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Existence of common fixed points for linear combinations of contractive maps in enhanced probabilistic metric spaces*

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Abstract. In this paper, we introduce the concept of enhanced probabilistic metric space (briefly EPM-space) as a type of probabilistic metric space. Also, we investigate the existence of fixed points for a (finite or infinite) linear combination of different types of contractive mappings in EPM-spaces. Furthermore, we investigate about the convergence of sequences (generated by a finite or infinite family of contractive mappings) to a common fixed point. The useful application of this research is the study of the stability of switched dynamic systems, where we study the conditions under which the iterative sequences generated by a (finite or infinite) linear combination of mappings (contractive or not), converge to the fixed point. Also, some examples are given to support the obtained results. In the end, a number of figures give us an overview of the examples.

Keywords: enhanced probabilistic metric space, fixed point, linear combination, convergent sequence, dynamic systems.

1 Introduction

In recent times, fixed point theory has become an important tool in pure and applied sciences, such as biology [4], chemistry [25], economics [10], engineering and physics, to

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cite just a few. The Banach's fixed point theory, widely known as the contraction principle, is an important tool in the theory of metric spaces [14, 22]. Moreover, since the location of the fixed point can be obtained by means of an iteration process, it can be implemented on a computer to find the fixed point of contraction mappings easily. Several authors have extended the Banach's fixed point theorem in various ways. For instance, a fixed point theorem for multivalued contractive mappings was established by Nadler [28]. Rich showed that a contractive mapping from complete metric space X into the family $K(X)$ of all nonempty compact subsets of X has a fixed point [29]. You can see similar works in [11, 30]. However, it is important to notice that some results presented in the literature as generalizations to well-known theorems are not actually real generalizations in the sense that they can be obtained from previous exisiting results, (see, for instance, [6,19,20, 31]). The concept of metric space was introduced by the French mathematician Maurice Frechet in 1906, and since then several generalizations of it have been proposed in the literature, see [1,5]. One of the important generalizations is the probabilistic metric space (PM space), which was introduced by the Austrian mathematician K. Menger in 1942 [26] by using the notion of distribution functions in place of nonnegative real numbers. Menger probabilistic metric spaces are a special class of the wide class of probabilistic metric spaces, which are endowed with a triangular norm [33].

A generalization of the Banach contraction principle on a complete Menger space was worked by Sehgal and Bharuch-Reid in 1972 [33]. After this initial work, the fixed point theory in probabilistic metric spaces has been developed in many works such as [13,21]. In 1984, Khan et al. introduced the concept of altering distance function [23]. A φ-function is the extension of altering distance function and has been worked by many authors, [24, 27, 34]. For instance, the concepts of (α, ψ)-type contractive and α-admissible mappings were introduced by Gopal et al. [17], who also established some fixed point theorems for these mappings in complete Menger spaces. After that, Shams and Jafari generalized this concept to (α, β, ψ)-contractive and α–β-admissible mappings and proved some fixed point theorems for such maps [34].

The family of contraction mappings was introduced by Ciric [12] and Taskovic [35]. The existence of a common fixed point for a finite or infinite family of self-mappings and contractive maps is worked by many authors [2, 7]. In this way, the existence of a fixed point for infinite families of self-mappings of a complete metric space satisfying some new conditions of common contractivity was studied by Allahyari et al. in [3]. Also, the study on fixed point and convergence of sequences generated by a family of k-contractions and φ-contractions in Menger spaces was worked by De la Sen et al. in [15].

In this paper, we introduce the concept of enhanced probabilistic metric space (briefly, EPM-space) as a type of PM-space and show that it is a real generalization of PM-spaces in the sense of [19]. Also, motivated by the definition of contractive mappings introduced in [16] and [34] in Menger PM-spaces, we investigate the existence of common fixed points for a (finite or infinite) linear combination of these contractive mappings in EPM-spaces. Furthermore, we investigate about the convergence of sequences (generated by a family of contractive mappings) to a common fixed point. Although there exist many works related to the existence of a fixed point for different types of contractive maps in Menger PM-spaces [9, 16, 17], this is the first time that this research is being investigated on linear combinations of these contractive maps. The useful potential application of this research is the study of the stability of switched dynamic systems, where we study the conditions under which the iterative sequences generated by (finite or infinite) linear combination of mappings (contractive or not), converge to the fixed point. Also, some examples and application to switched dynamic systems are given to support the obtained results.

The rest of the paper is organized as follows: in the first section, we introduce some previous and new notions employed in the article. The second section is our main result, which is related to the existence of a fixed point for a linear combination of contractive maps and about the convergence of sequences generated by these combinations. The final section contains some numerical examples that illustrate the main results.

2 Background results

In this section, we bring some notions, definitions and known results, which are related to our work in order to make the paper self-contained. For more details on PM-space, we refer the reader to [18]. Also, we introduce the notion of enhanced probabilistic metric space (briefly, EPM-space) and bring some properties of this space. We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}_+ = \{z \in \mathbb{Z}\colon\ z > 0\}$, $\mathbb{Z}_{0+} = \mathbb{Z}_+ \cup \{0\}$ and f^m is composition of f for m times.

Definition 1. A distribution function is a function $F : (-\infty, \infty) \to [0, 1]$, that is non-decreasing and left-continuous on $\mathbb{R}$. Moreover, $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. The set of all the distribution functions is denoted by D, and the set of those distribution functions such that $F(0) = 0$ is denoted by D^+. We will denote the specific Heaviside distribution function by:

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

Definition 2. An enhanced probabilistic metric space (briefly, EPM-space) is an ordered pair (X, F), where X is a nonempty set, and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(EPM1) $F_{x,y}(t) = H(t)$ if and only if $x = y$.
(EPM2) $F_{x,y}(t) = F_{y,x}(t)$.
(EPM3) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $s, t \geqslant 0$.
(EPM4) $1 / F_{\sum_{i=1}^{\infty} \lambda_i x_i,\, \sum_{i=1}^{\infty} \lambda_i y_i}(t) \leqslant \sum_{i=1}^{\infty} \lambda_i / F_{x_i,y_i}(t)$ for all $\lambda_i \in [0, 1]$, $\sum_{i=1}^{\infty} \lambda_i = 1$, and for all $t \geqslant 0$ and $x_i, y_i \in X$, $i \in \mathbb{Z}_+$.

We can see that an EPM-space is a particular case of a PM-space. Hence, all properties and previous results in PM-spaces (in this section) hold in EPM-space as follows.

Definition 3. A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if the following conditions hold:

(a) T is commutative and associative.
(b) T is continuous.
(c) $T(a,1) = a$ for all $a \in [0,1]$.
(d) $T(a,b) \leqslant T(c,d)$ whenever $a \leqslant c$ and $b \leqslant d$ for $a,b,c,d \in [0,1]$.

The following are three basic continuous t-norms [32]:

(i) The minimum t-norm T_M defined by $T_M(a,b) = \min\{a,b\}$.
(ii) The product t-norm T_p defined by $T_p(a,b) = ab$.
(iii) The Lukasiewicz t-norm T_L defined by $T_L(a,b) = \max\{a+b-1, 0\}$.

These t-norms are related in the following way: $T_L \leqslant T_P \leqslant T_M$.

Definition 4. A Menger enhanced probabilistic metric space (briefly, Menger EPM-space) is a triple (X,F,T), where (X,F) is a EPM-space, and T is a continuous t-norm such that, for all $x,y,z \in X$ and $s,t \geqslant 0$,

$$F_{x,y}(t+s) \geqslant T\big(F_{x,z}(t), F_{z,y}(s)\big).$$

It is straightforward to check that a Menger EPM-space is also a Menger PM-space. However, the converse is not true in general as the following example shows.

Example 1. Let $X = \mathbb{R}$, $T = T_p$, and

$$F_{x,y}(t) = \begin{cases} \frac{\min\{|x|,|y|\}+t}{\max\{|x|,|y|\}+t}, & x,y \in [0,\infty) \text{ or } x,y \in (-\infty,0], \\ \frac{t}{|x|+t}\frac{t}{|y|+t} & \text{elsewhere} \end{cases}$$

for all $t > 0$. Clearly, (X,F,T_p) is a Menger PM-space. We show that (X,F,T_p) is not a EPM-space since does not satisfy (EPM4). To see this, consider $\lambda_i = (1/2)^i$, $x_i = 1/((3/2)^i\lambda_i)$ and $y_i = 1/\lambda_i$. It is easy to see, $\sum_{i=1}^{\infty}\lambda_i = 1$, $\sum_{i=1}^{\infty}\lambda_i x_i = 2$, $\sum_{i=1}^{\infty}\lambda_i y_i = \infty$. Thus, $1/F_{\sum_{i=1}^{\infty}\lambda_i x_i, \sum_{i=1}^{\infty}\lambda_i y_i}(t) = \infty$ and $\sum_{i=1}^{\infty}\lambda_i/F_{x_i,x_i}(t) = 3 + 3/5t$. So (X,F,T_p) is not a EPM-space.

Definition 5. Let (X,F,T) be a Menger EPM-space. Then

(i) A sequence x_n in X is said to be convergent to x if, for every $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geqslant N$.
(ii) A sequence x_n in X is called a Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n,m \geqslant N$.
(iii) A Menger EPM-space is said to be complete if and only if every Cauchy sequence in X converges to a point in X.
(iv) A sequence x_n is called G-Cauchy if $\lim_{n\to\infty} F_{x_n,x_{n+m}}(t) = 1$ for each $m \in \mathbb{Z}_+$ and $t > 0$.
(v) The space (X,F,T) is called G-complete if every G-Cauchy sequence in X is convergent to a point in X.

According to [7], the (ϵ, λ)-topology in (X, F, T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = \{N_x(\epsilon, \lambda)\colon\ \epsilon > 0,\ \lambda \in (0, 1)\},$$

where

$$N_x(\epsilon, \lambda) = \{y \in X\colon\ F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The (ϵ, λ)-topology is a Hausdorff topology. Therefore, in this topology, a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$ for every sequence $x_n \to x_0$, [7].

Definition 6. (See [8].) A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$.
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$.
(iii) $\phi(t)$ is left-continuous in $t \in (0, \infty)$.
(iv) $\phi(t)$ is continuous at $t = 0$.

In the sequel, the class of all Φ-functions will be denoted by Φ. Also, we denote by Ψ the class of all continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$.

Definition 7. (See [34].) Let (X, F, T) be a Menger PM-space, $f : X \to X$ be a given mapping, and $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ be two functions. We say that f is α–β-admissible if

(i) $x, y \in X$, for all $t > 0$, $\alpha(x, y, t) \geqslant 1 \ \Rightarrow\ \alpha(fx, fy, t) \geqslant 1$;
(ii) $x, y \in X$, for all $t > 0$, $\beta(x, y, t) \leqslant 1 \ \Rightarrow\ \beta(fx, fy, t) \leqslant 1$.

Now we introduce the following definition that is a particular case of Definition 7. So all properties and previous results related to α–β-admissible hold for enhanced α–β-admissible.

Definition 8. Let (X, F, T) be a Menger EPM-space, $\{f_i\}_{i\in N}$ be a family of self-maps on X, and $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ be two bilinear functions. We say that $\{f_i\}_{i\in N}$ is enhanced α–β-admissible if

(i) $x, y \in X$, for all $t > 0$, $\alpha(x, y, t) \geqslant 1 \ \Rightarrow\ \alpha(f_i x, f_j y, t) \geqslant 1$, $i, j \in \mathbb{Z}_+$;
(ii) $x, y \in X$, for all $t > 0$, $\beta(x, y, t) \leqslant 1 \ \Rightarrow\ \beta(f_i x, f_j y, t) \leqslant 1$, $i, j \in \mathbb{Z}_+$.

Definition 9. (See [34].) Let (X, F, T) be a Menger EPM-space, $f : X \to X$ be a given mapping. We say that f is an (α, β, ψ)-contractive mapping if there exist two functions $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ and $\psi \in \Psi$ satisfying the following inequality:

$$\alpha(x, y, t)\left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1\right) \leqslant \beta(x, y, t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \tag{1}$$

for all $x, y \in X$, $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$, where $c \in (0, 1)$ and $\varphi \in \Phi$.

Remark 1. If $\alpha(x,y,t) = \beta(x,y,t) = 1$, then we have the following theorem in PM-space [16] that also hold in EPM-space.

Theorem 1. *Let (X,F,T) be a G-complete Menger EPM-space, and $f : X \to X$ be a mapping satisfying the following inequality*:

$$\frac{1}{F_{fx,fy}(\varphi(ct))} - 1 \leqslant \psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right), \tag{2}$$

where $c \in (0,1)$, $\varphi \in \phi$ and $\psi \in \Psi$. Then f has a fixed point, i.e., there exists a point $u \in X$ such that $fu = u$.

Remark 2. We say that the mapping $f : X \to X$ is ψ-contractive if it satisfies condition (2).

Theorem 2. (*See* [34].) *Let (X,F,T) be a G-complete Menger EPM-space, and let $f : X \to X$ be a (α,β,ψ)-contractive mapping satisfying the following conditions*:

(i) *f is α–β-admissible.*
(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geqslant 1$ and $\beta(x_0, fx_0, t) \leqslant 1$ for all $t > 0$.*
(iii) *There exists a sequence $\{x_n\}$ in X fulfilling this condition*: *if $\alpha(x_n, x_{n+1}, t) \geqslant 1$ and $\beta(x_n, x_{n+1}, t) \leqslant 1$ for all $n \in \mathbb{Z}_+$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, t) \geqslant 1$ and $\beta(x_n, x, t) \leqslant 1$ for all $n \in \mathbb{Z}_+$ and for all $t > 0$.*

Then f has a fixed point, i.e., there exists a point $u \in X$ such that $fu = u$.

Theorem 3. *With the same hypotheses of Theorem* 2, *if for all $x \in X$ and for all $t > 0$, there exists $z \in X$ such that $\alpha(x,z,t) \geqslant 1$ and $\beta(x,z,t) \leqslant 1$, then f has a unique fixed point.*

Remark 3. It should be noticed that the convergent point x of the sequence (x_n) in (iii) can be different from the fixed point u.

Proposition 1. *In order to calculate the fixed point of f in Theorems* 1 *and* 2, *we can set up the sequence $x_{n+1} = f(x_n)$ with $x_0 \in X$ arbitrary. Thus, the convergence points of this iteration are the fixed points of f. We can see this result in the proof of Theorems* 1 *and* 2 *in* [16] *and* [34], *respectively.*

Lemma 1. (*See* [15].) *The following properties hold*:

(i) *If for all $n \in \mathbb{Z}_+$, $f_n : X \to X$ are continuous and $\{f_n\}$ uniformly converges to $\{f\}$, then $\{f_n^m\}$ uniformly converge to $\{f^m\}$ for all $m \in \mathbb{Z}_+$.*
(ii) *If $\{f_n\}$ uniformly converges to f and f_n commutes with f in X for all $n \in \mathbb{Z}_{0+}$, then $\{f_n^m\}$ uniformly converge to $\{f^m\}$ for all $m \in \mathbb{Z}_+$.*

3 Main results

In this section, we prove that a linear combination of a finite or infinite family of (α,β,ψ)-contractive and ψ-contractive mappings have a fixed point in a Menger EPM-space. Also, we discuss about the convergence of iterative sequences generated by linear combinations of (α,β,ψ)-contractive and ψ-contractive maps. Moreover, we will show the application of this fixed point theory to switched dynamic systems.

The subsequent result shows that when the family of operators $\{f_n\}_{n\in N}$ are not contractive but converge to a contractive map f, then the iterated sequence generated by f_n, converge to the fixed point of f. We denote by $\mathrm{Fix}(f)=\{x\in X\colon f(x)=x\}$ the set of fixed points of f.

Theorem 4. *Let (X,F,T) be a complete Menger EPM-space, and let $\{f_n\}$ be a sequence of operators $f_n: X\to X$ such that $\mathrm{Fix}(f_n)=\{x_n^*\}$ for all $n\in\mathbb{Z}_+$, and $\{f_n\}$ uniformly converges to f, where $f: X\to X$ is a (α,β,ψ)-contractive map with $f(x^*)=x^*$. Consider the sequence $\{x_n\}\subset X$ generated by the iterated scheme $x_{n+1}=f_n(x_n)$ for any given $x_1\in X$ and $n\in\mathbb{Z}_+$. Assume that some of the conditions below holds*:

(i) *$\{f_n^m\}$ uniformly converge to $\{f^m\}$ for all $m\in\mathbb{Z}_+$.*
(ii) *$\{f_n\}$ uniformly converges to f and either $f_n: X\to X$ is continuous for all $n\in\mathbb{Z}_{0+}$ or $\{f_n\}$ commutes with f for all $n\in\mathbb{Z}_{0+}$.*

Then $\{x_n\}\to x^*$.

Proof. Since $\{f_n\}$ uniformly converges to $\{f\}$ for any given $\lambda\in(0,1)$ and $t>0$, there exists $N_1(\in\mathbb{Z}_{0+})=N_1(\lambda,t)$ such that $F_{f_nx_n,fx_n}(\varphi(ct))>1-\lambda$, where $\varphi\in\Phi$ and $c\in(0,1)$. The proof is performed by contradiction. Assume that $\{x_n\}$ does not converge to x^*. Then there exists some real constant $\lambda_0\in(0,1)$ such that for some subsequence $\{f_n^m\}$ of $\{f_n\}$, which generates the sequence $x_{n_j+1}=f_{n_j}x_{n_j}$ for some given $x_1\in X$ for all $j\in\mathbb{Z}_{0+}$, we have:

$$
\begin{aligned}
&\frac{1}{1-\lambda_0}-1\\
&\quad\leqslant\frac{1}{\overline{\lim}_{j\to\infty}(\overline{\lim}_{m\to\infty}F_{x_{nj+m},x^*}(\varphi(ct)))}-1\\
&\quad=\frac{1}{\overline{\lim}_{j\to\infty}(\overline{\lim}_{m\to\infty}F_{f_{nj}^m x_{nj},x^*}(\varphi(ct)))}-1\\
&\quad\leqslant\frac{1}{\overline{\lim}_{j\to\infty}(\overline{\lim}_{m\to\infty}T(F_{f_{nj}^m x_{nj},f^m x_{nj}}(\frac{\varphi(ct)}{2}),F_{f^m x_{nj},f^m x^*}(\frac{\varphi(ct)}{2})))}-1\\
&\quad\leqslant\frac{1}{T(\underline{\lim}_{j\to\infty}\overline{\lim}_{m\to\infty}F_{f_{nj}^m x_{nj},f^m x_{nj}}(\frac{\varphi(ct)}{2}),\ \lim_{m\to\infty}F_{f^m x_{nj},f^m x^*}(\frac{\varphi(ct)}{2}))}\\
&\qquad-1.
\end{aligned}\tag{3}
$$

From condition (i) and according to Lemma 1, $\{f^m_{nj}\}$ uniformly converge to f^m. Hence, from (3) we get

$$\begin{aligned}\frac{1}{1-\lambda_0}-1 &\leqslant \frac{1}{T(1, \lim_{m\to\infty} F_{f^m x_{nj}, f^m x^*}(\frac{\varphi(ct)}{2}))}-1\\ &= \frac{1}{\lim_{m\to\infty} F_{f^m x_{nj}, f^m x^*}(\frac{\varphi(ct)}{2})}-1. \qquad (4)\end{aligned}$$

Since f is (α, β, ψ)-contractive and $\psi^n(a_n) \to 0$ whenever $a_n \to 0$, from (4) we deduce that

$$\begin{aligned}\alpha(x,y,t)\left(\frac{1}{1-\lambda_0}-1\right) &\leqslant \alpha(x,y,t)\left(\frac{1}{\lim_{m\to\infty} F_{f^m x_{nj}, f^m x^*}(\frac{\varphi(ct)}{2})}-1\right)\\ &\leqslant \lim_{m\to\infty} \beta(x,y,t)\psi^m\left(\frac{1}{F_{x_{nj}, x^*}(\frac{\varphi(\frac{t}{c^m})}{2})}-1\right) = 0. \qquad (5)\end{aligned}$$

Note that (5) yields $\alpha(x,y,t)(1/(1-\lambda_0)-1) \leqslant 0$ implying $1-\lambda_0 \geqslant 1$ and $\lambda_0 \leqslant 0$, which is the contradiction. Hence, $\{x_{nj}\} \to x^*$ and then $\{x_n\} \to x^*$. So the theorem is proved. □

The following theorem shows that a linear combination of a family of (α, β, ψ)-contractive maps that satisfy Theorem 2 has a common fixed point.

Theorem 5. *Let $\{f_i\}_{i=1}^m$ be a finite family of self-maps on X that is enhanced α–β-admissible satisfying all conditions of Theorem 2, $\bigcap_{i=1}^m \mathrm{Fix}(f_i) = \{u_1, u_2, \dots, u_m\}$, and also f_i are continuous at the common fixed points. Let $f = \sum_{i=1}^m \lambda_i f_i$, where $\lambda_i \in (0,1)$ and $\sum_{i=1}^m \lambda_i = 1$. Then $\{u_1, u_2, \dots, u_m\}$ are also fixed points of f.*

Moreover, for each $u \in \bigcap_{i=1}^m \mathrm{Fix}\{f_i\}$, let $\{x_n^i\}$ be the sequence associated to each f_i such that $\{x_n^i\} \to u$ as $n \to \infty$ and $x_{n+1}^i = f_i(x_n^i)$ with $x_0^i \in X$ arbitrary. Then the following results hold:

- (i) *Consider $\{t_n\}$ to be the corresponding sequence of f such that $t_n = \sum_{i=1}^m \lambda_i x_n^i$. Then there exists a sequence $\{w_n\}$ satisfying $w_n = f(t_n)$ such that $\{w_n\} \to u$ as $n \to \infty$.*
- (ii) *Consider $\{z_n\}$ be one of the sequences associated to f such that $z_{n+1} = f(z_n)$ with $z_0 \in X$ arbitrary. Then $\{z_n\} \to u$ as $n \to \infty$.*

Proof. We show that u is a fixed point of f (linear combination of f_i), where $u \in \bigcap_{i=1}^m \mathrm{Fix}(f_i)$. Afterward, we prove that the sequences $\{w_n\}$ and $\{z_n\}$ converges to u.

Since u is a common fixed point of f_i, we have

$$\begin{aligned}f(u) &= \lambda_1 f_1(u) + \lambda_2 f_2(u) + \dots + \lambda_m f_m(u) = \lambda_1 u + \lambda_2 u + \dots + \lambda_m u\\ &= (\lambda_1 + \lambda_2 + \dots + \lambda_m)u = u\end{aligned}$$

since $\sum_{i=1}^m \lambda_i = 1$. Hence, $f(u) = u$ and u is a fixed point of f.

Now, in case (i), we want to prove that the sequence $\{w_n\}$ converges to u as $n \to \infty$.

According to Proposition 1, we know that each sequence $\{x_n^i\}$ converges to u as $n \to \infty$. Also, we have $w_n = f(t_n)$. Hence,

$$\begin{aligned}\lim_{n\to\infty} w_n &= \lim_{n\to\infty} f(t_n)\\ &= \lim_{n\to\infty} \bigl(\lambda_1 f_1(t_n) + \lambda_2 f_2(t_n) + \cdots + \lambda_m f_m(t_n)\bigr).\end{aligned} \tag{6}$$

On the other hand,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{i=1}^{m} \lambda_i x_n^i = \sum_{i=1}^{m} \Bigl(\lim_{n\to\infty} \lambda_i x_n^i\Bigr) = \left(\sum_{i=1}^{m} \lambda_i\right) u = u. \tag{7}$$

From (6), (7) and because of continuity of f_i at the fixed point, we get

$$\lim_{n\to\infty} w_n = \lambda_1 f_1(u) + \lambda_2 f_2(u) + \cdots + \lambda_m f_m(u) = \left(\sum_{i=1}^{m} \lambda_i\right) u = u.$$

This means the sequence $\{w_n\}$ converges to the fixed point u.

Now in case (ii), we prove that the iterative sequence $\{z_n\}$ converges to the fixed point u. According to (EPM4)-property, we have

$$\begin{aligned}\frac{1}{F_{fx,fy}(\varphi(ct))} &= \frac{1}{F_{\sum_{i=1}^{m} \lambda_i f_i x, \sum_{i=1}^{m} \lambda_i f_i y}(\varphi(ct))} \leqslant \sum_{i=1}^{m} \lambda_i \frac{1}{F_{f_i x, f_i y}(\varphi(ct))}\\ &= \sum_{i=1}^{m} \lambda_i \left(\frac{1}{F_{f_i x, f_i y}(\varphi(ct))} - 1\right) + 1.\end{aligned} \tag{8}$$

From (8) and since each f_i is (α, β, ψ)-contractive, we get

$$\begin{aligned}\alpha(x,y,t)\left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1\right) &\leqslant \sum_{i=1}^{m} \lambda_i \alpha(x,y,t)\left(\frac{1}{F_{f_i x, f_i y}(\varphi(ct))} - 1\right)\\ &\leqslant \sum_{i=1}^{m} \lambda_i \beta(x,y,t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right)\\ &= \beta(x,y,t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right),\end{aligned}$$

this means that f is (α, β, ψ)-contractive. Also, f is $\alpha{-}\beta$-admissible because

$$\alpha(fx, fy, t) = \alpha\left(\sum_{i=1}^{m} \lambda_i f_i x, \sum_{i=1}^{m} \lambda_i f_i y, t\right) = \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i \lambda_j \alpha(f_i x, f_j y, t) \tag{9}$$

since $\{f_i\}_{i=1}^{m}$ is enhanced $\alpha{-}\beta$-admissible. Hence, from (9) we get

$$\alpha(fx, fy, t) \geqslant \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i \lambda_j \geqslant 1.$$

Also, by the same proof we have $\beta(fx, fy, t) \leqslant 1$. It is easy to check that f satisfy all conditions of Theorem 2. Hence, according to Proposition 1, the sequence $\{z_n\}$ defined by $z_{n+1} = f(z_n)$, converges to the fixed point u. So the theorem is proved. $\square$

We show that the linear combination of contractive maps in PM-space is not necessary contractive and Theorem 5 is not applicable in PM-space.

Example 2. Consider the Menger PM-space (X, F, T_p), where $X = [-6, 6]$ and $F_{x,y}(t)$ is the same as Example 1. Let $\lambda_1 = \lambda_2 = 1/2$, and $f_i : X \to X$, $i = 1, 2$, defined by $f_1(x) = x$ and $f_2(x) = -4$. We define the functions $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ by

$$\alpha(x,y,t) = \begin{cases} 1, & x, y \in [-6, 0] \text{ or } x, y \in [0, 6], \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta(x,y,t) = \begin{cases} 2, & x, y \in [-6, 0] \text{ or } x, y \in [0, 6], \\ 1 & \text{otherwise}. \end{cases}$$

Let $\varphi(t) = \psi(t) = t$. Then f_1 is (α, β, ψ)-contractive map. To view this, suppose $x, y < 0$ and $x < y$, then $\alpha(x, y, t) = 1$ and $\beta(x, y, t) = 2$. So inequality (1) holds for all $c \in (1/2, 1)$. In the same way, we can show that for other cases, inequaility (1) holds, and f_1 is (α, β, ψ)-contractive map. It is easy to see that f_2 is (α, β, ψ)-contractive map. Also, f_1 and f_2 have common fixed point $x = -4$. We show that Theorem 5(ii) is not applicable in this case since $f = f_1/2 + f_2/2 = x/2 - 2$ is not (α, β, ψ)-contractive map. To see this, consider $x = 6$ and $y = 3$, then there is not any $c \in (0, 1)$ such that inequality (1) holds. Hence, f is not (α, β, ψ)-contractive map.

Remark 4. Notice that the previous result also holds when $\lambda_i \in [0, 1]$ instead of $\lambda_i \in (0, 1)$. In this case, if $\lambda_j = 0$ for some $j \in \{1, 2, \dots, m\}$ and $\lambda_i \neq 0$ for the remaining elements, Theorem 5 holds for $\bigcap_{\lambda_i \neq 0} \mathrm{Fix}\{f_i\}$.

Corollary 1. *In Theorem* 5*, if we have* $f = f_n = \sum_{i=1}^{m} \lambda_{in} f_i$ *with* $\lambda_{in} \in (0, 1)$ *and* $\sum_{i=1}^{m} \lambda_{in} = 1$ *for each* $n \geqslant 1$*, then the results of Theorem* 5 *hold as well.*

In fact, in equations (6) *and* (7) *of the proof of Theorem* 5*, we set up* λ_{in} *instead of* λ_i*, we get* $\lim_{n\to\infty} \lambda_{in} x_n^i$ *as follows*: *Since* $\lim_{n\to\infty} x_n^i = u$*, there exists* k*:* $n \geqslant k \Rightarrow |x_n^i - u| \leqslant \delta$*. Thus we have*

$$\begin{aligned} \left| \sum_{i=1}^{m} \lambda_{in} x_n^i - u \right| &= \left| \sum_{i=1}^{m} \lambda_{in} x_n^i - \sum_{i=1}^{m} \lambda_{in} u \right| = \left| \sum_{i=1}^{m} \lambda_{in} \left(x_n^i - u \right) \right| \\ &\leqslant \left| \sum_{i=1}^{m} \lambda_{in} \delta \right| \leqslant \delta. \end{aligned}$$

This means $\lim_{n\to\infty} \sum_{i=1}^{m} \lambda_{in} x_n^i = u$*. Hence, the results hold in the case of time-varying linear combinations of contractive maps.*

The subsequent result shows that a linear combination of ψ-contractive maps that satisfy Theorem 1, has a common fixed point.

Proposition 2. *Let $\{f_i\}_{i=1}^m$ be a finite family of self-maps on X satisfying all conditions of Theorem 1, $\bigcap_{i=1}^m \operatorname{Fix}(f_i) = \{u_1, u_2, \dots, u_m\}$ and also continuous at the common fixed points. Let $f = \sum_{i=1}^m \lambda_{in} f_i$, where $\lambda_{in} \in (0,1)$ and $\sum_{i=1}^m \lambda_{in} = 1$. Then $\{u_1, u_2, \dots, u_m\}$ are also fixed points of f.*

Moreover, for each $u \in \bigcap_{i=1}^m \operatorname{Fix}\{f_i\}$, let $\{x_n^i\}$ be the sequence associated to each f_i such that $\{x_n^i\} \to u$ and $x_{n+1}^i = f_i(x_n^i)$ with $x_0^i \in X$ arbitrary. Then the following conditions hold:

(i) *Consider $\{t_n\}$ to be the corresponding sequence of f such that $t_n = \sum_{i=1}^m \lambda_{in} x_n^i$. Then there exists a sequence $\{w_n\}$ satisfying $w_n = f(t_n)$ such that $\{w_n\}$ converges to u as $n \to \infty$.*

(ii) *Consider $\{z_n\}$ be one of the sequences associated to f such that $z_{n+1} = f(z_n)$ with $z_0 \in X$ arbitrary. Then $\{z_n\}$ converges to u as $n \to \infty$.*

Proof. The proof of this result follows the same steps as the proof of Theorem 5. □

Theorem 6. *Let $\{f_i\}_{i=1}^m$ be a finite family of self-maps on X that satisfy all conditions of Theorem 2, and $\{g_n^i\}_{i=1}^k$ be a finite family of sequences of continuous operators $g_n^i : X \to X$ such that $\{g_n^i\}$ uniformly convergence to g, where $g : X \to X$ is a (α, β, ψ)-contractive map. Also, assume that $\operatorname{Fix}(f_i) \cap \operatorname{Fix}(g_n^i) = \{u_1, u_2, \dots, u_m\}$ and $\{f_i\}$ is continuous in the common fixed points. Let $f = \sum_{i=1}^m \lambda_i f_i + \sum_{i=1}^k \mu_i g_n^i$, where $\lambda_i, \mu_i \in (0,1)$ and $\sum_{i=1}^m \lambda_i + \sum_{i=1}^k \mu_i = 1$. Then $\{u_1, u_2, \dots, u_m\}$ are also fixed points of f.*

Moreover, for each $u \in \operatorname{Fix}(f_i) \cap \operatorname{Fix}(g_n^i)$, let $\{x_n^i\}$ be the sequence associated to each f_i with $x_0^i \in X$ arbitrary, and let $\{y_n^i\}$ be the sequence associated to g_n^i with $y_0^i \in X$ arbitrary such that $\{x_n^i\} \to u$ and $\{y_n^i\} \to u$ as $n \to \infty$ and $x_{n+1}^i = f_i(x_n^i)$, $y_{n+1}^i = g_n^i(y_n^i)$. Consider $\{t_n\}$ to be the corresponding sequence of f such that $t_n = \sum_{i=1}^m \lambda_i x_n^i + \sum_{i=1}^k \mu_i y_n^i$. Then there exists a sequence $\{w_n\}$ satisfying $w_n = f(t_n)$ such that $\{w_n\}$ converges to u as $n \to \infty$.

Proof. We show that u is a fixed point of f (linear combination of f_i and g_n^i), where $u \in \operatorname{Fix}(f_i) \cap \operatorname{Fix}(g_n^i)$. Afterwards, we prove that the sequence $\{w_n\}$ converges to u. Since u is a common fixed point of f_i and g_n^i, we have

$$\begin{aligned} f(u) &= \left(\sum_{i=1}^m \lambda_i f_i + \sum_{i=1}^k \mu_i g_n^i\right) u \\ &= \lambda_1 f_1(u) + \lambda_2 f_2(u) + \dots + \lambda_m f_m(u) \\ &\quad + \mu_1 g_n^1(u) + \mu_2 g_n^2(u) + \dots + \mu_k g_n^k(u) \\ &= (\lambda_1 + \lambda_2 + \dots + \lambda_m + \mu_1 + \mu_2 + \dots + \mu_k) u. \end{aligned} \tag{10}$$

Since $\sum_{i=1}^m \lambda_i + \sum_{i=1}^k \mu_i = 1$, from (10) we get $f(u) = u$ and u is a fixed point of f.

Now we prove that the sequence $\{w_n\}$ converges to u. According to Proposition 1, we know that each sequence $\{x_n^i\}$ converges to u, and also the sequence $\{y_n^i\}$ converges to u by Theorem 4. We have $w_n = f(t_n)$. Hence,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} \sum_{i=1}^{m} \lambda_i f_i(t_n) + \lim_{n\to\infty} \sum_{i=1}^{k} \mu_i g_n^i(t_n). \tag{11}$$

On the other hand, we know

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \left(\sum_{i=1}^{m} \lambda_i x_n^i + \sum_{i=1}^{k} \mu_i y_n^i \right) = \sum_{i=1}^{m} \lambda_i(u) + \sum_{i=1}^{k} \mu_i(u) = u. \tag{12}$$

Now from (11), (12) and since the function f_i and g_n^i are continuous at the common fixed points, we get

$$\lim_{n\to\infty} w_n = \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{i=1}^{k} \mu_i g_n^i(u) = u.$$

So the theorem is proved. $\square$

Corollary 2. *In Theorem* 6*, if we have $f = \sum_{i=1}^{m} \lambda_{in} f_i + \sum_{i=1}^{k} \mu_{in} g_n^i$, where $\lambda_{in}, \mu_{in} \in (0,1)$ and $\sum_{i=1}^{m} \lambda_{in} + \sum_{i=1}^{k} \mu_{in} = 1$ for $n \geqslant 1$, then the results hold, too.*

Theorem 7. *Let $\{f_i\}_{i=1}^{m_1}$ be a finite family of (α, β, ψ)-contractive self-maps on X that satisfy Theorem* 2*, and $\{g_i\}_{i=1}^{m_2}$ be a family of operators from X to X. Also, assume that $\mathrm{Fix}(f_i) \cap \mathrm{Fix}(g_i) = \{u_1, u_2, \dots, u_m\}$ and $\{f_i\}$ are continuous in the common fixed poins. Let $f = \sum_{i=1}^{m_1} \lambda_{in} f_i + \sum_{i=m_1+1}^{m_1+m_2} \lambda_{in} g_i$, where $\lambda_{in} \in (0,1)$ and $\sum_{i=1}^{m_1+m_2} \lambda_{in} = 1$ for $n \geqslant 1$. Assume that we have the following condition*:

(i) *$\lambda_{in} \to \lambda_i^*$, $i = 1, 2, \dots, m_1$, and $\lambda_{in} \to 0$, $i = m_1+1, \dots, m_1+m_2$, as $n \to \infty$.*

Then $\{u_1, u_2, \dots, u_m\}$ are also fixed points of f. Moreover, for each $u \in \mathrm{Fix}(f_i) \cap \mathrm{Fix}(g_i)$, let $\{x_n^i\}$ and $\{y_n^i\}$ be the sequences associated to each f_i and g_i, respectively, such that $\{x_n^i\}$ and $\{y_n^i\}$ converge to u as $n \to \infty$ with $x_{n+1}^i = f_i(x_n^i)$ and $y_{n+1}^i = g_i(y_n^i)$. Consider $\{t_n\}$ to be the corresponding sequence of f such that $t_n = \sum_{i=1}^{m} \lambda_i x_n^i + \sum_{i=1}^{k} \mu_i y_n^i$. Then there exists a sequence $\{w_n\}$ satisfying $w_n = f(t_n)$ such that $\{w_n\}$ converges to u as $n \to \infty$.

Proof. We show that u is a fixed point of f:

$$\begin{aligned} f(u) &= \left(\sum_{i=1}^{m_1} \lambda_{in} f_i + \sum_{i=m_1+1}^{m_1+m_2} \lambda_{in} g_i \right) u = \sum_{i=1}^{m_1} \lambda_{in} f_i(u) + \sum_{i=m_1+1}^{m_1+m_2} \lambda_{in} g_i(u) \\ &= \left(\sum_{i=1}^{m_1+m_2} \lambda_{in} \right) u = u, \end{aligned}$$

hence, f has the fixed point u.

Now we want to show that the sequence $\{w_n\}$ converges to u. We have

$$\begin{aligned}\lim_{n\to\infty} w_n &= \lim_{n\to\infty} f(t_n)\\ &= \lim_{n\to\infty}\sum_{i=1}^{m_1}\lambda_{in} f_i(t_n) + \lim_{n\to\infty}\sum_{i=m_1+1}^{m_1+m_2}\lambda_{in} g_i(t_n).\end{aligned} \tag{13}$$

Since $\lim_{n\to\infty}\lambda_{in} = \lambda_i^*$, $i = 1, 2, \dots, m_1$, and $\lim_{n\to\infty}\lambda_{in} = 0$, $i = m_1+1, \dots, m_1+m_2$, in (13), we get

$$\begin{aligned}\lim_{n\to\infty} w_n &= \lim_{n\to\infty} f(t_n)\\ &= \lim_{n\to\infty}\sum_{i=1}^{m_1}\lambda_i^* f_i(t_n) + \underbrace{\lim_{n\to\infty}\sum_{i=m_1+1}^{m_1+m_2}\lambda_{in} g_i(t_n)}_{0}.\end{aligned} \tag{14}$$

On the other hand,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\left(\sum_{i=1}^{m_1}\lambda_{in} x_n^i + \sum_{i=m_1+1}^{m_1+m_2}\lambda_{in} y_n^i\right), \tag{15}$$

We know $\lim_{n\to\infty}\lambda_{in} = 0$, $i = m_1+1, \dots, m_1+m_2$, and also, according to Proposition 1, we know that each sequence $\{x_n^i\}$ converges to u. Hence, in (15), we get

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\sum_{i=1}^{m_1}\lambda_{in} x_n^i + \underbrace{\lim_{n\to\infty}\sum_{i=m_1+1}^{m_1+m_2}\lambda_{in} y_n^i}_{0} = \left(\sum_{i=1}^{m_1}\lambda_i^*\right) u = u. \tag{16}$$

Now from (16) and since the function f_i is continuous in u, in (14) we conclude that

$$\lim_{n\to\infty} w_n = \sum_{i=1}^{m_1}\lambda_i^* f_i(u) = \sum_{i=1}^{m_1}\lambda_i^* u = u.$$

This means the sequence $\{w_n\}$ converges to the fixed point of f. So the theorem is proved. □

The subsequent result will show that a linear combination of an infinite family of contractive maps in Menger EPM-space have a common fixed point. Also, we prove that the iterative sequences generated by this linear combination converges to the fixed point.

Theorem 8. *Let $\{f_i\}_{i\in N}$ be an infinite family of self-maps on X that is enhanced α–β-admissible satisfying Theorem 2, and $\cap\,\mathrm{Fix}(f_i) = \{u_1, u_2, u_3, \dots\}$. Consider $f = \sum_{i=1}^{\infty}\lambda_i f_i$, where $\lambda_i \in (0,1)$ and $\sum_{i=1}^{\infty}\lambda_i = 1$ for all $i \in N$. Then $\{u_1, u_2, u_3, \dots\}$ are also fixed points of f.*

Moreover, for each $u \in \bigcap_{i=1}^{\infty} \mathrm{Fix}\{f_i\}$, let $\{x_n^i\}$ be the sequence associated to each f_i with $x_0^i \in X$ arbitrary such that $\{x_n^i\} \to u$ and $x_{n+1}^i = f_i(x_n^i)$. Consider $\{z_n\}$ be one of the sequences associated to f such that $z_{n+1} = f(z_n)$ with $z_0 \in X$ arbitrary. Then $\{z_n\} \to u$ as $n \to \infty$.

Proof. It is easy to see that u is a fixed points of f. Now we show that the iterative sequence $\{z_n\}$ converges to the fixed point u. According to (EPM4)-property, we have

$$\begin{aligned}\frac{1}{F_{fx,fy}(\varphi(ct))} &= \frac{1}{F_{\sum_{i=1}^{\infty}\lambda_i f_i x, \sum_{i=1}^{\infty}\lambda_i f_i y}(\varphi(ct))} \leqslant \sum_{i=1}^{\infty}\lambda_i \frac{1}{F_{f_i x, f_i y}(\varphi(ct))} \\ &= \sum_{i=1}^{\infty}\lambda_i\left(\frac{1}{F_{f_i x, f_i y}(\varphi(ct))} - 1\right) + 1, \end{aligned} \tag{17}$$

From (17) and because each f_i is (α, β, ψ)-contractive, we get

$$\begin{aligned}\alpha(x,y,t)\left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1\right) &\leqslant \sum_{i=1}^{\infty}\lambda_i\alpha(x,y,t)\left(\frac{1}{F_{f_i x, f_i y}(\varphi(ct))} - 1\right) \\ &\leqslant \sum_{i=1}^{\infty}\lambda_i\beta(x,y,t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \\ &= \beta(x,y,t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right),\end{aligned}$$

this means that f is (α, β, ψ)-contractive map. Also, by the same proof of Theorem 5, f is $\alpha{-}\beta$-admissible. It is easy to check that f satisfy all conditions of Theorems 2. Thus, according to Proposition 1, the sequence $z_{n+1} = f(z_n)$ converges to the fixed point u. So the theorem is proved. □

Corollary 3. *In Theorem 8, if we have $f = f_n = \sum_{i=1}^{\infty}\lambda_{in} f_i$, where $\lambda_{in} \in (0,1)$ and $\sum_{i=1}^{\infty}\lambda_{in} = 1$ for $n \geqslant 1$, then the results hold, too.*

Notice that in Theorems 5 and 8, we prove that, when $\{f_i\}_{i=1}^m$ is a family of enhanced $\alpha{-}\beta$-admissible self-maps, then the linear combination of $\{f_i\}_{i=1}^m$ is $\alpha{-}\beta$-admissible. Now an open question arises: is the linear combination of $\{f_i\}_{i=1}^m$, $\alpha{-}\beta$-admissible when each element of the $\{f_i\}_{i=1}^m$ family is $\alpha{-}\beta$-admissible?

4 Numerical examples

In this section, we consider some numerical examples obtained by simulation, which illustrate the main results discussed in the previous sections. The first, second and third examples are concerned with a finite linear combination of contractive mappings with constant and time-varying coefficients. The other examples illustrate the results previously established in Theorem 8 and Corollary 3.

Example 3. Let $X = \mathbb{R}$, $F_{x,y}(t) = t/(t+|x-y|)$ for all $x, y \in X$, $t > 0$, and the t-norm T is defined as $T(a,b) = ab$. Then (X, F, T) is a G-complete Menger EPM-space.

Consider the dynamic system given by $z_{n+1} = f(z_n)$ with $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$, $\lambda_1 = 1/2$, $\lambda_2 = 1/5$, $\lambda_3 = 31/0$ and $f_i : X \to X$, $i = 1, 2, 3$, by

$$f_1(x) = \begin{cases} \frac{x}{4}, & x \in [0,1], \\ 0 & \text{otherwise}, \end{cases} \qquad f_2(x) = \frac{1}{4}\sin(x), \qquad f_3(x) = \begin{cases} \frac{x}{5}, & x \in [0,1], \\ \frac{x^2}{4} & \text{otherwise}. \end{cases}$$

We define the functions $\alpha, \beta : X \times X \times (0,\infty) \to [0,\infty)$ by

$$\alpha(x,y,t) = \begin{cases} 1, & x, y \in [0,1], \\ 0 & \text{otherwise}, \end{cases} \qquad \beta(x,y,t) = \begin{cases} \frac{1}{2}, & x, y \in [0,1], \\ 2 & \text{otherwise}. \end{cases}$$

Consider $c = 3/4$ and $\varphi, \psi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \psi(t) = t$. We want to prove that mappings f_i and f satisfy the hypotheses of Theorem 5. It is easy to see that f_i and f satisfy all conditions of Theorem 2 and $x = 0$ is a common fixed point of f and f_i. Define the sequence $\{x_n^i\}$ such that $x_{n+1}^i = f_i(x_n^i)$, $i = 1, 2, 3$, with $x_0^1 = 1/4$, $x_0^2 = \pi/4$ and $x_0^3 = 1/2$. Hence, we get $(x_n^1) = 1/4^n$, $(x_n^2) = (1/4^n)\sin^n(\pi/4)$ and $(x_n^3) = (1/2)(1/5^n)$. It is easy to see these sequences converge to zero as $n \to \infty$. Now let $\{t_n\}$ be the corresponding sequence of f such that $t_n = \sum_{i=1}^{3} \lambda_i x_n^i$. We have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{i=1}^{3} \lambda_i x_n^i = 0$. On the other hand, since function f_i is continuous in $x = 0$. Hence, we get

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} \sum_{i=1}^{3} \lambda_i f_i(t_n) = 0.$$

Now we define the sequence (z_n) by $z_{n+1} = f(z_n)$ with $z_0 = 1$. Then we get $z_n = (9/20)^n + (1/4^n)\sin^n(1/4)$. Hence, $\lim_{n\to\infty} z_n = 0$, i.e., the sequence $\{z_n\}$ converges to the fixed point. For a better understanding of the above example, see Fig. 1 in which sequences $\{t_n\}$ converges to the fixed point u. Also, see Fig. 2 in which the sequence $\{w_n\}$ converges to u and Fig. 3, where the sequence $\{z_n\}$ converges to u, as predicted by Theorem 5.

Example 4. Let $X = \mathbb{R}$, $F_{x,y}(t) = t/(t + |x-y|)$ for all $x, y \in X$, $t > 0$, and the t-norm T is defined as $T(a,b) = ab$. Then (X, F, T) is a G-complete Menger EPM-space. Consider the switched dynamic system $z_{n+1} = f_n(z_n)$ given by $f_n = \lambda_{1n} f_1 + \lambda_{2n} f_2 + \lambda_{3n} f_3$ with $\lambda_{1n} = 1/2$, $\lambda_{2n} = 1/5$, $\lambda_{3n} = 3/10$.

Define: $\lambda_{1n+1} = \lambda_{3n}$, $\lambda_{2n+1} = \lambda_{1n}$, $\lambda_{3n+1} = \lambda_{2n}$. Let the functions f_i, α_i, β_i and φ, ψ be the same as Example 3. Then Corollary 1 holds and the switched dynamic system is stable. See Fig. 4, where the iterated sequence converge to the fixed (equilibrium) point $x = 0$. Also, if we define: $\lambda_{1n+1} = \lambda_{3n} - 1$, $\lambda_{2n+1} = \lambda_{1n} + 1/2$, $\lambda_{3n+1} = \lambda_{2n} + 1/2$ for $n \in N$, then the results hold, and the switched dynamic system is stable. This is an example of how these techniques are useful to study switched dynamic systems and prove that the stability holds independent of the switching law.

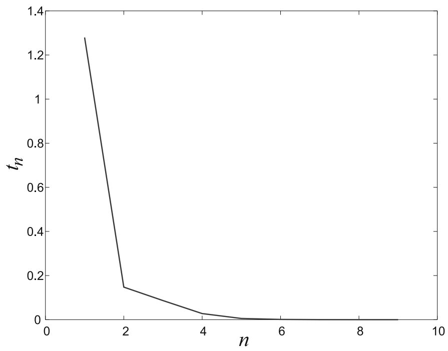

Figure 1. Convergence of the sequence $\{t_n\}$ defined by $t_n = \sum_{i=1}^m \lambda_i x_n^i$, where $\{x_n^i\}$ converge to u.

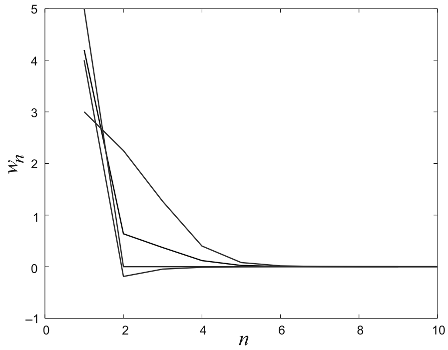

Figure 2. Convergence of the sequence $\{w_n\}$ defined by $w_n = f(t_n)$ to the fixed point.

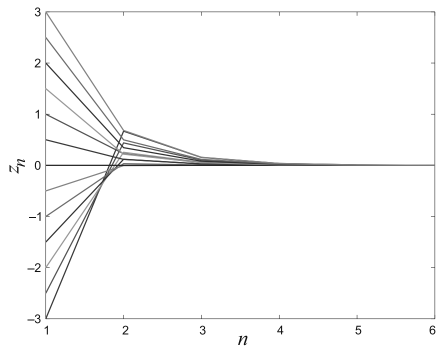

Figure 3. Convergence of the sequence $\{z_n\}$ defined by $z_{n+1} = f(z_n)$, where $f = \sum_{i=1}^m \lambda_i f_i$ for different initial conditions.

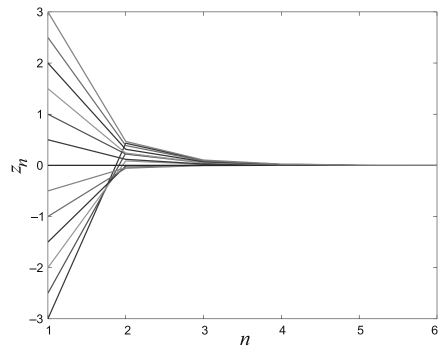

Figure 4. Convergence of the sequence $\{z_n\}$ defined by $z_{n+1} = f_n(z_n)$, where $f_n = \sum_{i=1}^m \lambda_{in} f_i$ for different initial conditions.

Example 5. Let $X = [0,1]$, $F_{x,y}(t) = t/(t + |x - y|)$ for all $x, y \in X$, $t > 0$, and the t-norm T is defined as $T(a,b) = ab$. Then (X, F, T) is a G-complete Menger EPM-space. Consider the dynamic system $z_{n+1} = f(z_n)$ given by $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ with $\lambda_1 = 1/4$, $\lambda_2 = 2/4$, $\lambda_3 = 1/4$ and $f_i : X \to X$ by $f_1(x) = x/2$, $f_2(x) = x$ and $f_3(x) = x/4$ for all $x \in X$. We consider $c = 1/2$ and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \varphi(t) = t$. It is easy to check that each f_i is ψ-contractive map and $x = 0$ is common fixed point for f_i. Now $fx = x/8 + x/2 + x/16 = 11x/16$, and $x = 0$ is a fixed point of f. Consider $z_0 = 1$. Then we get $\lim_{n\to\infty} z_n = \lim_{n\to\infty} (11/16)^n = 0$, i.e., the iterative sequence $\{z_n\}$ converges to the fixed point as predicted by Proposition 2, and the dynamic system is stable (asymptotically stable).

Example 6. Let $X = [0,1]$, $F_{x,y}(t) = t/(t + |x - y|)$ for all $x, y \in X$, $t > 0$, and the t-norm T is defined as $T(a,b) = ab$. Then (X, F, T) is a G-complete Menger EPM-space. Consider the switched dynamic system given by $z_{n+1} = f(z_n)$ with $f = \sum_{i=1}^\infty \lambda_i f_i(x)$ with $\{\lambda_i\} = 1/2^i$ and $f_i : X \to X$ by $f_i(x) = x^2/(10i)$. We define $\varphi, \psi : [0,\infty) \to$

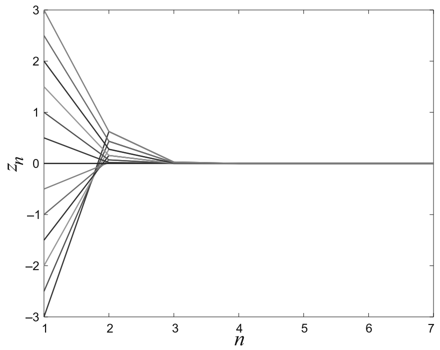

Figure 5. Convergence of the sequence $\{z_n\}$ defined by $z_{n+1} = f(z_n)$, where $f = \sum_{i=1}^{\infty} \lambda_i f_i$ for different initial conditions.

$[0, \infty)$ by $\varphi(t) = \psi(t) = t$ and $\alpha_i, \beta_i : X \times X \times (0, \infty) \to [0, \infty)$ by $\alpha_i(x, y, t) = 2$ and $\beta_i(x, y, t) = 1/2$ for all $x, y \in X$. It is easy to see that f_i and f satisfy the hypotheses of Theorem 2 for $c = [2/(5i), 1)$, and hence, f_i have the fixed point $x = 0$. We have $f = \sum_{i=1}^{\infty} \lambda_i f_i(x) = \sum_{i=1}^{\infty} (1/2^i)(x^2/(10i))$. According to Theorem 8, f has a common fixed point. In fact, $x = 0$ is common fixed point for f. Now we define the sequence $\{z_n\}$ by $z_{n+1} = f(z_n)$ with $z_0 = 1$. Then we get $z_n = (\sum_{i=1}^{\infty} (1/2^i)(1/(10i))^n$. Hence, $\lim_{n\to\infty} z_n = 0$. i.e., the sequence $\{z_n\}$ converge to the fixed point. See Fig. 5, where the iterative sequence of infinite linear combination of contractive maps converges to the fixed point.

Example 7. Let $X = [0, 1]$, $F_{x,y}(t) = t/(t + |x - y|)$ for all $x, y \in X$, $t > 0$, and the t-norm T is defined as $T(a, b) = ab$. Then (X, F, T) is a G-complete Menger EPM-space. Consider the switched dynamic system given by $z_{n+1} = f_n(z_n)$ with $f_n = \sum_{i=1}^{\infty} \lambda_{in} f_i(x)$ with $\{\lambda_{in}\} = 1/2^i$. Define $\lambda_{in+1} = \lambda_{in}$. Let the functions f_i, α_i, β_i and φ, ψ be the same as Example 3. Then Corollary 3 holds, and the switched dynamic system is stable. The fixed point theory is a useful tool in this content.

5 Conclusion

In this paper, we investigate the existence of fixed points for a (finite or infinite) linear combination of (α, β, ψ)-contractive and ψ-contractive mappings. Also, we investigate about the convergence of sequences (generated by a linear combination of (α, β, ψ)-contractive and ψ-contractive mappings) to the fixed point and prove that these sequences converge to the common fixed point. Also, we study about linear combination of mappings such that some of them are contractive and some of them are not. In this case, we provide the condition that ensure that these linear combinations have common fixed points.

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References

1. M.S. Adamo, C. Vetro, Fixed point and homotopy results for mixed multi-valued mappings in 0-complete partial metric spaces, *Nonlinear Anal. Model. Control*, **20**(2):159–174, 2015.
2. C. Alaca, A common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type, *Albanian J. Math.*, **3**(1):13–17, 2009.
3. R. Allahyari, R. Arab, A.S. Haghighi, Common fixed point theorems for infinite families of contractive maps, *Math. Sci., Springer*, **9**(4):199–203, 2015.
4. A.B. Amar, A. Jeribi, M. Mnif, Some fixed point theorems and application to biological model, *Numer. Funct. Anal. Optim.*, **29**(1–2):1–23, 2008.
5. H. Aydi, M. Jellali, E. Karapınar, On fixed point results for α-implicit contractions in quasi-metric spaces and consequences, *Nonlinear Anal. Model. Control*, **21**(1):40–56, 2016.
6. H. Aydi, E. Karapinar, B. Samet, Remarks on some recent fixed point theorems, *Fixed Point Theory Appl.*, **2012**(1):76, 2012.
7. N. Buong, N.T.Q. Anh, An implicit iteration method for variational inequalities over the set of common fixed points for a finite family of nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, **2011**(1):276859, 2011.
8. B.S. Choudhury, K. Das, A new contraction principle in Menger spaces, *Acta Math. Sin., Engl. Ser.*, **24**(8):1379, 2008.
9. B.S. Choudhury, K. Das, S.K. Bhandari, Fixed point theorem for mappings with cyclic contraction in Menger spaces, *Int. J. Pure Appl. Sci. Technol.*, **4**(1):1–9, 2011.
10. K. Chwastek, The applications of fixed-point theorem in optimisation problems, *Arch. Electr. Eng.*, **61**(2):189–198, 2012.
11. L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**(2): 267–273, 1974.
12. L.B. Ćirić, On a family of contractive maps and fixed points, *Publ. Inst. Math.*, **17**(37):45–51, 1974.
13. L.B. Ćirić, On fixed points of generalized contractions on probabilistic metric spaces, *Publ. Inst. Math., Nouv. Sér.*, **18**(32):71–78, 1975.
14. L.B. Ćirić, *Fixed Point Theory. Contraction Mapping Principle*, FME Press, Beograd, 2003.
15. M. De la Sen, A. Ibeas, J. Herrera, On fixed points and convergence results of sequences generated by uniformly convergent and point-wisely convergent sequences of operators in Menger probabilistic metric spaces, *SpringerPlus*, **5**(1):557, 2016.
16. P.N. Dutta, B.S. Choudhury, K. Das, Some fixed point results in Menger spaces using a control function, *Surv. Math. Appl.*, **4**:41–52, 2009.
17. D. Gopal, M. Abbas, C. Vetro, Some new fixed point theorems in Menger PM-spaces with application to Volterra type integral equation, *Appl. Math. Comput.*, **232**:955–967, 2014.
18. O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic, Dordrecht, 2001.
19. R.H. Haghi, Sh. Rezapour, N. Shahzad, Some fixed point generalizations are not real generalizations, *Nonlinear Anal., Theory Methods Appl.*, **74**(5):1799–1803, 2011.

20. R.H. Haghi, Sh. Rezapour, N. Shahzad, Be careful on partial metric fixed point results, *Topology Appl.*, **160**(3):450–454, 2013.
21. X. Hu, X. Ma, Coupled coincidence point theorems under contractive conditions in partially ordered probabilistic metric spaces, *Nonlinear Anal., Theory Methods Appl.*, **74**(17):6451–6458, 2011.
22. M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley & Sons, New York, 2011.
23. M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30**(1):1–9, 1984.
24. V. La Rosa, P. Vetro, Common fixed points for α-ψ-ϕ-contractions in generalized metric spaces, *Nonlinear Anal. Model. Control*, **19**(1):43–54, 2014.
25. L.D. Marks, Fixed-point optimization of atoms and density in DFT, *J. Chem. Theory Comput.*, **9**(6):2786–2800, 2013.
26. K. Menger, Statistical metrics, *Proc. Natl. Acad. Sci. USA*, **28**(12):535–537, 1942.
27. D. Miheţ, Altering distances in probabilistic Menger spaces, *Nonlinear Anal., Theory Methods Appl.*, **71**(7–8):2734–2738, 2009.
28. S.B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.*, **30**(2):475–488, 1969.
29. S. Reich, Fixed points of contractive functions, *Boll. in. Mar. Ital.*, **5**:26–42, 1972.
30. S. Reich, Some fixed point problems, *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat.*, **57**(3–4):194–198, 1974.
31. B. Samet, E. Karapinar, H. Aydi, V.C. Rajic, Discussion on some coupled fixed point theorems, *Fixed Point Theory Appl.*, **2013**(1):50, 2013.
32. B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, 1983.
33. V.M. Sehgal, A.T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, *Math. Syst. Theory*, **6**(1):97–102, 1972.
34. M. Shams, S.H. Jafari, Some fixed point theorem for cyclic (α, β, ψ)-contractive mapping in probabilistic Menger space, *Int. J. Pure Appl. Math.*, **106**(3):727–740, 2016.
35. M.R. Tasković, On a family of contractive maps, *Bull. Aust. Math. Soc.*, **13**(2):301–308, 1975.