Hopf Bifurcation Analysis in a Delayed Kaldor-Kalecki Model of Business Cycle

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Abstract. In this paper, we analyze the model of business cycle with time delay set forth by A. Krawiec and M. Szydłowski [1]. Our goal in this model is to introduce the time delay into capital stock and gross product in capital accumulation equation. The dynamics are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exist as the delay (taken as a parameter of bifurcation) cross some critical value. Additionally we conclude with an application.

Keywords: Kaldor-Kalecki business cycle, delayed differential equations, Hopf bifurcation, periodic solutions.

1 Introduction and mathematical models

Great attention has been paid to equations with delay, which have significant economical and biological background (see for example [2–9]). In most application of delay differential equations in investment processes, the need of incorporation of a time delay is often the result of the time interval required between investment decision and installation of investment capital [10, 11]. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since time delay could cause a stable equilibrium to become unstable and cause the system to fluctuate.

In this paper, we consider the Kaldor-Kalecki model of business cycle with time delay as follows:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t), K(t)\big) - S\big(Y(t), K(t)\big)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau), K(t-\tau)\big) - \delta K(t), \end{cases} \tag{1}$$

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, δ is the depreciation rate of capital stock, $I(Y, K)$ is the investment function, $S(Y, K)$ is the saving and τ is the time delay needed for new capital to be installed.

Clearly, introducing time delay into capital stock and gross product in capital accumulation equation is more reasonable, because the change in the capital stock is due to the past investment decisions (see [12, p. 103]).

The first model in this optic is proposed by Kalecki (in 1935 [10]). The main characteristic feature of his model is the distinction between investment decisions and implementation, i.e. there is a time delay after which capital equipment is available for production.

Besides the influence of Keynes (in 1936 [13]) and Kalecki (in 1937 [14]), Kaldor (in 1940 [15]) presented a nonlinear model of business cycle by an ordinary differential equations as follows:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t), K(t)\big) - S\big(Y(t), K(t)\big)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t), K(t)\big). \end{cases} \tag{2}$$

In this model the nonlinearity of investment and saving function leads to limit cycle solution (see also [16–18] for more information).

Based on the Kaldor model of business cycle and the Kalecki's idea on time delay, Krawiec and Szydłowski (in 1999, [1]) proposed the following Kaldor-Kalecki model of business cycle:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t), K(t)\big) - S\big(Y(t), K(t)\big)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau), K(t)\big) - \delta K(t). \end{cases} \tag{3}$$

The fundamental characteristics of this model is the nonlinearity of investment function and the inclusion of time delay into the gross product in capital accumulation equation.

In ([1] and [6], 2000), Krawiec and Szydłowski investigated the stability and Hopf bifurcation of a positive equilibrium E^* of system (3) in the special case of small time delay. In ([12], 2001), they showed that for a small time delay parameter the Kaldor-Kalecki model assumes the form of the Lienard equation. In ([19], 2005), they investigate the stability of limit cycle. Zhang and Wei ([9], 2004) investigated local and global existence of Hopf bifurcation for (3).

In this work, the dynamics of the system (1) are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exist as the delay (taken as a parameter of bifurcation) cross some critical value. A numerical illustrations is given to compare our results and the ones (3) of Krawiec-Szydłowski model [1].

2 Steady state and stability analysis

As in [6], we consider some assumptions on the investment and saving functions:

$$I(Y,K) = I(Y) - \beta K,$$

and

$$S(Y,K) = \gamma Y,$$

where $\beta > 0$ and $\gamma \in (0,1)$. For economic justification of this simple mathematical formulation, see [20–23].

Then system (1) becomes:

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = \alpha\big[I\big(Y(t)\big) - \beta K(t) - \gamma Y(t)\big], \\ \dfrac{\mathrm{d}K}{\mathrm{d}t} = I\big(Y(t-\tau)\big) - \beta K(t-\tau) - \delta K(t). \end{cases} \tag{4}$$

2.1 Steady state

In the following proposition, we give a sufficient conditions for the existence and uniqueness of positive equilibrium E^* of the system (4).

Proposition 1. *Suppose that*

(i) *there exists a constant* $L > 0$ *such that* $|I(Y)| \le L$ *for all* $Y \in \mathbb{R}$*;*

(ii) $I(0) > 0$*;*

(iii) $I'(Y) - \gamma < \frac{\gamma\beta}{\delta}$ *for all* $Y \in \mathbb{R}$.

Then there exists a unique equilibrium $E^* = (Y^*, K^*)$ *of system* (4)*, where* Y^* *is the positive solution of*

$$I(Y) - \frac{(\beta+\delta)\gamma}{\delta} Y = 0 \tag{5}$$

and K^* *is determined by*

$$K^* = \frac{\gamma}{\delta} Y^*. \tag{6}$$

Proof. (Y, K) is a steady-state of (4) if

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\mathrm{d}K}{\mathrm{d}t} = 0,$$

that is

$$\begin{cases} I(Y) - \beta K - \gamma Y = 0, \\ I(Y) - (\beta + \delta) K = 0. \end{cases} \tag{7}$$

Let us assume that $Y > 0$ and $K > 0$ satisfy (7). Then

$$K = \frac{\gamma}{\delta} Y, \tag{8}$$

and

$$I(Y) - \frac{(\beta + \delta)\gamma}{\delta} Y = 0. \tag{9}$$

In view of hypotheses (i), (ii) and (iii) of Proposition 1, it's clear that equation (9) has a unique solution $Y^* > 0$. This concludes the proof. $\square$

2.2 Local stability analysis

Let $y = Y - Y^*$ and $k = K - K^*$. Then by linearizing system (4) around (Y^*, K^*) we have

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \alpha\big(I'(Y^*) - \gamma\big) y(t) - \alpha\beta k(t), \\ \dfrac{\mathrm{d}k}{\mathrm{d}t} = I(Y^*) y(t - \tau) - \beta k(t - \tau) - \delta k(t). \end{cases} \tag{10}$$

The characteristic equation associated to system (10) is

$$\lambda^2 + a\lambda + b\lambda \exp(-\lambda\tau) + c + d \exp(-\lambda\tau) = 0, \tag{11}$$

where

$$a = \delta - \alpha(I'(Y^*) - \gamma),$$
$$b = \beta,$$
$$c = -\alpha\delta\big(I'(Y^*) - \gamma\big),$$

and

$$d = \alpha\beta\gamma.$$

The local stability of the steady state E^* is a result of the localization of the roots of the characteristic equation (11). In order to investigate the local stability of the steady state, we begin by considering the case without delay $\tau = 0$. In this case the characteristic equation (11) reads as

$$\lambda^2 + (a + b)\lambda + c + d = 0, \tag{12}$$

hence, according to the Hurwitz criterion, we have the following lemma.

Lemma 1. *For $\tau = 0$, the equilibrium E^* is locally asymptotically stable if and only if* $I'(Y^*) - \gamma < min(\frac{\gamma\beta}{\delta}, \frac{\delta+\beta}{\alpha})$.

We now return to the study of equation (11) with $\tau > 0$.

Theorem 1. *Let the hypotheses*

(H1) $|I'(Y^*) - \gamma| < \frac{\gamma\beta}{\delta}$

and

(H2) $I'(Y^*) - \gamma < \frac{\delta+\beta}{\alpha}$.

Then there exists $\tau_0 > 0$ such that, when $\tau \in [0, \tau_0)$ the steady state E^* is locally asymptotically stable, when $\tau > \tau_0$, E^* is unstable and when $\tau = \tau_0$, equation (11) *has a pair of purely imaginary roots $\pm i\omega_0$, with*

$$\begin{aligned}\omega_0^2 = &-\frac{1}{2}\big(\alpha^2\big(I'(Y^*) - \gamma\big)^2 + \delta^2 - \beta^2\big) + \frac{1}{2}\big[(\alpha^2(I'(Y^*) - \gamma)^2 + \delta^2 - \beta^2)^2 \\ &- 4\big(\alpha^2\delta^2\big(I'(Y^*) - \gamma\big)^2 - \beta^2\gamma^2\big)\big]^{1/2}\end{aligned} \tag{13}$$

and

$$\tau_0 = \frac{1}{\omega_0}\arctan\frac{\alpha\big[\gamma\delta - (\alpha\gamma - \delta)\big(I'(Y^*) - \gamma\big)\big]\omega_0 + \omega_0^3}{\big(\alpha I'(Y^*) - \delta\big)\omega_0^2 + \alpha^2\gamma\delta\big(I'(Y^*) - \gamma\big)}. \tag{14}$$

Proof. From the hypotheses (H1) and (H2), the characteristic equation (11) has negative real parts for $\tau = 0$ (see Lemma 1). By Rouché's theorem [24, p. 248], it follows that if instability occurs for a particular value of the delay τ, a characteristic root of (11) must intersect the imaginary axis. Suppose that (11) has a purely imaginary root $i\omega$, with $\omega > 0$. Then, by separating real and imaginary parts in (11), we have

$$\begin{cases}-\omega^2 - \alpha\delta\big(I'(Y^*) - \gamma\big) + \beta\omega\sin(\omega\tau) + \alpha\beta\gamma\cos(\omega\tau) = 0,\\ \big(\delta - \alpha(I'(Y^*) - \gamma)\big)\omega + \beta\omega\cos(\omega\tau) - \alpha\beta\gamma\sin(\omega\tau) = 0.\end{cases} \tag{15}$$

Hence,

$$\omega^4 + \big(\alpha^2\big(I'(Y^*) - \gamma\big)^2 + \delta^2 - \beta^2\big)\omega^2 + \alpha^2\big(\delta^2\big(I'(Y^*) - \gamma\big)^2 - \beta^2\gamma^2\big) = 0. \tag{16}$$

It's roots are

$$\begin{aligned}\omega_\pm^2 = &-\frac{1}{2}\big(\alpha^2\big(I'(Y^*) - \gamma\big)^2 + \delta^2 - \beta^2\big) \pm \frac{1}{2}\big[\big(\alpha^2\big(I'(Y^*) - \gamma\big)^2 + \delta^2 - \beta^2\big)^2 \\ &- 4\big(\alpha^2\delta^2\big(I'(Y^*) - \gamma\big)^2 - \beta^2\gamma^2\big)\big]^{1/2}\end{aligned} \tag{17}$$

Clearly, the hypothesis (H1) implies that $\omega_0 = \omega_+$ makes sense.

From equation (15), we obtain the following set of values of τ for which there are imaginary roots:

$$\tau_{n,1} = \frac{\theta_1}{\omega_+} + \frac{2n\pi}{\omega_+},$$

where $0 \le \theta_1 < 2\pi$, and

$$\cos\theta_1 = \frac{\big(\alpha I'(Y^*) - \delta\big)\omega_0^2 + \alpha^2\gamma\delta\big(I'(Y^*) - \gamma\big)}{\beta(\omega_0^2 + \alpha^2\gamma^2)},$$
$$\sin\theta_1 = \frac{\alpha\big[\gamma\delta - (\alpha\gamma - \delta)\big(I'(Y^*) - \gamma\big)\big]\omega_0 + \omega_0^3}{\beta(\omega_0^2 + \alpha^2\gamma^2)},$$

where $n = 0, 1, 2, \ldots$.

We set $\tau_0 = \tau_{0,1}$. Thus, from (H1) and (H2), we have:

For $\tau \in [0, \tau_0)$, E^* is locally asymptotically stable.

For $\tau > \tau_0$, E^* is unstable.

For $\tau = \tau_0$, equation (11) has a purely imaginary roots $\lambda_0 = \pm i\omega_0$ where ω_0 is given by (13).

Theorem 2. *Assume that*

(H3) $I'(Y^*) - \gamma \le \min(-\frac{\beta\gamma}{\delta}, \frac{\delta^2 - \beta^2}{\alpha^2})$.

Then E^* is locally asymptotically stable for all $\tau \ge 0$.

Proof. From Lemma 1, (H3) implies that the characteristic equation (11) has all roots with negative real parts for $\tau = 0$ and no purely imaginary roots for $\tau > 0$. Thus, E^* is locally asymptotically stable for all $\tau \ge 0$. □

3 Hopf bifurcation occurrence

According to the Hopf bifurcation theorem [25], we establish sufficient conditions for the local existence of periodic solutions.

Theorem 3. *Under hypotheses* (H1) *and* (H2) *of Theorem* 1*, a Hopf bifurcation of periodic solutions of system* (4) *occurs at E^* when $\tau = \tau_0$.*

Proof. For the proof of this theorem we apply the Hopf bifurcation theorem introduced in [25]. From Lemma 1, the characteristic equation (11) has a pair of imaginary roots $\pm i\omega_0$ at $\tau = \tau_0$. In the first, lets show that $i\omega_0$ is simple: Consider the branch of characteristic roots $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$, of equation (11) bifurcating from $i\omega_0$ at $\tau = \tau_0$. By derivation of (11) with respect to the delay τ, we obtain

$$\begin{aligned}&\big\{2\lambda + \delta - \alpha\big(I'(Y^*) - \gamma\big) + \big[\beta - \tau(\beta\lambda + \alpha\beta\gamma)\big]\exp(-\lambda\tau)\big\}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\\ &\quad = (\beta\lambda + \alpha\beta\gamma)\lambda\exp(-\lambda\tau).\end{aligned} \tag{18}$$

If we suppose, by contradiction, that $i\omega_0$ is not simple, the right hand side of (18) gives

$$\alpha\gamma + i\omega_0 = 0,$$

and leads a contradiction with the fact that α and γ are positive.

Lastly we need to verify the transversally condition,

$$\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_0} \neq 0.$$

From (18), we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{\big(2\lambda + \delta - \alpha\big(I'(Y^*) - \gamma\big)\big)\exp(\lambda\tau) + \beta}{\lambda(\beta\lambda + \alpha\beta\gamma)} - \frac{\tau}{\lambda}.$$

As,

$$\mathrm{Sign}\,\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_0} = \mathrm{Sign}\,\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau_0}.$$

Then

$$\mathrm{Sign}\,\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_0} = \mathrm{Sign}\,\mathrm{Re}\frac{\big(-2i\omega_0 + \beta + \delta + \alpha\big(I'(Y^*) - \gamma\big)\big)\exp(i\omega_0\tau_0)}{-i\alpha\beta I'(Y^*)\omega_0}.$$

From (11), we have

$$\exp(\lambda\tau) = -\frac{\beta\lambda + \alpha\beta\gamma}{\lambda^2 + \delta - \alpha\big(I'(Y^*) - \gamma\big)\lambda - \alpha\delta\big(I'(Y^*) - \gamma\big)}. \tag{19}$$

So, by (H1) and (13) we obtain

$$\begin{aligned}&\mathrm{Sign}\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}\bigg|_{\tau_0}\\ &= \mathrm{Sign}\Big(\big[\big(\alpha^2\big(I'(Y^*) - \gamma\big)^2 + \delta^2 - \beta^2\big)^2 - 4\big(\alpha^2\delta^2\big(I'(Y^*) - \gamma\big)^2 - \beta^2\gamma^2\big)\big]^{1/2}\Big).\end{aligned}$$

Consequently,

$$\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}(\tau_0) > 0.$$

4 Application

4.1 Effect of additional delay

Let's compare the principal results of systems (3) (see Krawiec-Szydłowski model in [1]) and (4) by a numerical illustration. Consider the following Kaldor-type investment function:

$$I(Y) = \frac{\exp(Y)}{1 + \exp(Y)}.$$

Theorems 1 and 3 implie:

Proposition 2. *If*

$$\alpha = 3; \quad \beta = 0.2; \quad \delta = 0.1; \quad \gamma = 0.2.$$

Then systems (3) *and* (4) *have the following positive equilibrium*

$$E^* = (1.31346, 2.62699).$$

Furthermore, the critical delay and the period of oscillations corresponding to (4) *(resp.* (3)*) are* $\tau_0 = 2.9929$ *and* $P_0 = 48.2646$ *(resp.* $\tau_c = 5.3312$ *and* $P_c = 38.0053$*) (see Zhang* [9] *for more details).*

The following numerical simulations are given for system (4) for $\tau = 2$, and $\tau = 3$ and for system (3) for $\tau = 3$.

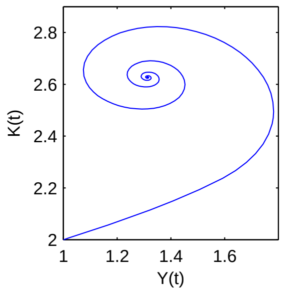

Fig. 1. The steady state E^* of (4) is stable when $\tau = 2$.

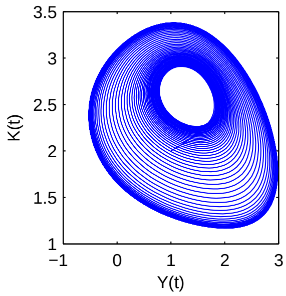

Fig. 2. The steady state E^* of (4) is unstable when $\tau = 3$.

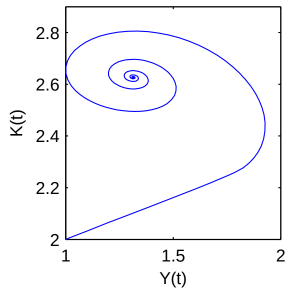

Fig. 3. The steady state E^* of the Krawiec-Szydłowski model [1] is stable when $\tau = 3$.

As $\tau_0 < \tau_c$, we think that it's more interesting to introduce the delay τ into both gross product and capital stock (see also [12, p. 103]).

4.2 Effect of changing parameters

Now, let's show how the critical delay τ_0 and the period of oscillations P_0 change as the model parameters move.

In Fig. 4, we construct the family of curves $\tau_0(\alpha, \beta, \gamma, \delta)$ assuming that three of parameters α, β, γ and δ are fixed. For values of γ which are less than a critical value $\gamma_c = 0.004$, the condition of existence of equilibrium is violated (see Fig. 4(a)). For values of β (resp. δ) which are less (resp. greater) than a critical value $\beta_c = 0.07$ (resp. $\delta_c = 0.28$), the system will not exhibit a Hopf bifurcation (see Fig. 4(b)) (resp. (see Fig. 4(d))). Additionally the family of curves $P_0(\alpha, \beta, \gamma, \delta)$ are presented in Fig. 5.

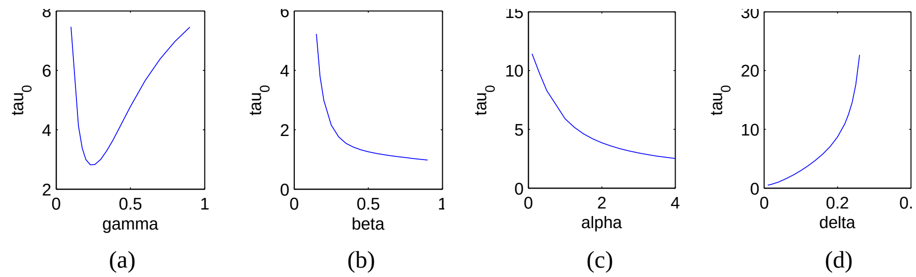

Fig. 4. The dependence of the critical value of delay τ_0 on the model parameters: (a) $\alpha = 3,\ \beta = 0.2,\ \delta = 0.1$ and $\gamma \in (0.004, 1]$; (b) $\alpha = 3,\ \gamma = 0.2,\ \delta = 0.1$ and $\beta \in (0.07, 1]$; (c) $\beta = 0.2,\ \gamma = 0.2,\ \delta = 0.1$ and $\alpha \in (0, 4]$; (d) $\alpha = 3,$ $\beta = 0.2,\ \gamma = 0.2$ and $\delta \in [0, 0.28)$.

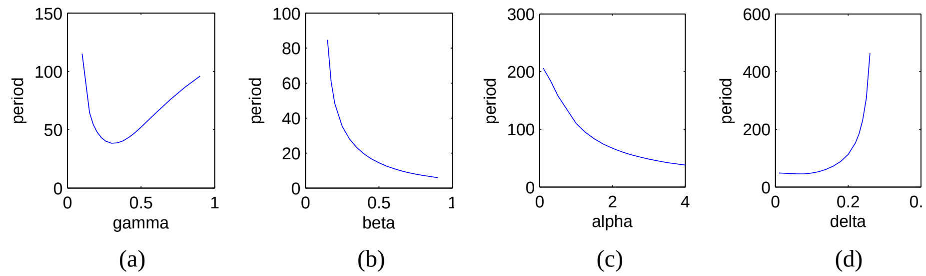

Fig. 5. The dependence of the period of oscillations P_0 on the model parameters: (a) $\alpha = 3,\ \beta = 0.2,\ \delta = 0.1$ and $\gamma \in (0.004, 1]$; (b) $\alpha = 3,\ \gamma = 0.2,$ $\delta = 0.1$ and $\beta \in (0.07, 1]$; (c) $\beta = 0.2,\ \gamma = 0.2,\ \delta = 0.1$ and $\alpha \in (0, 4]$; (d) $\alpha = 3,\ \beta = 0.2,\ \gamma = 0.2$ and $\delta \in [0, 0.28)$.

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