Analysis of a Vaccination Model for Carrier Dependent Infectious Diseases with Environmental Effects

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Abstract. We have proposed and analyzed a nonlinear mathematical model for the spread of carrier dependent infectious diseases in a population with variable size structure including the role of vaccination. It is assumed that the susceptibles become infected by direct contact with infectives and/or by the carrier population present in the environment. The density of carrier population is assumed to be governed by a generalized logistic model and is dependent on environmental and human factors which are conducive to the growth of carrier population. The model is analyzed using stability theory of differential equations and numerical simulation. We have found a threshold condition, in terms of vaccine induced reproduction number $R(\phi)$ which is, if less than one, the disease dies out in the absence of carriers provided the vaccine efficacy is high enough, and otherwise the infection is maintained in the population. The model also exhibits backward bifurcation at $R(\phi) = 1$. It is also shown that the spread of an infectious disease increases as the carrier population density increases. In addition, the constant immigration of susceptibles makes the disease more endemic.

Keywords: carrier dependent, infectious diseases, vaccination, environmental discharge, modified carrying capacity, backward bifurcation.

1 Introduction

Many infectious diseases are spread by direct contact between susceptibles and infectives. Other diseases are spread in the environment and are transmitted to the human population by insects or other vectors. Here we develop and analyze a model for diseases that are transmitted in both ways. This is the case for typhoid fever and other enteric diseases. There are many carrier dependent infectious diseases which afflict human population around the world. However, the third world countries are most affected by such diseases due to lack of sanitation, wide occurrence of carriers such as flies, ticks, mites, etc. which

are generally present in the environment [1]. For example, air-borne carriers or bacteria spread diseases such as tuberculosis and measles; while water-borne carriers or bacteria are responsible for the spread of dysentery, gastroenteritis, diarrhea, etc. [2–4]. These carriers transport infectious agents of diseases from infectives to susceptibles and thus spread such diseases in human population. In this paper, we have used the term carriers as a mode of transmission only, which transmit infectious agents of diseases from infectives to susceptibles, without having clinical symptoms.

The modeling and analysis of infectious diseases have been done by many workers, see for example [5–11]. In particular, Hethcote [10] discussed an epidemic model in which carrier population is assumed to be constant. But, in general, the size of the carrier population varies and depends on the natural conditions of the environment as well as on various human related factors. The effect of variable carrier population has not been considered in these studies, however the spread of such diseases is very much dependent on the carrier population, the density of which increases due to environmental factors such as temperature, humidity, rain, vegetation, etc. in the habitat [3, 12–14]. In particular, Ghosh et al. [13] studied the spread of carrier dependent infectious diseases with environmental effects using variable carrier population. The density of carrier population further increases as the human population density increases. With increase in human population density, the effects of human population related factors like discharge of household wastes, open sewage drainage, industrial effluents in residential areas, open water storage tanks and ponds etc. leads to further growth of carrier population density. This provides a very conducive environment for the growth of these carriers which enhances the chance of carrying more bacteria from infectives to the susceptibles in the population leading to fast spread of carrier dependent infectious diseases. Thus, unhygienic environmental conditions in the habitat caused by human population become responsible for the fast spread of an infectious disease. It is, therefore, reasonable to assume that the carrier population density is governed by a generalized logistic model. The per capita growth rate and the modified carrying capacity of carrier population are taken to be functions of human population density and assumed to increase as the human population density increases [3, 14–16]. In particular, Singh et al. [16] studied the spread of malaria by taking into account mosquito population density governed by a generalized logistic model.

It may be noted that the outbreak of infectious diseases cause mortality of millions of people as well as expenditure of enormous amount of money in health care and disease control. It is, therefore, essential that adequate attention must be paid to stop spreading of such diseases by taking control measures. Vaccination is an important control measure to reduce spreading of such diseases. Various modeling studies have been made to study the role of vaccination on the spread of infectious diseases [17–22]. In particular, Shulgin et al. [21] studied a simple SIR epidemic model with pulse vaccination and showed that pulse vaccination leads to epidemic eradication if certain conditions regarding the magnitude of vaccination proportion and on the period of pulses are satisfied. Kribs-Zaleta and Velasco-Hernandez [20] presented a simple two dimensional SIS model with vaccination exhibiting backward bifurcation. Farrington [18] analyzed the impact of vaccination program on the transmission potential of the infection in large populations and derived relation between vaccine efficacy against transmission, vaccine coverage and reproduction

numbers. Gumel and Moghadas [19] proposed a model for the dynamics of an infectious disease in the presence of a preventive vaccine considering non-linear incidence rate and found the optimal vaccine coverage threshold needed for disease control and eradication.

In the case of carrier dependent infectious diseases like cholera, measles, etc., the vaccination can be an important tool to help control the spread of such diseases especially when the density of carrier population increases with human population density related factors. It is pointed out here that in above models, vaccination has been studied without considering the effective role of variable carrier population which depends on human population related factors and is responsible for spreading the infectious diseases. In this paper, we extend the model presented by Singh et al. [14] and Ghosh et. al [15] by incorporating the effect of vaccination on the spread of carrier dependent infectious diseases and assuming a generalized logistic model governing the growth of carrier population. In addition, we use more realistic standard mass action type interaction for direct contact between susceptibles and infectives instead of simple mass action. However, we assume that susceptibles are infected by carriers in direct proportion to the density of carrier population (bilinear interaction). The model is analyzed qualitatively to determine the stability of its associated equilibria and the optimal vaccine coverage level needed to control effectively or eradicate the disease. The numerical simulation of the model is also given to see the influence of certain key parameters on the spread of the disease.

2   Mathematical model

We consider the human population $N(t)$ at time t with immigration of susceptibles at a constant rate A. The total population is divided into three subclasses: the susceptibles $X(t)$, the infectives $Y(t)$ and the vaccinated individuals $V(t)$. In the modeling process, it is assumed that the susceptibles are infected by the direct interaction with infectives and also by the carrier population of density $C(t)$, which is governed by a generalized logistic model. It is further assumed that the susceptibles are vaccinated at a constant rate and some of them may again become infected while coming in contact with infectives or with carriers due to inefficacy of vaccines. It is also considered that the infected individuals, after being recovered, may again become susceptible. The block diagram of the model is given in Fig. 1.
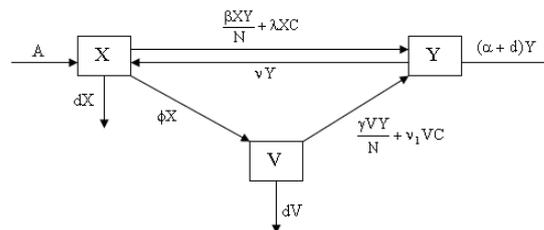

Fig. 1. Block diagram of the model.

Thus, by assuming standard mass action interaction for direct contact between susceptibles and infectives and simple mass action interaction between susceptibles and carrier population density, the model dynamics is governed by following system of nonlinear ordinary differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= A - \frac{\beta XY}{N} - \lambda XC + \nu Y - dX - \phi X, \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= \frac{\beta XY}{N} + \lambda XC - (\nu + \alpha + d)Y + \frac{\gamma VY}{N} + \nu_1 VC, \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \phi X - \frac{\gamma VY}{N} - \nu_1 VC - dV, \\
\frac{\mathrm{d}N}{\mathrm{d}t} &= A - dN - \alpha Y, \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= s(N)C - \frac{s_0 C^2}{L(N)} - s_1 C, \\
X + Y + V &= N, \\
X(0) = X_0 &> 0, \ \ Y(0) = Y_0 \geq 0, \ \ V(0) = V_0 \geq 0, \\
N(0) = N_0 &> 0, \ \ C(0) = C_0 \geq 0,
\end{aligned}
\tag{1}
$$

where β and λ are transmission coefficients due to infectives and carrier population respectively. The parameters ϕ, ν, and d represent the vaccination coverage (of susceptibles), therapeutic treatment coverage (of infected individuals) and natural deaths respectively, α is the disease related death constant, γ and ν_1 denote the transmission coefficient of vaccinated individuals due to interaction with infectives and carrier population respectively. However, the rate with which vaccinated persons become infected is very small as compared to the rate with which susceptibles get infected i.e., $\gamma \ll \beta$ and $\nu_1 \ll \lambda$. The constant s_1 is the death rate coefficient of carriers due to natural factors as well as by control measures. Here, $s(N)$ denotes the growth rate per capita of the carrier population density such that $s(N) - s_1$ is its intrinsic growth rate. It may be noted that if the growth rate and death rate due to natural as well as control measures of carrier population are balanced, then it may tend to zero. Similarly, $L(N)$ is the modified carrying capacity of the carrier population and its value is $L(N)[\frac{s(N)-s_1}{s_0}]$ as compared to usual logistic model.

It has been pointed out in the introduction, that as the human population increases, the effects of human population related factors/activities enhance the chances of growth of carrier population. Thus, in the model, $s(N)$ and $L(N)$ are taken to be functions of total human population instead of infective population. Since we assume that the growth rate per capita increases as the human population density increases, we have

$$
s(0) = s_0 \ \text{ and } \ s'(N) \geq 0,
\tag{2}
$$

where s_0 is the value of $s(N)$ at $N = 0$ and $()'$ denotes the derivative of the function with respect to its argument. We also assume that the modified carrying capacity increases

with human population density, so that

$$L(0) = L_0 > 0 \quad \text{and} \quad L'(N) \geq 0, \tag{3}$$

where L_0 is the value of $L(N)$ when $N = 0$.

From equations (1), (2) and (3), we see that even if human population related factors are absent, carrier population density increases in its natural environment and it tends to $L_0(1 - \frac{s_1}{s_0})$ which may become zero if $s_1 \to s_0$. In the model, all the dependent variables and parameters are assumed to be non-negative.

3 Equilibrium analysis

It is sufficient to consider the reduced system of model (1) (since $X + Y + V = N$) as follows:

$$
\begin{aligned}
\frac{dY}{dt} &= \frac{\beta(N - Y - V)Y}{N} + \lambda(N - Y - V)C - (\nu + \alpha + d)Y + \frac{\gamma VY}{N} + \nu_1 VC, \\
\frac{dV}{dt} &= \phi(N - Y - V) - \frac{\gamma VY}{N} - \nu_1 VC - dV, \\
\frac{dN}{dt} &= A - dN - \alpha Y, \\
\frac{dC}{dt} &= s(N)C - \frac{s_0 C^2}{L(N)} - s_1 C.
\end{aligned}
\tag{4}
$$

The equilibrium analysis of the model system (4) has been carried out and the results are given as follows. There exist following three non-negative equilibria of the system (4).

1. Disease free equilibrium, $E_0(0, \frac{\phi A}{d(\phi+d)}, \frac{A}{d}, 0)$ exists, without any condition. The existence of E_0 is obvious.

2. Carrier free equilibrium, $E_1(\overline{Y}, \overline{V}, \overline{N}, 0)$.

This equilibrium may be obtained by solving the following algebraic equations,

$$\beta(N - Y) - (\beta - \gamma)V - (\nu + \alpha + d)N = 0, \tag{5}$$

$$Y = \frac{A - dN}{\alpha}, \tag{6}$$

$$V = \frac{\phi N[(\alpha + d)N - A]}{\alpha(\phi + d)N + \gamma(A - dN)}. \tag{7}$$

Using equations (6) and (7) in equation (5), we get an algebraic equation in single variable N, i.e., $F(N) = 0$, where $F(N)$ is given by the following equation,

$$F(N) = [\beta - (\nu + \alpha + d)]N - (\beta - \gamma)V - \beta Y. \tag{8}$$

Keeping in mind equations (6) and (7), we note that $\overline{Y}$ and $\overline{V}$ will be positive only when $F(N) = 0$ has a root in the interval $(\frac{A}{\alpha+d}, \frac{A}{d})$. From equation (8) it is easy to observe

that, $F(\frac{A}{\alpha+d}) < 0$ and $F(\frac{A}{d}) > 0$, if $\frac{\beta d+\gamma\phi}{(\nu+\alpha+d)(\phi+d)} > 1$ (i.e., $R(\phi) > 1$, see Section 4). Thus, there exists a root $\overline{N}$ of $F(N) = 0$ in $\frac{A}{\alpha+d} < N < \frac{A}{d}$. Also $F'(N) > 0$ in $\frac{A}{\alpha+d} < N < \frac{A}{d}$. Hence, there exists a unique positive root $\overline{N}$ given by $F(N) = 0$. Knowing the value of $\overline{N}$, we can compute the values of $\overline{Y}$ and $\overline{V}$ from equations (6) and (7), respectively.

Thus there exists a unique carrier-free equilibrium $E_1(\overline{Y}, \overline{V}, \overline{N}, 0)$, provided the condition $R(\phi) > 1$ is satisfied.

3. The endemic equilibrium, $E_2(Y^*, V^*, N^*, C^*)$.

The endemic equilibrium E_2 is given by the solution of the following set of algebraic equations,

$$(\beta Y + \lambda C N)(N - Y - V) + (\gamma Y + \nu_1 C N)V - (\nu + \alpha + d)YN = 0, \tag{9}$$

$$Y = \frac{A - dN}{\alpha}, \tag{10}$$

$$V = \frac{\phi N[(\alpha + d)N - A]}{\alpha N(\phi + d + \nu_1 C) + \gamma(A - dN)}, \tag{11}$$

$$C = \frac{L(N)[s(N) - s_1]}{s_0}, \tag{12}$$

We may reduce equation (9) in a single variable N i.e., $F(N) = 0$ by using equations (10), (11) and (12), where

$$F(N) = (\beta Y + \lambda C N)(N - Y - V) + (\gamma Y + \nu_1 C N)V - (\nu + \alpha + d)YN. \tag{13}$$

It is clear from equation (13) that $F(\frac{A}{\alpha+d}) < 0$ and $F(\frac{A}{d}) > 0$. This implies that there exists a root N of $F(N) = 0$ in $\frac{A}{\alpha+d} < N < \frac{A}{d}$. Also, $F'(N) > 0$, provided, $[\alpha\nu_1 NC' - \gamma A] > 0$. Hence, there exists a unique positive root N^* given by $F(N) = 0$ in $\frac{A}{\alpha+d} < N < \frac{A}{d}$. Knowing the value of N^*, the values of Y^*, V^* and C^* can be computed from equations (10), (11) and (12), respectively. Thus, the equilibrium E_2 exists provided $s(N^*) - s_1 > 0$ and $F'(N) > 0$. From equation (13) it is easy to note that $F(\overline{N}) > 0$. Keeping in mind the above analysis of equation (13) we have $\overline{N} > N^*$. From equation (12), it may be noted that $dC^*/dN^* > 0$ in view of equations (2) and (3). Thus, the equilibrium value of carrier population density increases with increase in the equilibrium value of human population.

4 Stability analysis

Now, we analyze the stability of equilibria E_0, E_1 and E_2. The local stability results of these equilibria are stated in the following theorem:

Theorem 1. *The equilibrium E_0 is unstable whenever E_1 or E_2 exists, E_1 is unstable whenever E_2 exists and the equilibrium E_2 is locally asymptotically stable provided the following conditions are satisfied,*

$$\alpha \left[\frac{(\beta - \gamma)Y^*}{N^*} + (\lambda - \nu_1)C^* \right]^2$$

$$< \frac{1}{4} b_1 b_2^2 b_3 \min \left\{ \frac{b_1}{4b_3 \left(\phi + \frac{\gamma V^*}{N^*}\right)^2}, \frac{d}{3\left(\phi + \frac{\gamma V^* Y^*}{N^*}\right)}, \frac{k_3 s_0 C^*}{3\nu_1^2 V^{*2} L(N^*)} \right\}, \quad (14)$$

$$3\alpha L^2(N^*)[\lambda(N^* - Y^* - V^*) + \nu_1 V^*]^2 \left[s'(N^*) + \frac{s_0 C^* L'(N^*)}{L^2(N^*)} \right]^2 < \frac{8}{27} b_1 b_3 d s_0^2, \quad (15)$$

where

$$b_1 = \left[\frac{\lambda N^* C^*}{Y^*} - \frac{(\lambda - \nu_1)V^* C^*}{Y^*} + \frac{\beta Y^*}{N^*} \right],$$

$$b_2 = \left(\phi + d + \frac{\gamma Y^*}{N^*} + \nu_1 C^* \right),$$

$$b_3 = \left[\frac{\beta Y^*}{N^*} + \frac{(\beta - \gamma)V^* Y^*}{N^{*2}} + \lambda C^* \right].$$

Proof. The general variational matrix M for the system (4) is given as follows:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ -\left(\phi + \frac{\gamma V}{N}\right) & -\left(\phi + d + \nu_1 C + \frac{\gamma V}{N}\right) & \phi + \frac{\gamma VY}{N^2} & -\nu_1 V \\ -\alpha & 0 & -d & 0 \\ 0 & 0 & s'(N)C + \frac{s_0 C^2}{L^2(N)} L'(N) & s(N) - s_1 - \frac{2s_0 C}{L(N)} \end{bmatrix},$$

where

$$m_{11} = \frac{\beta(N - Y - V)}{N} - \frac{\beta Y}{N} - \lambda C + \frac{\gamma V}{N} - (\nu + \alpha + d),$$

$$m_{12} = -\frac{\beta Y}{N} - \lambda C + \frac{\gamma V}{N} + \nu_1 C,$$

$$m_{13} = \frac{\beta Y}{N} - \frac{\beta(N - Y - V)Y}{N^2} + \lambda C - \frac{\gamma VY}{N^2},$$

$$m_{14} = \lambda(N - Y - V) + \nu_1 V.$$

The variational matrix M_0 (M evaluated at E_0) of model (4) is given by,

$$M_0 = \begin{bmatrix} \frac{\beta d + \gamma \phi}{\phi + d} - (\nu + \alpha + d) & 0 & 0 & \frac{A(\lambda d + \nu_1 \phi)}{d(\phi + d)} \\ -\frac{\phi(\gamma + \phi + d)}{\phi + d} & -(\phi + d) & \phi & -\frac{\nu_1 \phi A}{d(\phi + d)} \\ -\alpha & 0 & -d & 0 \\ 0 & 0 & 0 & s(A/d) - s_1 \end{bmatrix}.$$

The eigenvalues of M_0 are $\psi_1 = \frac{\beta d + \gamma \phi}{\phi + d} - (\nu + \alpha + d)$, $\psi_2 = -(\phi + d)$, $\psi_3 = -d$ and $\psi_4 = s(A/d) - s_1$. Since all the model parameters are assumed to be nonnegative, it follows that $\psi_2, \psi_3 < 0$. Thus, the stability of E_0 will depend on the sign of ψ_1 and ψ_4.

We define a threshold parameter $R(\phi) = \frac{\beta d + \gamma \phi}{(\phi + d)(\nu + \alpha + d)}$ (say vaccine induced reproduction number). The disease free equilibrium (DFE) is locally asymptotically stable if $R(\phi) < 1$ and $s(A/d) - s_1 < 0$. Since $s(N)$ is an increasing function of N, so $\psi_4 < 0$, this always implies that $s(N) - s_1 < 0$. Biologically $s(A/d) - s_1 < 0$ implies the absence of carrier population. Thus, DFE may be stable only in the absence of carrier population because otherwise disease still persists even if there is no direct interaction of susceptibles with infectives. Thus, E_0 is unstable if either $R(\phi) > 1$ or $s(A/d) - s_1 > 0$, keeping in mind that the necessary condition for the existence of E_1 is $R(\phi) > 1$ and for the existence of E_2 is $s(N^*) - s_1 > 0$. Thus, E_0 is unstable whenever E_1 or E_2 exists. Similarly one of the eigen values of the variational matrix M_1 (M evaluated at E_1) is $s(\overline{N}) - s_1$. Now keeping in mind that $\overline{N} > N^*$, we have that E_1 is unstable whenever E_2 exists.

When $R(\phi) = 1$, there exists backward bifurcation of the model system (4) without carriers as explained below [20, 23, 24].

From equations (5), (6) and (7), we get ϕ as a function of Y alone

$$\phi(Y) = \frac{d\{[\beta - (\nu + \alpha + d)](A - \alpha Y) - \beta d Y\}(A - (\alpha - \gamma)Y)}{\{[(\nu + \alpha + d) - \gamma](A - \alpha Y) + \gamma d Y\}(A - \alpha Y)}. \tag{16}$$

From equation (16), we note that

$$\phi(0) = \frac{d[\beta - (\nu + \alpha + d)]}{(\nu + \alpha + d) - \gamma} = d\frac{R_0 - 1}{1 - R_1}.$$

The expression for $\phi(0)$ is the threshold vaccination rate given by the term ϕ_{C1} in equation (33).

We have already shown the uniqueness of $\overline{Y}$ for $R(\phi) > 1$ in Section 3. Now here we show the bifurcation analysis for the disease free equilibrium E_0 when $R(\phi) = 1$. We note that for $R(\phi) = 1$, one eigenvalue of M_0 is 0, whereas other eigenvalues are negative provided $s(A/d) - s_1 < 0$.

From equation (16), we have $\phi'(0) > 0$ provided

$$(R_0 - 1)\left(R_1 - \frac{d}{(\nu + \alpha + d)(1 - R_1)}\right) > \frac{d}{\nu + \alpha + d}. \tag{17}$$

When d is very small, ν and α are large, R_1 is far from both 0 and 1 and R_0 is large. Then there is a backward bifurcation at $R(\phi) = 1$ for the model (4) as above inequality (17) is satisfied.

For one set of parameters we have made a graph of Y^* versus $R(\phi)$ showing backward bifurcation. In Fig. 2 we have shown the backward bifurcation of the model (4) at $R(\phi) = 1$ for the parameter values $A = 1.0$, $\alpha = 1.0$, $\beta = 5.0$, $d = 0.1$, $\nu = 0.35$, $\gamma = 0.6$. Fig. 2 also shows that there may exist two endemic equilibria of system (4)
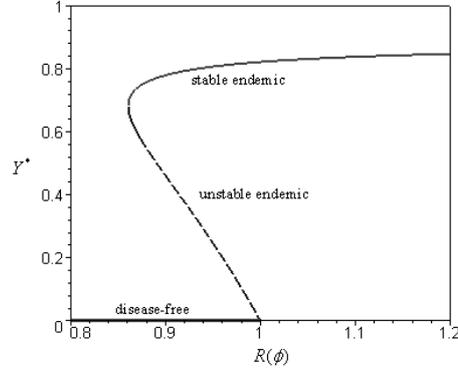
Fig. 2. Bifurcation diagram of the model system (4).

for $R(\phi) < 1$. Out of these two equilibria one will be stable (solid line), whereas second one will be unstable (dashed line).

To establish the local stability of endemic equilibrium E_2, we consider the following positive definite function,

$$U_1 = \frac{1}{2}\left(k_0 y^2 + k_1 v^2 + k_2 n^2 + k_3 c^2\right), \tag{18}$$

where k_i $(i = 0, 1, 2, 3)$ are positive constants to be chosen appropriately and y, v, n and c are small perturbations about E_2, as follows

$$Y = Y^* + y, \quad V = V^* + v, \quad N = N^* + n \quad \text{and} \quad C = C^* + c.$$

Differentiating (18), with respect to t, using the linearized system corresponding to E_2, we get,

$$
\begin{aligned}
\frac{\mathrm{d}U_1}{\mathrm{d}t} = &- k_0\left[\beta\frac{Y^*}{N^*} + \frac{\lambda N^* C^*}{Y^*} + \frac{(\lambda - \nu_1)V^* C^*}{Y^*}\right]y^2 \\
&- k_1\left(\phi + d + \gamma\frac{Y^*}{N^*} + \nu_1 C^*\right)v^2 - k_2 dn^2 - \frac{k_3 s_0 C^*}{L(N^*)}c^2 \\
&- \left\{k_0\left[(\beta - \gamma)\frac{Y^*}{N^*} + (\lambda - \nu_1)C^*\right] + k_1\left(\phi + \gamma\frac{V^*}{N^*}\right)\right\}vy \\
&+ k_1\left(\phi + \gamma\frac{V^* Y^*}{N^{*2}}\right)vn - k_1\nu_1 V^* vc \\
&+ \left[k_0\left(\beta\frac{Y^{*2}}{N^{*2}} + (\beta - \gamma) + \frac{V^* Y^*}{N^{*2}} + \lambda C^*\right) - \alpha k_2\right]ny \\
&+ k_0\left[\lambda\left(N^* - V^* - Y^*\right) + \nu_1 V^*\right]yc + k_3\left[s'(N^*)C^* + \frac{s_0 C^{*2}}{L^2(N^*)}L'(N^*)\right]nc.
\end{aligned}
$$

Now $\frac{dU_1}{dt}$ will be negative definite under the following conditions,

$$k_0\left[(\beta-\gamma)\frac{Y^*}{N^*}+(\lambda-\nu_1)C^*\right]^2$$
$$< k_1\left(\phi+d+\gamma\frac{Y^*}{N^*}+\nu_1 C^*\right)\left[\frac{\lambda N^*C^*}{Y^*}-\frac{(\lambda-\nu_1)V^*C^*}{Y^*}+\frac{\beta Y^*}{N^*}\right], \quad (19a)$$

$$k_1\left(\phi+\frac{\gamma V^*}{N^*}\right)^2$$
$$< k_0\left(\phi+d+\gamma\frac{Y^*}{N^*}+\nu_1 C^*\right)\left[\frac{\lambda N^*C^*}{Y^*}-\frac{(\lambda-\nu_1)V^*C^*}{Y^*}+\frac{\beta Y^*}{N^*}\right], \quad (19b)$$

$$\left\{k_0\left[\frac{\beta Y^{*2}}{N^{*2}}+\lambda C^*+\frac{(\beta-\gamma)V^*Y^*}{N^{*2}}\right]-\alpha k_2\right\}^2$$
$$< \frac{2k_0 k_2 d}{3}\left[\frac{\lambda N^*C^*}{Y^*}-\frac{(\lambda-\nu_1)V^*C^*}{Y^*}+\frac{\beta Y^*}{N^*}\right], \quad (19c)$$

$$k_0\left[\lambda\left(N^*-Y^*-V^*\right)+\nu_1 V^*\right]^2$$
$$< \frac{2k_3}{3}\frac{s_0 C^*}{L(N^*)}\left[\frac{\lambda N^*C^*}{Y^*}-\frac{(\lambda-\nu_1)V^*C^*}{Y^*}+\frac{\beta Y^*}{N^*}\right], \quad (19d)$$

$$k_1\left(\phi+\frac{\gamma V^*Y^*}{N^{*2}}\right)^2 < \frac{k_2}{3}d\left(\phi+d+\gamma\frac{Y^*}{N^*}+\nu_1 C^*\right), \quad (19e)$$

$$k_1\nu_1^2 V^{*2} < \frac{k_3}{3}\frac{s_0 C^*}{L(N^*)}\left(\phi+d+\gamma\frac{Y^*}{N^*}+\nu_1 C^*\right), \quad (19f)$$

$$k_3\left[s'(N^*)C^*+\frac{s_0 C^{*2}}{L(N^*)}L'(N^*)\right]^2 < \frac{4k_2 d s_0 C^*}{9L(N^*)}. \quad (19g)$$

Choosing $k_0 = \frac{\alpha}{\frac{\beta Y^{*2}}{N^{*2}}+\frac{(\beta-\gamma)V^*Y^*}{N^*}+\lambda C^*}$, $k_2 = 1$, we can choose k_1 and k_3 such that

$$\frac{4\alpha\left[\frac{(\beta-\gamma)Y^*}{N^*}+(\lambda-\nu_1)C^*\right]^2}{b_1 b_2 b_3} < k_1$$
$$< b_2 \min\left\{\frac{b_1}{4b_3\left(\phi+\frac{\gamma V^*}{N^*}\right)^2}, \frac{d}{3\left(\phi+\frac{\gamma V^*Y^*}{N^{*2}}\right)}, \frac{k_3 s_0 C^*}{3\nu_1^2 V^{*2}L(N^*)}\right\},$$
$$\frac{3\alpha L(N^*)[\lambda(N^*-Y^*-V^*)+\nu_1 V^*]^2}{2s_0 C^* b_1 b_3} < k_3$$
$$< \frac{4ds_0 C^*}{9L(N^*)\left[s'(N^*)C^*+\frac{s_0 C^{*2}L'(N^*)}{L^2(N^*)}\right]^2},$$

where

$$b_1 = \left[\frac{\lambda N^* C^*}{Y^*} - \frac{(\lambda - \nu_1)V^* C^*}{Y^*} + \frac{\beta Y^*}{N^*} \right],$$

$$b_2 = \left(\phi + d + \frac{\gamma Y^*}{N^*} + \nu_1 C^* \right),$$

$$b_3 = \left[\frac{\beta Y^{*2}}{N^{*2}} - \frac{(\beta - \gamma)V^* Y^*}{N^{*2}} + \lambda C^* \right].$$

The stability conditions are then obtained as given in the theorem. Hence, $\frac{\mathrm{d}U_1}{\mathrm{d}t}$ is a negative definite under the conditions (14) and (15) as stated in the statement of the theorem, showing that E_2 is locally asymptotically stable. $\qquad\square$

To study the nonlinear asymptotic stability of endemic equilibrium E_2, we require the bounds of dependent variables. For this, we state the following lemma giving the region of attraction, without proof.

Lemma. *The region of attraction for the system* (4) *is given by,*

$$\Omega = \left\{ (Y, N, V, C) \colon 0 \leq Y \leq N \leq A/d, \ 0 \leq V \leq \frac{\phi A}{d(\phi + d)}, \ 0 \leq C \leq C_m \right\} \quad (20)$$

which attracts all solutions initiating in the positive orthant, where $C_m = \frac{L(A/d)}{s_0} \times [s(A/d) - s_1]$.

Theorem 2. *In addition to assumptions* (2) *and* (3), *let $s(N)$ and $L(N)$ satisfy $0 \leq s'(N) \leq p$ and $0 \leq L'(N) \leq q$ for some positive constants p and q in Ω, then E_2 is nonlinearly asymptotically stable in Ω provided the following inequalities are satisfied:*

$$\alpha N^* \left[\frac{\beta - \gamma}{N^*} + \frac{\lambda C_m}{Y^*} \right] < \frac{1}{5} d\beta, \tag{21}$$

$$\alpha N^* L^2(N^*) \left[\lambda \left(N^* - Y^* - V^* \right) + \nu_1 V^* \right]^2 \left[p + \frac{s_0 q C_m}{L_0^2} \right]^2 < \frac{4}{45} \beta d s_0^2 Y^{*2}, \tag{22}$$

$$N^* \left[\frac{\beta - \gamma}{N^*} + \frac{(\lambda - \nu_1) C_m}{Y^*} \right]^2 < \frac{1}{5} \beta \left(\phi + d + \frac{\gamma Y^*}{N^*} \right)^2$$

$$\times \min \left\{ \frac{\beta}{5 \left(\phi + \frac{\gamma \phi A}{d(\phi + d)N^*} \right)^2}, \ \frac{d}{4 \left(\phi + \frac{\gamma A}{dN^*} \right)^2}, \ \frac{m_3 s_0}{3 \nu_1^2 V^{*2} L(N^*)} \right\}. \tag{23}$$

It is clear from (22) that in the absence of human related factors, i.e., $p = q = 0$, the inequality is automatically satisfied. This implies that human population related factors, conducive to the growth of carrier population, have a destabilizing effect on the system. Here we also note that due to presence of a vaccinated class, a condition (23) is required for the nonlinear stability which further destabilizes the system.

Proof. Consider the following positive definite function,

$$
U_2 = m_0 \left(Y - Y^* - Y^* \ln \frac{Y}{Y^*} \right) + \frac{m_1}{2} (V - V^*)^2
$$
$$
+ \frac{m_2}{2} (N - N^*)^2 + m_3 \left(C - C^* - C^* \ln \frac{C}{C^*} \right),
$$

(24)

where the coefficients m_0, m_1, m_2 and m_3 can be chosen suitably. Differentiating (24) with respect to t and using (4), we get,

$$
\begin{aligned}
\frac{\mathrm{d}U_2}{\mathrm{d}t} = & - \left[\frac{m_0 \lambda N C}{Y Y^*} + \frac{m_0 (\lambda - \nu_1) V C}{Y Y^*} \right] (Y - Y^*)^2 - m_1 \nu_1 C (V - V^*)^2 \\
& - \left\{ \frac{m_0 \beta}{5 N^*} (Y - Y^*)^2 + \frac{m_2 d}{4} (N - N^*)^2 \right. \\
& \quad \left. - m_0 \left[\frac{\lambda C}{Y^*} + \frac{\beta (Y + V)}{N N^*} - \frac{\gamma V}{N N^*} \right] (Y - Y^*)(N - N^*) \right\} \\
& - \left\{ \frac{m_0 \beta}{5 N^*} (Y - Y^*)^2 + \frac{m_2 d}{4} (N - N^*)^2 + m_2 \alpha (Y - Y^*)(N - N^*) \right\} \\
& - \left\{ \frac{m_0 \beta}{5 N^*} (Y - Y^*)^2 + \frac{m_1}{4} \left(\phi + d + \frac{\gamma Y^*}{N^*} \right) (V - V^*)^2 \right. \\
& \quad \left. - m_0 \left[\frac{(\lambda - \nu_1) C}{Y^*} + \frac{\beta - \gamma}{N^*} \right] (Y - Y^*)(V - V^*) \right\} \\
& - \left\{ \frac{m_0 \beta}{5 N^*} (Y - Y^*)^2 + \frac{m_1}{4} \left(\phi + d + \frac{\gamma Y^*}{N^*} \right) (V - V^*)^2 \right. \\
& \quad \left. + m_1 \left(\phi + \frac{\gamma V}{N^*} \right) (Y - Y^*)(V - V^*) \right\} \\
& - \left\{ \frac{m_0 \beta}{5 N^*} (Y - Y^*)^2 + \frac{m_3 S_0}{3 L(N^*)} (C - C^*)^2 \right. \\
& \quad \left. - m_0 \frac{\lambda (N^* - Y^* - V^*) + \nu_1 V^*}{Y^*} (Y - Y^*)(C - C^*) \right\} \\
& - \left\{ \frac{m_1}{4} \left(\phi + d + \frac{\gamma Y^*}{N^*} \right) (V - V^*)^2 + \frac{m_2 d}{4} (N - N^*)^2 \right. \\
& \quad \left. - m_1 \left(\phi + \frac{\gamma V Y}{N N^*} \right) (V - V^*)(N - N^*) \right\} \\
& - \left\{ \frac{m_1}{4} \left(\phi + d + \frac{\gamma Y^*}{N^*} \right) (V - V^*)^2 + \frac{m_3 s_0}{3 L(N^*)} (C - C^*)^2 \right. \\
& \quad \left. + m_1 \nu_1 V^* (V - V^*)(C - C^*) \right\} \\
& - \left\{ \frac{m_2 d}{4} (N - N^*)^2 + \frac{m_3 s_0}{3 L(N^*)} (C - C^*)^2 \right. \\
& \quad \left. - m_3 [f(N) + s_0 C g(N)] (C - C^*)(N - N^*) \right\},
\end{aligned}
$$

(25)

where $f(N)$ and $g(N)$ are defined as follows,

$$f(N) = \begin{cases} \dfrac{s(N) - s(N^*)}{N - N^*}, & N \neq N^*, \\ \dfrac{\mathrm{d}s}{\mathrm{d}N}, & N = N^* \end{cases} \tag{26}$$

$$g(N) = \begin{cases} \dfrac{L(N) - L(N^*)}{(N - N^*)L(N)L(N^*)}, & N \neq N^*, \\ \dfrac{1}{L^2(N^*)}\dfrac{\mathrm{d}L}{\mathrm{d}N}, & N = N^*. \end{cases} \tag{27}$$

Then by using the assumptions of the theorem and the mean value theorem, we have,

$$|f(N)| \leq p \quad \text{and} \quad |g(N)| \leq \frac{q}{L_0^2}. \tag{28}$$

After choosing $m_0 = 1$, $m_2 = \frac{1}{\alpha}$, we choose m_1 and m_3 such that:

$$\frac{5N^*\left[\frac{\beta}{N^*} + \frac{(\lambda - \nu_1)C_m}{Y^*}\right]^2}{\beta\left(\phi + d + \frac{\gamma Y^*}{N^*}\right)} < m_1$$

$$< \left(\phi + d + \frac{\gamma Y^*}{N^*}\right)\min\left\{\frac{\alpha}{5\left(\phi + \frac{\gamma\phi A}{d(\phi + d)N^*}\right)^2}, \frac{d}{4\left(\phi + \frac{\gamma A}{dN^*}\right)^2}, \frac{m_3 s_0}{3\nu_1^2 V^{*2}L(N^*)}\right\}, \tag{29a}$$

$$\frac{N^* L(N^*)[\lambda(N^* - Y^* - V^*) + \nu_1 V^*]^2}{s_0 \beta Y^{*2}} < m_3 < \frac{4}{45}\frac{\mathrm{d}s_0}{\alpha L(N^*)}\frac{1}{\left[p + \frac{s_0 q C_m}{L_0^2}\right]^2}. \tag{29b}$$

The stability conditions can then be easily obtained, as given in the statement of the theorem. Thus, $\frac{\mathrm{d}U_1}{\mathrm{d}t}$ is negative definite under the conditions (21)–(23). Hence proof. $\quad\square$

The above theorem implies that under appropriate conditions, if the carrier population density increases, then the number of infectives in human population also increases leading to fast spread of carrier dependent infectious diseases.

5 Vaccine induced reproduction number

We define $R(\phi)$, the vaccine induced reproduction number as, (see Section 4) .

$$R(\phi) = \frac{\beta d + \gamma\phi}{(\phi + d)(\nu + \alpha + d)} = \frac{\beta}{\nu + \alpha + d}\left[1 - \frac{(\beta - \gamma)\phi}{\beta(\phi + d)}\right], \tag{30}$$

where

$$R'(\phi) = -\frac{(\beta - \gamma)d}{(\beta + d)^2(\nu + \alpha + d)} < 0 \quad (\text{since } \beta \gg \gamma). \tag{31}$$

Thus, $R(\phi)$ is a decreasing function in $\phi \geq 0$. This indicates the impact of vaccination in reducing the vaccine induced reproduction number. Moreover, in the absence of vaccination i.e.,

$$\phi = 0, \quad R(\phi) = \frac{\beta}{\nu + \alpha + d} = R_0. \tag{32}$$

From the definition of $R(\phi)$ and R_0, it is clear that the introduction of vaccination implies $R(\phi) \leq R_0$ and, consequently, if $R_0 < 1$ then $R(\phi) < 1$ when $\phi > 0$. Thus E_0 is locally asymptotically stable as long as $R(\phi)$ is less than one.

As indicated earlier that $\beta \gg \gamma$, therefore, we can write $\gamma = \sigma\beta$, where $0 \leq \sigma < 1$. From equation (30), we can deduce that $\sigma R_0 = \frac{\sigma\beta}{\nu + \alpha + d} \leq R(\phi) \leq R_0$. Thus if $\frac{\sigma\beta}{\nu + \alpha + d} > 1$ then $\sigma > \sigma_c \equiv \frac{\nu + \alpha + d}{\beta}$. This implies that $R(\phi) > 1$ and, therefore, no amount of vaccination can bring $R(\phi)$ below one. Hence σ_c defines the critical value for the vaccine-related reduction rate of infection.

Also $\lim_{\phi \to \infty} R(\phi) = \frac{\gamma}{\nu + \alpha + d} = R_1$ which implies that $R_1 < R_0$ as $\beta \gg \gamma$. Thus, if the vaccination rate is sufficiently high then R_1 can be made less than one if $\gamma \to 0$. Furthermore, we can write $R(\phi) = \frac{dR_0 + \phi R_1}{\phi + d}$ using R_0 and R_1. Setting $R(\phi) = 1$ and solving for ϕ, we get a threshold vaccination rate, ϕ_{C1}, given by (see Fig. 3)

$$\phi_{C1} = \frac{d(R_0 - 1)}{1 - R_1}. \tag{33}$$

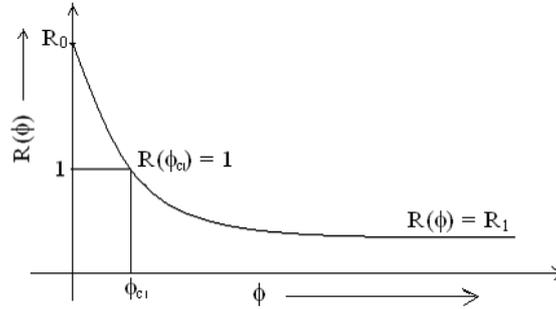Fig. 3. Variation of vaccine induced reproduction number $R(\phi)$ with vaccination coverage ϕ.

Now consider $R_1 < 1 < R_0$, we get ϕ_{C1} positive. Here $R_0 < 1$ and $R_1 > 1$ is not admissible as $\beta \gg \gamma$. If $\phi > \phi_{C1}$, then $R(\phi) < 1$ as $R(\phi)$ is a decreasing function for $\phi \geq 0$. Thus, if the vaccination coverage level ϕ exceeds the threshold ϕ_{C1} then the disease can be eradicated provided vaccine efficacy is high enough, i.e. $\gamma = 0$.

From equations (10)–(12), we also find that,

$$\phi = \frac{dV^*[(d + \nu_1 C^*)(A - \alpha Y^*) + \gamma d Y^*]}{(A - \alpha Y^*)[(A - \alpha Y^*) - d(V^* + Y^*)]}. \tag{34}$$

The critical vaccination level that ensures disease eradication, when carrier population remains at its equilibrium is obtained as,

$$\phi_{C2} = \frac{dV^*[d + \nu_1 C^*]}{A - dV^*}. \tag{35}$$

Therefore, if critical vaccination level ϕ is such that $\phi > \max(\phi_{C1}, \phi_{C2})$, then disease eradication is possible in the population.

6 Numerical simulation

It is noted here that our aim is to study, through a non-linear model and its qualitative analysis, the role of vaccination on the spread of carrier dependent infectious diseases. It is, therefore, desirable that we must show the existence of equilibrium values of variables of the model as well as the feasibility of stability conditions numerically for a set of parameters.

To study the dynamical behaviour of the model, numerical simulation of the system (4) is done by MAPLE 7.0 using the parameters [9, 25]: $\beta = 0.001$, $\lambda = 0.001$, $\gamma = 0.0001$, $\nu_1 = 0.00003$, $\phi = 0.65$, $\alpha = 0.45$, $\nu = 0.4$, $d = 1/60$, $A = 1000$, $s_0 = 0.8$, $s_1 = 0.75$, $L_0 = 10000$, $a = 0.0001$, $b = 0.01$, which satisfy stability conditions.

In the model, $s(N)$ and $L(N)$ are the growth rates and modified carrying capacity of carrier population and are functions of human population density. Thus, for numerical simulation it is assumed that $s(N)$ and $L(N)$ are linear function of N i.e., $s(N) = s_0 + aN$ and $L(N) = L_0 + bN$, satisfying conditions (2) and (3).

The equilibrium values are computed as follows:

$N^* = 3603.244658$, $Y^* = 2088.768716$, $V^* = 1198.816861$, $C^* = 5147.537066$.

The eigen values corresponding to variational matrix of endemic equilibrium are:

-6.199634202, $\quad -0.5019151699$, $\quad -0.1509347167$, $\quad -0.0001247976042$.

Since all the eigen values are found to be negative, therefore, endemic equilibrium is locally asymptotically stable for the above set of parameters.

The computer simulations are performed for different initial starts in the following four cases and displayed graphically in Figs. 4, 5.   In these figures the variation of infectives and vaccinated population with total human population respectively is shown. The trajectories starting with different initial starts reach equilibrium point E_2. Hence we infer that the system (4) may be nonlinearly asymptotically stable about this equilibrium point E_2 for the above set of parameters.

(i) $Y(0) = 3200$, $V(0) = 300$, $N(0) = 4000$, $C(0) = 5000$,

(ii) $Y(0) = 1000$, $V(0) = 100$, $N(0) = 2000$, $C(0) = 5000$,

(iii) $Y(0) = 500$, $V(0) = 2000$, $N(0) = 3000$, $C(0) = 5000$,

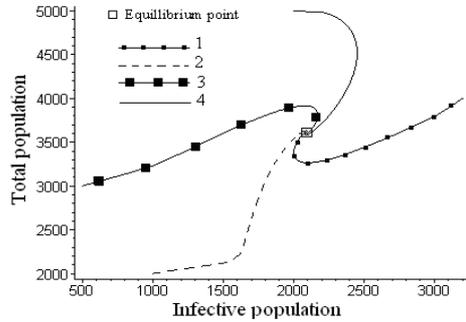(iv) $Y(0) = 2000$, $V(0) = 2500$, $N(0) = 5000$, $C(0) = 5000$.

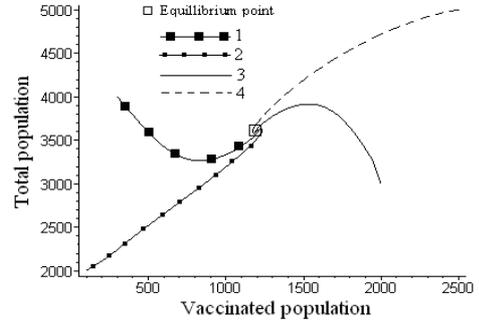Fig. 4. Variation of total human population with infective population.

Fig. 5. Variation of total human population with vaccinated population.

The results of numerical simulation are displayed graphically in Figs. 6–14. In Figs. 6, 7, the variation of infective and vaccinated population is shown respectively for different values of vaccination rate ϕ. It is found that as vaccination rate (ϕ) increases, the infective population decreases, whereas the vaccinated population increases tremendously. This signifies that only by increasing the vaccination rate, spread of carrier dependent infectious disease cannot be significantly controlled.
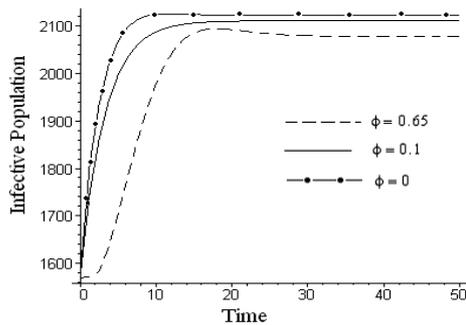
Fig. 6. Variation of infective population for different values of vaccination rate ϕ.
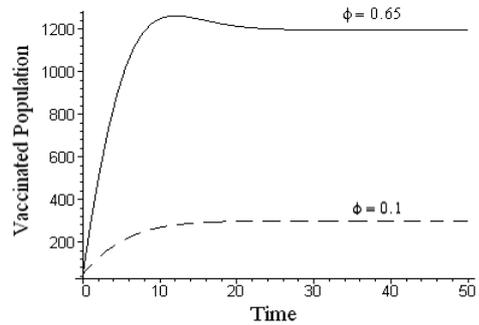
Fig. 7. Variation of vaccinated population for different values of vaccination rate ϕ.

Figs. 8, 9 depict the role of decay coefficient (s_1) of carrier population on carriers and infectives. When there is a rise in the decay coefficient either due to natural factors or control measures, carrier population density decreases significantly, and consequently the infective population declines. This decline in infective population is not much significant. It seems, it is due to the fact that disease spreads not only through carriers but also through direct interaction of susceptibles with infectives. It is, therefore, speculated that not only the growth of carrier population be curbed using effective control mechanism but the direct interaction of susceptibles with infectives be also restricted. This is also clear from

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Fig. 10, where variation of infective population is shown for different values of contact rates of susceptibles with carrier population density. The infective population increases with the increase in the contact rate of susceptibles with carrier population.
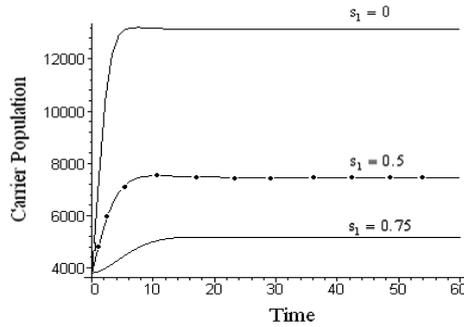

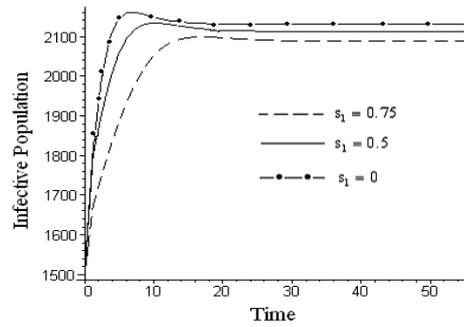Fig. 8. Variation of carrier population for different s_1, the control parameter.

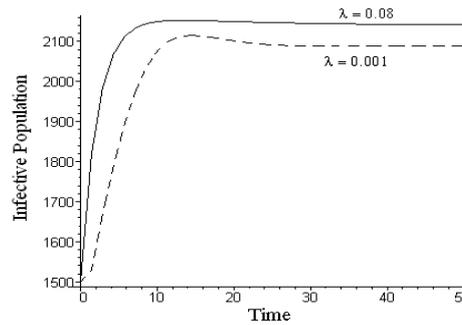Fig. 9. Variation of infective population for different s_1, the control parameter.

Fig. 10. Variation of infective population for different values of contact rate λ.

As is pointed out earlier that due to inefficacy of vaccines, some of the vaccinated population may again become infected due to interaction with infectives as well as with carriers. In Figs. 11, 12, the effect of γ and ν_1, the contact rate of vaccinated population with infectives and with carrier population density respectively, is shown on vaccinated and infective populations. It is found that with increase in transmission rate γ and ν_1, the vaccinated population decreases, which in turn, increases the infective population. Thus, we conclude that vaccine efficacy should be high enough so that vaccinated individuals do not get infected either by direct interaction with infectives or with carrier population density (see Figs. 11, 12 for $\gamma = 0$, $\nu_1 = 0$). In Fig. 13, the effect of disease induced death rate is shown and it is found that with increase in α, the infective population also decreases. Fig. 14 shows the effect of immigration rate of susceptibles and it is seen that the infective population increases with increase in the rate of immigration. Thus, the constant migration of susceptibles make the disease more endemic.

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Finally from the above discussion, we infer that in order to keep the spread of carrier dependent infectious diseases under control, a proper vaccination campaign (as discussed in Section 5) dependent on the critical vaccination rate, be introduced. Moreover, the vaccine efficacy be high enough to ensure that vaccinated persons while coming in contact with infectives or carriers, are not infected again. A suitable control mechanism like elimination of carrier breeding sites, larvaciding, adulticiding, etc. may be devised so that the carrier population density is diminished which otherwise increases due to human population related factors and other unhygienic environmental conditions leading to fast spread of infectious diseases.
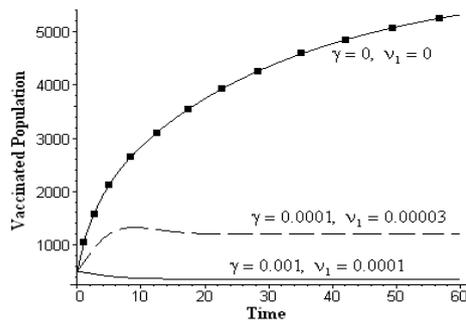

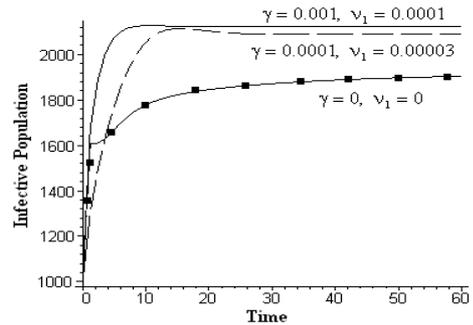Fig. 11. Variation of vaccinated population for different values of γ and ν_1.

Fig. 12. Variation of infective population for different values of γ and ν_1.
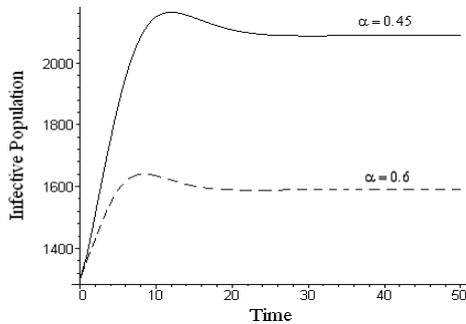

Fig. 13. Variation of infective population for different values of disease induced-death rate α.
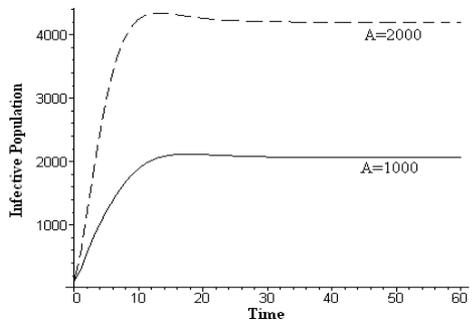

Fig. 14. Variation of infective population for different rates of immigration A.

7 Conclusions

In this paper, a nonlinear vaccination model is proposed and analyzed to study the spread of carrier dependent infectious diseases with vaccination in a population with variable

size structure. It is assumed that the disease spreads by direct contact of susceptibles with infectives and by carrier population density present in the environment. The density of carrier population, which increases by environmental and human population related factors, is assumed to be governed by a generalized logistic model. The growth rate per capita and the modified carrying capacity of the carrier population are also assumed to increase as the human population density increases. Some inferences have been drawn regarding the spread of the disease by establishing local and global stability results and numerical simulation. The model exhibits three equilibria, namely disease free, carrier free and endemic equilibrium. The first two equilibria are found to be unstable, whereas third equilibrium is locally stable. It is also found to be nonlinearly asymptotically stable under certain conditions. We have found a threshold condition in terms of vaccine induced reproduction number $R(\phi)$ which is, if less than one, the disease dies out, in the absence of carrier population otherwise the infection is maintained in the population. And at $R(\phi) = 1$, the model exhibits backward bifurcation. The results show that in the absence of carrier population into the community, if vaccination rate is above the critical vaccination level (to ensure $R(\phi) < 1$) then the disease can be eradicated. The presence of human population related factors, causing the growth of carrier population, have a destabilizing effect on the system. Moreover, if the vaccination rate is above a critical level ϕ_{C2}, then carrier dependent infectious disease does not take a form of epidemic provided the density of carrier population remains at its equilibrium level.

Thus, three main control strategies against the spread of carrier dependent infectious diseases in the human population are carrier (such as flies, ticks, mites etc.) reduction policy, vaccination against the disease and the vaccine efficacy. The carrier reduction strategies may include elimination of carrier breeding sites, larvaciding, adulticiding, keeping surroundings clean and hygienic. On the other hand, vaccinating the susceptibles such that vaccination level is maintained above the critical vaccination level ϕ_C i.e., $\max(\phi_{C1}, \phi_{C2})$ and the vaccine efficacy be high enough so that vaccinated individuals are not infected again either by infectives or by carriers.

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