$$

Then we get from (3), (4), (H$^*_g$)$_1$, (F*)$_2$, (29) and Hölder's inequality that

$$
\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| \leqslant \|B^{-1}\|\sup_{t\in[0,t_1]} t^{1-\gamma}\|S_{\alpha,\beta}(t)(Bg(x_n))\|
$$

$$
\begin{aligned}
&+ \sup_{t\in[0,t_1]} \frac{\|B^{-1}\|Mt^{1-\gamma}\|x_n\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\
&\leqslant \frac{M\|B^{-1}\|}{\Gamma(\gamma)}\Psi(n) + \frac{\|B^{-1}\|M^{1-\gamma}}{\Gamma(\alpha)} n\sigma l^{\alpha-1/p}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}.
\end{aligned} \tag{35}
$$

Also, from $(\mathrm{H}^*)_1$ it follows for $i \in \mathbb{N}$ that

$$
\begin{aligned}
\sup_{t\in[t_i,s_i]} \|y_n(t)\| &= \sup_{t\in[t_i,s_i]} \|g_i(t, x_n(t_i^-))\| \\
&\leqslant \|B^{-1}\| \sup_{t\in[t_i,s_i]} \|B^{-1}Bg_i(s_i, x_n(t_i^-))\| \\
&\leqslant h\|B^{-1}\|(t_i - s_{i-1})^{1-\gamma}\|x_n(t_i^-)\| \\
&\leqslant \|B^{-1}\|h\|x_n\|_\infty \leqslant \|B^{-1}\|hn.
\end{aligned} \tag{36}
$$

Similarly, we get for $t \in (s_i, t_{i+1}]$, $i \in \mathbb{N}$,

$$
\begin{aligned}
&\sup_{t\in[s_i,t_{i+1}]} (t-s_i)^{1-\gamma}\|y_n(t)\| \\
&\leqslant \sup_{t\in[s_i,t_{i+1}]} \frac{\|B^{-1}\|M\|Bg_i(s_i, x_n(t_i^-))\|}{\Gamma(\gamma)} \\
&\quad + \frac{\|B^{-1}\|Ml^{1-\gamma}}{\Gamma(\alpha)} n\sigma l^{\alpha-1/p}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \\
&\leqslant \frac{\|B^{-1}\|Mhn}{\Gamma(\gamma)} + \frac{\|B^{-1}\|Ml^{1-\gamma}}{\Gamma(\alpha)} n\sigma l^{\alpha-1/p}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}.
\end{aligned} \tag{37}
$$

From (35), (36) and (37), it follows that

$$
\begin{aligned}
n &< \|y_n\|_\infty \\
&\leqslant \frac{M\|B^{-1}\|}{\Gamma(\gamma)}\Psi(n) + \frac{n\sigma l^{\alpha-\frac{1}{p}}(\frac{p-1}{p\alpha-1})^{(p-1)/p}\|B^{-1}\|Ml^{1-\gamma}}{\Gamma(\alpha)} \\
&\quad + \|B^{-1}\|hn + \frac{\|B^{-1}\|Mhn}{\Gamma(\gamma)}.
\end{aligned}
$$

By dividing both sides by n and passing to the limit as $n \to \infty$, we obtain

$$
1 \leqslant \|B^{-1}\|\left[\frac{M(h+w)}{\Gamma(\gamma)} + \frac{\|B^{-1}\|Ml^{1-\gamma+\alpha-1/P}}{\Gamma(\alpha)}\sigma\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} + h\right],
$$

which contradicts (30).

Thus we deduce that there is a natural number n_0 such that $R(B_{n_0}) \subseteq B_{n_0}$.

Step 3. R^* is χ^*-condensing. Let D be a bounded subset of $PC^0_{1-\gamma}([0,\infty),E)$ and $Z = R(D)$. By arguing as in Step 4 in the proof of Theorem 1, one can see that

$$\begin{aligned}
\chi_\infty(Z) &= \sup_{i\in\mathbb{N}} \chi_{PC_{1-\gamma}([0,t_i],E)}\{x_{|[0,t_i]}\colon y\in Z\} \\
&\leqslant \sup_{i\in\mathbb{N}} \chi_{PC_{1-\gamma}([0,t_i],E)}\{x_{|[0,t_i]}\colon x\in D\}\biggl[\frac{\kappa_1\|B^{-1}\|}{\Gamma(\gamma)} \\
&\quad + 2\frac{\|B^{-1}\|}{\Gamma(\alpha)} l^{\alpha-1/p}\biggl(\frac{p-1}{p\alpha-1}\biggr)^{(p-1)/p}\xi + \kappa_2\|B^{-1}\| + \frac{\|B^{-1}\|}{\Gamma(\gamma)}\kappa_2\biggr] \\
&< \chi_\infty(D)\biggl[\frac{(\kappa_1+\kappa_2)\|B^{-1}\|}{\Gamma(\gamma)} + 2\frac{\|B^{-1}\|}{\Gamma(\alpha)} l^{\alpha-1/p}\biggl(\frac{p-1}{p\alpha-1}\biggr)^{(p-1)/p}\xi \\
&\quad + \kappa_2\|B^{-1}\|\biggr].
\end{aligned} \tag{38}$$

It remains to estimate $d_\infty(Z)$, where $d_\infty(Z) = \lim_{m\to\infty}\sup_{y\in Z} d_m(y)$ and

$$d_m(y) = \max\Bigl\{\max_{k\geqslant m}\sup_{t\in J_k}(t-s_k)^{1-\gamma}\|y(t)\|_E,\ \max_{i\geqslant m}\sup_{t\in T_i}\|y(t)\|_E\Bigr\}.$$

Let $m\in\mathbb{N}$ be fixed and $y\in R^*(x)$, $x\in D$. We have

$$\begin{aligned}
\sup_{t\in T_m}\|y(t)\| &\leqslant \sup_{t\in T_m}\|g_m(t,x(t_m^-))\| \leqslant \sup_{t\in T_m}\|B^{-1}\|\|Bg_m(t,x(t_m^-))\| \\
&\leqslant \sup_{t\in T_m}\|B^{-1}\|(t_m-s_{m-1})^{1-\gamma}\|x(t_m^-)\|h_m \leqslant \|B^{-1}\|\|x\|_\infty h_m,
\end{aligned}$$

which yields

$$\max_{i\geqslant m}\sup_{t\in T_i}\|y(t)\|_E \leqslant \|B^{-1}\|\|x\|_\infty\sum_{i=m}^\infty h_i. \tag{39}$$

Also, we have

$$\begin{aligned}
&\sup_{t\in J_m}(t-s_m)^{1-\gamma}\|y(t)\| \\
&\leqslant \sup_{t\in J_m}\biggl[(t-s_m)^{1-\gamma}\|B^{-1}\|\|S_\alpha(t-s_m)\|\|Bg_m(s_m,x(t_m^-))\| \\
&\quad + \|B^{-1}\|(t-s_m)^{1-\gamma}\int_{s_m}^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\varphi(s)(s-s_m)^{1-\gamma}\|x(s)\|\,\mathrm{d}s\biggr] \\
&\leqslant \sup_{t\in J_m}\biggl[(t-s_m)^{1-\gamma}\|B^{-1}\|\frac{M(t-s_m)^{\gamma-1}}{\Gamma(\gamma)}h_m(t_m-s_{m-1})^{1-\gamma}\|x(x(t_m^-))\| \\
&\quad + \|B^{-1}\|\frac{M(t-s_m)^{1-\gamma}}{\Gamma(\alpha)}\sup_{t\in J_m}\bigl((t-s_m)^{1-\gamma}\|x(t)\|\bigr)\sup_{t\in J_m}\int_{s_m}^t(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s\biggr]
\end{aligned}$$

$$\begin{aligned}
&\leqslant \|B^{-1}\| \frac{M}{\Gamma(\gamma)} \|x\|_\infty h_m + \sup_{t\in J_m} \bigl((t-s_m)^{1-\gamma}\bigl\|x(t)\bigr\|\bigr) \|B^{-1}\| l^{1-\gamma} \\
&\quad \times \frac{M}{\Gamma(\alpha)} (t_{m+1}-s_m)^{\alpha-1/p} \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \sigma \\
&\leqslant \|B^{-1}\| \frac{M}{\Gamma(\gamma)} \|x\|_\infty h_m + \sup_{t\in J_m} \bigl((t-s_m)^{1-\gamma}\bigl\|x(t)\bigr\|\bigr) \|B^{-1}\| l^{1-\gamma+\alpha-1/p} \\
&\quad \times \frac{M}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \sigma,
\end{aligned}$$

which yields

$$\begin{aligned}
&\max_{k\geqslant m} \sup_{t\in J_k} (t-s_k)^{1-\gamma} \bigl\|y(t)\bigr\|_E \\
&\quad \leqslant \|B^{-1}\| \frac{M}{\Gamma(\gamma)} \|x\|_\infty \sum_{k=m}^{\infty} h_k + \sup_{t\in J_m} \bigl((t-s_m)^{1-\gamma}\bigl\|x(t)\bigr\|\bigr) \\
&\quad\quad \times \|B^{-1}\| l^{1-\gamma+\alpha-1/p} \frac{M}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \sigma.
\end{aligned} \tag{40}$$

It follows from the fact that $\sum_{k=1}^{\infty} h_k$ is convergent, (39) and (40) that

$$\begin{aligned}
d_\infty(Z) &= \lim_{m\to\infty} \sup_{y\in Z} d_m(y) \\
&= \lim_{m\to\infty} \max\Bigl\{ \max_{k\geqslant m} \sup_{t\in J_k} (t-s_k)^{1-\gamma} \|y(t)\|_E,\ \max_{i\geqslant m} \sup_{t\in T_i} \bigl\|y(t)\bigr\|_E \Bigr\} \\
&\leqslant \|B^{-1}\| l^{1-\gamma+\alpha-1/p} \frac{M}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \sigma d_\infty(D),
\end{aligned}$$

which yields with (38) that

$$\begin{aligned}
\chi^*(Z) \leqslant \|B^{-1}\| \biggl[& l^{1-\gamma+\alpha-1/p} \frac{M}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \sigma \\
& + \frac{\kappa_1+\kappa_2}{\Gamma(\gamma)} + \frac{2}{\Gamma(\alpha)} l^{\alpha-1/p} \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \xi\kappa_2 \biggr] \chi^*(D).
\end{aligned}$$

This inequality with (31) implies that R^* is χ^*-condensing.

Now arguing as in Step 3 in the proof of Theorem 1, one can see that the graph of R^* is closed. Moreover from the fact that R^* is χ^*-condensing with closed graph we have the compactness of the values of R^*. Now Lemma 3 guarantees the existence of a decay mild solution of (1) □

Remark 4. According to [7, Lemma 3.2], if B^{-1} is compact, then the operator $B^{-1}P_\alpha(t)$, $t > 0$, is compact, and hence we do not need $(\mathrm{F}^*)_3$ in Theorem 4.

To end this paper, we give an example illustrating our abstract results.

Example 1. Let $\alpha \in (0,1)$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $E = L^2(\Upsilon)$, where Υ is a bounded smooth domain in $\mathbb{R}^2$. Clearly, E is a separable Hilbert space. Set $s_i = 2i$, $i \in \{0\} \cup \mathbb{N}$, $t_k = 2k-1$, $k \in \mathbb{N}$. Define A and B by $A = \Delta$, the Laplacian operator, and $B = I - \Delta$, where for each domains A and B, $D(A) = D(B) = H^2(\Upsilon) \cap H_0^1(\Upsilon)$, and I is the identity operator. Let $\{\lambda_n\}$ be the eigenvalues of $-A$ with corresponding eigenvector $\{e_n\}_{n\geqslant 1}$. It is known that $0 < \lambda_1 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$ (see [18]). Moreover, A can be written as $Af = -\sum_{n=1}^{\infty} \lambda_n \langle f, e_n\rangle e_n$, and $Bf = \sum_{n=1}^{\infty}(1+\lambda_n)\langle f, e_n\rangle e_n$. Therefore $B^{-1}f = \sum_{n=1}^{\infty}\langle f, e_n\rangle e_n/(1+\lambda_n)$ and $AB^{-1}f = \sum_{n=1}^{\infty} -\lambda_n\langle f, e_n\rangle e_n/(1+\lambda_n)$.

This implies (see [18]) that the semigroup generated by AB^{-1} can be expressed as in (H$_1$). Moreover, for any $f \in L^2(\Upsilon)$, we have $T(t)(f) = \sum_{n=1}^{\infty} \exp(-\lambda_n/(1+\lambda_n)) \times t\langle f, e_n\rangle e_n$. Moreover, B^{-1} is compact, $\|B^{-1}\| \leqslant 1$ and $\|T(t)\| \leqslant \exp(-\lambda_1/(1+\lambda_1)) \times t \leqslant 1$ for all $t \geqslant 0$ (see [7, 18]). Hence we choose $M = 1$.

Let $F : [0,\infty) \times E \to P_{\mathrm{ck}}(E)$ be the multifunction defined by $F(t,v)(s) = \mathrm{co}\{f_1(t,v(s)), f_2(t,v(s)), \dots, f_m(t,v(s))\}$, where co denotes the convex hull, and $f_k : [0,\infty)\times\mathbb{R} \to \mathbb{R}$, $k = 1,2,\dots$, are continuous bounded such that for any $k = 1,2,\dots,m$, any $t \in (s_i, t_{i+1}]$, $i \in \{0\} \cup \mathbb{N}$,

$$\left|f_k(t,z)\right| \leqslant (t-s_i)^{1-\gamma}\varphi(t)|z| \quad \forall (t,z) \in [0,\infty) \times \mathbb{R}, \tag{41}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function, and $|\varphi(t)| \leqslant \sigma$ for all $t \in [0,\infty)$. Clearly, $F(t,v)$ is closed, bounded and contained in the finite dimensional subspace, which is spanned by $E_m = \{f_1(t,v(\cdot)), f_2(t,v(\cdot)), \dots, f_m(t,v(\cdot))\}$ and hence it is compact. From the continuity of f_k, $k = 1,2,\dots,m$, one can see that F satisfies (F*)$_1$. Moreover, if $t \in (s_i, t_{i+1}]$, $i \in \{0\} \cup \mathbb{N}$, $x \in PC^0_{1-\gamma}([0,\infty), E)$ and $y \in F(t, x(t))$, then $(\|y\|_E)^2 = \int_\Upsilon |y(s)|^2\,\mathrm{d}s = \int_\Upsilon |\sum_{k=1}^{k=m}\mu_k f_k(t, x(t)(s))|^2\,\mathrm{d}s$, where $\mu_k > 0$ and $\sum_{k=1}^{k=m}\mu_k = 1$. Then from (41) we have $\|y\|_E \leqslant (\int_\Upsilon(\sum_{k=1}^{k=m}\mu_k)^2 \times (t-s_i)^{2-2\gamma}\varphi^2(t)\|x(t)(s)\|^2\,\mathrm{d}s)^{1/2}$, and hence F satisfies (F*)$_2$. Moreover, it follows from Remark 4 that F satisfies (F*)$_3$ with $\varsigma(t) = 0$ for all $t \in [0,\infty)$. In order to define $g_i : [t_i, s_i] \times E \to D(B)$, $i \in \mathbb{N}$, let $K_i : \Upsilon \times \Upsilon \to \mathbb{R}$ be integrable functions such that K_i together with its second derivative with respect to the first argument belongs to $L^2(\Upsilon\times\Upsilon)$. We define $g_i : [t_i, s_i] \times E \to D(B)$ as $g_i(t,v)(x) = (t_i - s_{i-1})^{1-\gamma}\int_\Upsilon K_i(x,y)v(y)\,\mathrm{d}y$, $x, y \in \Upsilon$. Then $Bg_i(t,v)(x) = h_i(t_i - s_{i-1})^{1-\gamma}\int_\Upsilon k_i(x,y)v(y)\,\mathrm{d}y$, where $k_i(x,y) = K_i(x,y) - \Delta_x K_i(x,y)$, $(x,y) \in \Upsilon \times \Upsilon$. Then Bg_i is a Hilbert–Schmidt operator, and hence compact (see [18]), so, (H*)$_2$ is satisfied. In addition, $\|Bg_i(t,v)\|_E \leqslant h_i \times (t_i - s_{i-1})^{1-\gamma}\|v\|_E$, where $h_i = \|k_i\|_{L^2(\Upsilon\times\Upsilon)}$, so (H*)$_1$ is satisfied if we assume that $\sum_{i=1}^{\infty}\|k_i\|_{L^2(\Upsilon\times\Upsilon)} = h < \infty$.

In order to define the nonlocal function, let $G : [0,b] \times \Upsilon \times \Upsilon \to \mathbb{R}$ be an integrable function with $G(t,\cdot,\cdot)$, $\Delta_x G(t,\cdot,\cdot) \in L^2(\Upsilon \times \Upsilon)$ and $v \in H^2(\Upsilon)$. Put $g : PC^0_{1-\gamma}(J,E) \to E$, $g(w)(x) = v(x) + \int_0^b\int_\Upsilon G(s,x,y)w(s,y)\,\mathrm{d}y\,\mathrm{d}s$, $x \in \Upsilon$, where $w(s,y) = w(s)(y)$, the values of $w(s)$ at y. Then $Bg(w)(x) = v(x) - \Delta v(x) + \int_0^b\int_\Upsilon \widetilde{G}(s,x,y)w(s,y)\,\mathrm{d}y\,\mathrm{d}s$, where $\widetilde{G}(s,x,y) = (I - \Delta_x)G(s,x,y)$. It follows that $\|Bg(w)\|_{PC^0_{1-\gamma}(J,E)} \leqslant \|v\|_{H^2(\Upsilon)} + (\int_0^b \|\widetilde{G}(s,\cdot,\cdot)\|_{L^2(\Upsilon\times\Upsilon)}\,\mathrm{d}s)\|w\|_\infty$.

Then $(\mathrm{H}_g^*)_1$ is satisfied with $\Psi(r) = \|v\|_{H^2(\Upsilon)} + (\int_0^b \|\widetilde{G}(s,\cdot,\cdot)\|_{L^2(\Upsilon\times\Upsilon)}\,\mathrm{d}s)r$. Then $w = \int_0^b \|\widetilde{G}(s,\cdot,\cdot)\|_{L^2(\Upsilon\times\Upsilon)}\,\mathrm{d}s$. In addition, since $L : L^2(\Upsilon) \to L^2(\Upsilon)$ defined by $L(v)(x) = \int_\Upsilon \widetilde{G}(s,x,y)v(y)\,\mathrm{d}y$ is a Hilbert–Schmidt operator for fixed $s \in [0,b]$, we see that for any bounded set $D \subset PC^0_{1-\gamma}(J,E)$, $L(D(s))$ is relatively compact in E. It follows from [18, p. 1316] that the set $Bg(D)$ presented by $Bg(D) = Bv + \int_0^b L(D)(s)\,\mathrm{d}s$ is relatively compact. Therefore, $\chi_E(Bg(D)) \leqslant 4\int_0^b \chi L(D)(s)\,\mathrm{d}s = 0$, which implies that $(\mathrm{H}_g^*)_2$ is satisfied with $\kappa_1 = 0$.

From the above discussion and Theorem 4, the following nonlocal noninstantaneous impulsive fractional semilinear differential inclusion

$$
\begin{aligned}
&D^{\alpha,\beta}_{s_i^+}\big(x(t,y) - x_{yy}(t,y)\big) \in x_{yy}(t,y) + F\big(t,x(t,y)\big),\\
&\quad \text{a.e. } t \in (2i, 2i+1],\ i \in \{0\}\cup\mathbb{N},\\
&x(t_1^+,y) = g_1\big(t_1, x(t_1^-,y)\big), \quad x(t,y) = g_i\big(t, x(t_i^-,y)\big),\\
&\quad t \in [2i-1, 2i],\ y \in [0,\infty),\ i \in \mathbb{N},\\
&I^{1-\gamma}_{0^+}x(0,y) = x_0 + g(x)y, \quad I^{1-\gamma}_{s_i^+}x(s_1^+,y) = g_1\big(s_1, x(t_1^-,y)\big),\\
&\quad y \in \Upsilon,\ i \in \{0\}\cup\mathbb{N}.
\end{aligned}
$$

admits a decay solution, provided that

$$
\begin{aligned}
&\frac{w}{\Gamma(\gamma)} + \frac{\sigma}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} + h + \frac{h}{\Gamma(\gamma)} < 1,\\
&\frac{1}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}\sigma < 1.
\end{aligned}
\tag{42}
$$

Clearly, one can choose α, β, w, σ, p such that (42) is satisfied.

5 Conclusion

In this paper, we establish the existence of decay mild solution on an unbounded interval of nonlocal fractional (involving the Hilfer derivative) semilinear differential inclusions with noninstantaneous impulses. Note that in our paper, the lower limit in the Hilfer derivative is varying with some previous fixed points, so our approach is different than that in [18, 26]. We also generalize existence results [18, 26] to a more general case, and we consider the large time behavior of solutions of fractional evolution equations in a suitable weighted piecewise continuous functions space.

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