Analysis of an Antiplane Contact Problem with Adhesion for Electro-Viscoelastic Materials

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Abstract. We consider a mathematical model which describes the antiplane shear deformation of a cylinder in frictionless contact with a rigid foundation. The adhesion of the contact surfaces, caused by the glue, is taken into account. The material is assumed to be electro-viscoelastic and the foundation is assumed to be electrically conductive. We derive a variational formulation of the model which is given by a system coupling an evolutionary variational equality for the displacement field, a time-dependent variational equation for the electric potential field and a differential equation for the bonding field. Then we prove the existence of a unique weak solution to the model. The proof is based on arguments of evolution equations with monotone operators and fixed point.

Keywords: antiplane shear, electro-viscoelastic material, contact process, adhesion, fixed point, weak solution.

1 Introduction

Antiplane shear deformations are one of the simplest examples of deformations that solids can undergo: in antiplane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. For this reason, the antiplane problems play a useful role as pilot problems, allowing for various aspects of solutions in Solid Mechanics to be examined in a particularly simple setting. Considerable attention has been paid to the analysis of such kind of problems, see for instance [1–5]. In particular, the last two references deal with antiplane problems for piezoelectric materials.

Piezoelectric materials are characterized by the coupling between the mechanical and electrical properties. This coupling leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors and the reverse effect is used in actuators, in engineering control equipment. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials and those for which the mechanical properties are viscoelastic are called electro-viscoelastic materials. General models for electro-elastic materials can be found in [6–8]. Static frictional contact problems for electro-elastic materials were studied in [9–12] and contact problems for electro-viscoelastic materials were considered in [13, 14]. In all these references the foundation was assumed to be electrically insulated.

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between bodies, when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the literature. General models with adhesion can be found in [15–18]. In all these references the idea is the introduction of a surface internal variable, the bonding field $\beta \in [0,1]$, which describes the fractional density of active bonds on the contact surface. At a point on the contact surface, when $\beta = 1$ the adhesion is complete and all bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active.

Existence and uniqueness results in the study of models for adhesive contact were obtained by several authors, by using various functional methods. A partial list include [19–23], among other references. The method used in [19] is based on time-discretization and compactness arguments and the method used in [20] requires the application of a compactness lemma and the Faedo-Galerkin discretization. The existence of a solution for a delamination problem is obtained in [21] by using a regularized interface model and arguments of nonsmooth analysis; the lack of convexity of the functional governing this problem leads to a new and nonstandard mathematical model. Finally, the unique weak solvability of the adhesive problems studied in [22] and [23] is based on arguments of evolution equations with monotone operators and fixed point.

In this paper we study an antiplane frictionless contact problem with adhesion for electro-viscoelastic materials, in the framework of the Mathematical Theory of Contact Mechanics, when the foundation is electrically conductive. Our interest is to describe a physical process in which both antiplane shear, contact, adhesion and piezoelectric effect are involved, leading to a well posedness mathematical problem. Taking into account the piezoelectric effect, the conductivity of the foundation and the adhesion in the study of an antiplane problem for viscoelastic materials represents the main novelty of this work. We rarely actually load piezoelectric bodies so as to cause them to deform in antiplane shear. However, the governing equations and boundary conditions for antiplane shear problems are beautifully simple and the solution has many of the features of the more general case and may help us to solve the more complex problem too.

Our paper is structured as follows. In Section 2 we present the model of the antiplane frictionless adhesive contact for an electro-viscoelastic cylinder. In Section 3 we introduce the notation and list the assumptions on problem's data, derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 1. In Section 4 we provide a proof of the theorem which is carried out in several steps by constructing three intermediate problems for the displacement field, the electric potential and the bonding field, respectively. We prove the unique solvability of the intermediate problems, then we consider a contraction mapping whose unique fixed point leads us to construct the solution of the original problem.

2 The model

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measure of Γ_1 and Γ_a, denoted $meas\,\Gamma_1$ and $meas\,\Gamma_a$, are positive. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge of density q_b is prescribed on $\Gamma_b \times (-\infty, +\infty)$. The cylinder is in adhesive contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, the so called foundation.

Below in this paper the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable. Also, a dot above represents the time derivative. We use $\mathbb{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3; "$\cdot$" and $\|\cdot\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathbb{S}^3$, i.e.

$$
\begin{aligned}
&\mathbf{u}\cdot\mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2} \quad \text{for all} \quad \mathbf{u} = (u_i), \quad \mathbf{v} = (v_i) \in \mathbb{R}^3,\\
&\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \ \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \text{for all} \quad \boldsymbol{\sigma} = (\sigma_{ij}), \ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^3.
\end{aligned}
$$

We assume that

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t)\colon\ \Omega \times [0, T] \to \mathbb{R}, \tag{1}$$

$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t)\colon\ \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2}$$

$$q_0 = q_0(x_1, x_2, t)\colon\ \Omega \times [0, T] \to \mathbb{R}, \tag{3}$$

$$q_2 = q_2(x_1, x_2, t)\colon\ \Gamma_b \times [0, T] \to \mathbb{R}, \tag{4}$$

where $[0, T]$ denotes the time interval of interest, $T > 0$.

The forces (1), (2) and the electric charges (3), (4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and an electric potential field φ which are independent on x_3 and have the form

$$\mathbf{u} = (0,0,u) \quad \text{with} \quad u = u(x_1,x_2,t) \colon\ \Omega \times [0,T] \to \mathbb{R}, \tag{5}$$

$$\varphi = \varphi(x_1,x_2,t) \colon\ \Omega \times [0,T] \to \mathbb{R}. \tag{6}$$

Such kind of deformation, associated to a displacement field of the form (5), is called an antiplane shear, see for instance [2, 3] for details.

We denote by $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ the strain tensor and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the stress tensor; we also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}.$$

The material's behavior is modelled by an electro-viscoelastic constitutive law of the form

$$\boldsymbol{\sigma} = 2\theta\,\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \zeta\,tr\,\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\,\mathbf{I} + 2\mu\,\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,tr\,\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{I} - \mathcal{E}^*\mathbf{E}(\varphi), \tag{7}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) + \alpha\,\mathbf{E}(\varphi), \tag{8}$$

where ζ and θ are viscosity coefficients, λ and μ are the Lamé coefficients, $tr\,\varepsilon(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, α is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ its transpose. We assume that

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13}+\varepsilon_{31}) \\ e(\varepsilon_{23}+\varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall\,\boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^3, \tag{9}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, α and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\tau}\cdot\mathbf{v} = \boldsymbol{\tau}\cdot\mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\tau} \in \mathbb{S}^3$ and $\mathbf{v} \in \mathbb{R}^3$, it follows from (9) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall\,\mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{10}$$

In the antiplane context (5), (6), using the constitutive equations (7), (8) and equalities (9), (10), it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} \\ 0 & 0 & \theta\dot{u}_{,2} + \mu u_{,2} + e\varphi_{,2} \\ \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} & \theta\dot{u}_{,2} + \mu u_{,2} + e\varphi_{,2} & 0 \end{pmatrix}, \tag{11}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \alpha\varphi_{,1} \\ eu_{,2} - \alpha\varphi_{,2} \\ 0 \end{pmatrix}. \tag{12}$$

We assume that the process is mechanically dynamic and electrically static and therefore is governed by the balance equations

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}_0 = \rho\ddot{\mathbf{u}}, \quad D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B} \times (0,T), \tag{13}$$

where $\operatorname{div} \boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$ and ρ denotes the mass density, assumed to be independent on x_3. Taking into account (11), (12), (5), (6), (1) and (3), the balance equations (13) reduce to the following scalar equations

$$\operatorname{div}(\theta\nabla\dot{u} + \mu\nabla u + e\nabla\varphi) + f_0 = \rho\ddot{u} \quad \text{in } \Omega \times (0,T), \tag{14}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) = q_0 \quad \text{in } \Omega \times (0,T). \tag{15}$$

Here and below we use the notation

$$\operatorname{div} \boldsymbol{\tau} = \tau_{1,1} + \tau_{1,2} \quad \text{for} \quad \boldsymbol{\tau} = \big(\tau_1(x_1,x_2,t), \tau_2(x_1,x_2,t)\big),$$
$$\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu \nu = v_{,1}\nu_1 + v_{,2}\nu_2 \quad \text{for} \quad v = v(x_1,x_2,t).$$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1 \times (-\infty,+\infty)$ and the electric potential vanishes on $\Gamma_a \times (-\infty,+\infty)$; thus (5) and (6) imply that

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{16}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T). \tag{17}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty,+\infty)$. We have

$$\boldsymbol{\nu} = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i(x_1,x_2)\colon\ \Gamma \to \mathbb{R},\ i = 1,2. \tag{18}$$

For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}. \tag{19}$$

Also, for a given stress tensor $\boldsymbol{\sigma}$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{20}$$

From (11), (12) and (18) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0,0,\theta\partial_\nu\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi), \quad \mathbf{D}\cdot\boldsymbol{\nu} = e\partial_\nu u - \alpha\partial_\nu\varphi. \tag{21}$$

Therefore, taking into account (2), (4) and (21), the traction condition on $\Gamma_2\times(-\infty,+\infty)$ and the electric condition on $\Gamma_b\times(-\infty,+\infty)$ are given by

$$\theta\partial_\nu\dot{u}+\mu\partial_\nu u+e\partial_\nu\varphi=f_2\quad\text{on }\ \Gamma_2\times(0,T),\tag{22}$$

$$e\partial_\nu u-\alpha\partial_\nu\varphi=q_b\qquad\text{on }\ \Gamma_b\times(0,T).\tag{23}$$

We now continue with the boundary conditions on the contact surface $\Gamma_3\times(-\infty,+\infty)$ in which our interest is. First, from (5) and (18) we infer that the normal displacement vanishes, $u_\nu=0$, which shows that the contact is bilateral, i.e. is kept during all the process. Using now (5), (11), (18)–(20) we conclude that

$$\mathbf{u}_\tau=(0,0,u),\quad \boldsymbol{\sigma}_\tau=(0,0,\theta\partial_\nu\dot{u}+\mu\partial_\nu u+e\partial_\nu\varphi).\tag{24}$$

Since the contact is adhesive, following [22, 23] we assume that the tangential tangential stress $\boldsymbol{\sigma}_\tau$ satisfies

$$-\boldsymbol{\sigma}_\tau=p(\beta)\,\mathbf{R}(\mathbf{u}_\tau)\quad\text{on }\ \Gamma_3\times(-\infty,+\infty)\times(0,T).\tag{25}$$

Here p is a given function, β is the bonding field and $\mathbf{R}$ is a truncation operator defined by

$$\mathbf{R}(\mathbf{v})=\begin{cases}\mathbf{v} & \text{if}\quad \|\mathbf{v}\|\le L,\\ L\,\dfrac{\mathbf{v}}{\|\mathbf{v}\|} & \text{if}\quad \|\mathbf{v}\|>L\end{cases}\tag{26}$$

with $L>0$ being a characteristic length of the bond, beyond which there is no any additional traction (see, e.g. [18]). It follows from (25) that the shear of the contact surface depends on the bonding field and on the tangential displacement, but only up to the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. Using now (24) and assuming that p does not depend on x_3, it is straightforward to see that the tangential boundary condition (25) implies

$$-(\theta\partial_\nu\dot{u}+\mu\partial_\nu u+e\partial_\nu\varphi)=p(\beta)\,R(u)\quad\text{on }\ \Gamma_3\times(0,T),\tag{27}$$

where R is the real valued function defined by

$$R(v)=\begin{cases}-L & \text{if}\quad v<-L,\\ \ \ v & \text{if}\quad -L\le v\le L,\\ \ \ L & \text{if}\quad v>L.\end{cases}\tag{28}$$

Since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D}\cdot\boldsymbol{\nu}=k(\varphi-\varphi_F)\quad\text{on }\ \Gamma_3\times(-\infty,+\infty)\times(0,T),\tag{29}$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient, both assumed to be independent on x_3. We use (21) and (29) to obtain

$$e\partial_\nu u - \alpha\partial_\nu\varphi = k(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T). \tag{30}$$

We describe the evolution of the bonding field β is by the first order ordinary differential equation

$$\dot{\beta} = -\big(\beta\left(\delta S(u_\nu)^2 + \gamma\,\|\mathbf{R}(\mathbf{u}_\tau)\|^2\right) - \epsilon_a\big)_+ \quad \text{on } \Gamma_3 \times (-\infty,+\infty) \times (0,T), \tag{31}$$

already used in [22, 23]. Here δ, γ and ϵ_a are given adhesion coefficients which depend only on x_1 and x_2, $\boldsymbol{R}$ is defined by (26), $S\colon \mathbb{R} \to \mathbb{R}$ is a truncation operator such that $S(0) = 0$ and $r_+ = \max\{r, 0\}$. We note that the adhesive process is irreversible; indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \le 0$. Replacing the differential equation (31) with a condition which allows the adhesive process for rebonding will represent an important extension of the results in this paper and will be consider in a further paper. Using now equalities $u_\nu = 0$, $S(0) = 0$, $\boldsymbol{u}_\tau = (0,0,u)$ and the definitions (26) and (28) of the operators $\mathbf{R}$ and R, it is straightforward to see that (31) implies

$$\dot{\beta} = -\big(\gamma\beta R(u)^2 - \epsilon_a\big)_+, \quad \text{on } \Gamma_3 \times (0,T). \tag{32}$$

In (32) and below we use the simplified notation $R(u)^2$ for the square of $R(u)$, i.e. $R(u)^2 = (R(u))^2$.

Finally, we prescribe the initial displacement, velocity and bonding fields, i.e.

$$u(0) = u_0 \quad \text{in } \Omega, \tag{33}$$

$$\dot{u}(0) = v_0 \quad \text{in } \Omega, \tag{34}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3, \tag{35}$$

where u_0, v_0 and β_0 are given.

We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictionless adhesive contact with a conductive foundation.

Problem P**.** *Find a displacement field* $u\colon \Omega \times [0,T] \to \mathbb{R}$*, an electric potential* $\varphi\colon\ \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\beta\colon\ \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that* (14)–(17), (22), (23), (27), (30), (32)–(35) *hold.*

Note that once the displacement field u and the electric potential φ which solve Problem P are known, then the stress tensor $\boldsymbol{\sigma}$ and the electric displacement field $\mathbf{D}$ can be obtained by using (11) and (12), respectively.

3 Variational formulation

In this section we derive a variational formulation of the Problem P. To this end we introduce the function spaces

$$V=\big\{v\in H^1(\Omega):\ v=0\ \text{ on }\ \Gamma_1\big\},\quad W=\big\{\psi\in H^1(\Omega):\ \psi=0\ \text{ on }\ \Gamma_a\big\},$$

where, here and below, we write w for the trace on Γ of a function $w\in H^1(\Omega)$. Since $meas\,\Gamma_1>0$ and $meas\,\Gamma_a>0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u,v)_V=\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x\quad \forall\, u,v\in V,\qquad (\varphi,\psi)_W=\int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathrm{d}x\quad \forall\,\varphi,\psi\in W.$$

Moreover, the associated norms

$$\|v\|_V=\|\nabla v\|_{L^2(\Omega)^2}\quad \forall\, v\in V,\quad \|\psi\|_W=\|\nabla\psi\|_{L^2(\Omega)^2}\quad \forall\,\psi\in W \tag{36}$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. Also, by Sobolev's trace theorem we deduce that there exists positive constants $c_V>0$, $c_W>0$ such that

$$\|v\|_{L^2(\Gamma_3)}\le c_V\|v\|_V\quad \forall\, v\in V,\qquad \|\psi\|_{L^2(\Gamma_3)}\le c_W\|\psi\|_W\quad \forall\,\psi\in W. \tag{37}$$

We suppose that the mass density satisfies

$$\rho\in L^\infty(\Omega)\ \text{ and there exists }\ \rho^*>0\ \text{ such that }\ \rho(\mathbf{x})\ge\rho^*\ \text{ a.e. }\ \mathbf{x}\in\Omega. \tag{38}$$

We use a modified inner product on $H=L^2(\Omega)$, given by

$$(u,v)_H=(\rho u,v)^{\frac12}_{L^2(\Omega)}\quad \forall\, u,v\in H, \tag{39}$$

that is, it is weighted with ρ, and let $\|\cdot\|_H$ be the associated norm, i.e.

$$\|v\|_H=(\rho v,v)^{\frac12}_{L^2(\Omega)}\quad \forall\, v\in H. \tag{40}$$

It follows from assumptions (38) that $\|\cdot\|_H$ and $\|\cdot\|_{L^2(\Omega)}$ are equivalent norms on H, and the inclusion mapping of $(V,\|\cdot\|_V)$ into $(H,\|\cdot\|_H)$ is continuous and dense. We denote by $(V',\|\cdot\|_{V'})$ the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V\subset H\subset V'.$$

We use the notation $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between V' and V and we recall that

$$\langle u,v\rangle_{V'\times V}=(u,v)_H\quad \forall\, u\in H,\ v\in V. \tag{41}$$

For a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty$, $k = 1, 2, \ldots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, respectively, with the norms

$$\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X,$$
$$\|u\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X + \max_{t\in[0,T]} \|\dot{u}(t)\|_X.$$

Finally, we will use the set

$$\mathcal{Z} = \big\{\theta \in C\big([0,T];L^2(\Gamma_3)\big) :\ 0 \le \theta(t) \le 1 \quad \forall t \in [0,T],\ \text{ a.e. on } \Gamma_3\big\}.$$

We now list the assumptions on the rest of the problem's data. We assume that the viscosity coefficient and the electric permittivity coefficient satisfy

$$\theta \in L^\infty(\Omega) \quad \text{and} \quad \exists\, \theta^* > 0 \quad \text{such that} \ \ \theta(\mathbf{x}) \ge \theta^* \ \ \text{a.e. } \mathbf{x} \in \Omega, \tag{42}$$
$$\alpha \in L^\infty(\Omega) \quad \text{and} \quad \exists\, \alpha^* > 0 \quad \text{such that} \ \ \alpha(\mathbf{x}) \ge \alpha^* \ \ \text{a.e. } \mathbf{x} \in \Omega. \tag{43}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad \text{and} \quad \mu(\mathbf{x}) \ge 0 \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{44}$$
$$e \in L^\infty(\Omega). \tag{45}$$

The tangential function p satisfies

$$\begin{cases} \text{(a)}\ \ p\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b)}\ \ \text{There exists } L > 0 \text{ such that} \\ \qquad |p(\mathbf{x},\beta_1) - p(\mathbf{x},\beta_2)| \le L|\beta_1 - \beta_2| \quad \forall\, \beta_1, \beta_2 \in \mathbb{R},\ \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c)}\ \ \text{There exists } M > 0 \text{ such that } |p(\mathbf{x},\beta)| \le M \quad \forall\, \beta \in \mathbb{R},\ \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(d)}\ \ \text{The mapping } \mathbf{x} \mapsto p(\mathbf{x},\beta) \text{ is measurable on } \Gamma_3 \quad \forall\, \beta \in \mathbb{R}. \end{cases} \tag{46}$$

The adhesion coefficients γ and ϵ_a satisfy the conditions

$$\gamma \in L^\infty(\Gamma_3) \quad \text{and} \quad \gamma(\mathbf{x}) \ge 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3, \tag{47}$$
$$\epsilon_a \in L^2(\Gamma_3) \quad \text{and} \quad \epsilon_a(\mathbf{x}) \ge 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \tag{48}$$

The forces, tractions, volume and surface free charges densities have the regularity

$$f_0 \in L^2\big(0,T;L^2(\Omega)\big), \qquad f_2 \in L^2\big(0,T;L^2(\Gamma_2)\big), \tag{49}$$
$$q_0 \in W^{1,2}\big(0,T;L^2(\Omega)\big), \quad q_b \in W^{1,2}\big(0,T;L^2(\Gamma_b)\big). \tag{50}$$

The electric conductivity coefficient satisfies

$$k \in L^\infty(\Gamma_3) \quad \text{and} \quad k(\mathbf{x}) \ge 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \tag{51}$$

Finally, we assume that the electric potential of the foundation and the initial data are such that

$$\varphi_F \in W^{1,2}\big(0,T;L^2(\Gamma_3)\big), \tag{52}$$

$$u_0 \in V, \quad v_0 \in L^2(\Omega), \tag{53}$$

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0(\mathbf{x}) \le 1 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \tag{54}$$

Next, we define bilinear forms $a_\theta\colon V\times V \to \mathbb{R}$, $a_\mu\colon V\times V \to \mathbb{R}$, $a_e\colon V\times W \to \mathbb{R}$, $a_e^*\colon W\times V \to \mathbb{R}$ and $a_\alpha\colon W\times W \to \mathbb{R}$ by equalities

$$a_\theta(u,v) = \int\limits_\Omega \theta\, \nabla u\cdot \nabla v \,\mathrm{d}x, \tag{55}$$

$$a_\mu(u,v) = \int\limits_\Omega \mu\, \nabla u\cdot \nabla v \,\mathrm{d}x, \tag{56}$$

$$a_e(u,\varphi) = \int\limits_\Omega e\, \nabla u\cdot \nabla \varphi \,\mathrm{d}x = a_e^*(\varphi,u), \tag{57}$$

$$a_\alpha(\varphi,\psi) = \int\limits_\Omega \alpha\, \nabla \varphi\cdot \nabla \psi \,\mathrm{d}x + \int\limits_{\Gamma_3} k\,\varphi\psi\,\mathrm{d}x, \tag{58}$$

for all $u, v \in V$, φ, $\psi \in W$. Assumptions (42)–(45), (51) imply that the integrals above are well defined and, using (36) and (37), it follows that the forms a_θ, a_μ, a_e, a_e^* and a_α are continuous; moreover, the forms a_θ, a_μ and a_α are symmetric and, in addition, the form a_θ is V- elliptic and the form a_α is W- elliptic, since

$$a_\theta(v,v) \ge \theta^*\,\|v\|_V^2 \quad \forall\, v\in V, \tag{59}$$

$$a_\alpha(\psi,\psi) \ge \alpha^*\,\|\psi\|_W^2 \quad \forall\, \psi\in W. \tag{60}$$

Assumptions (49) allows us, for a.e. $t\in(0,T)$, to define $f(t)\in V'$ by

$$\langle f(t), v\rangle_{V'\times V} = \int\limits_\Omega f_0(t)v\,\mathrm{d}x + \int\limits_{\Gamma_2} f_2(t)v\,\mathrm{d}a \quad \forall\, v\in V, \tag{61}$$

and, moreover, yields

$$f\in L^2(0,T;V'). \tag{62}$$

We also define the mappings $q\colon [0,T] \to W$ and $j\colon L^2(\Gamma_3)\times V\times V \to \mathbb{R}$, respectively, by

$$(q(t),\psi)_W = \int\limits_\Omega q_0(t)\psi\,\mathrm{d}x - \int\limits_{\Gamma_b} q_b(t)\psi\,\mathrm{d}a + \int\limits_{\Gamma_3} k\,\varphi_F\psi\,\mathrm{d}a, \tag{63}$$

$$j(\beta, u, v) = \int_{\Gamma_3} p(\beta) R(u)\, v \, \mathrm{d}a, \tag{64}$$

for all $v \in V$, $\psi \in W$, $\beta \in L^2(\Gamma_3)$ and $t \in [0, T]$. For the convenience of the reader we recall that here and below R is the real valued function defined by (28). The definition of q is based on Riesz's representation theorem; moreover, it follows from assumptions (50)–(52) that the integrals in (63) are well defined and

$$q \in W^{1,2}(0, T; W). \tag{65}$$

Performing integrals par parts, using notation (55)–(58), (61), (63)–(64) and recalling (39), (41), we obtain the following variational formulation of the antiplane contact Problem P.

Problem PV**.** *Find a displacement field* $u\colon [0, T] \to V$, *an electric potential field* $\varphi\colon\ [0, T] \to W$ *and a bonding field* $\beta\colon\ [0, T] \to L^2(\Gamma_3)$ *such that, for a.e.* $t \in (0, T)$,

$$\begin{aligned} \langle \ddot{u}(t), w\rangle_{V'\times V} + a_\theta\big(\dot{u}(t), w\big) + a_\mu\big(u(t), w\big) + a_e^*\big(\varphi(t), w\big) \\ + j\big(\beta(t), u(t), w\big) = \langle f(t), w\rangle_{V'\times V} \quad \forall\, w \in V, \end{aligned} \tag{66}$$

$$a_\alpha\big(\varphi(t), \psi\big) - a_e\big(u(t), \psi\big) = \big(q(t), \psi\big)_W \quad \forall\, \psi \in W, \tag{67}$$

$$\dot{\beta}(t) = -\big(\gamma\beta(t) R\big(u(t)\big)^2 - \epsilon_a\big)_+, \tag{68}$$

and

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \beta(0) = \beta_0. \tag{69}$$

The main existence and uniqueness result in the study Problem PV, that we state here and prove in the next section, is the following.

Theorem 1. *Assume that* (42)–(54) *hold. Then, there exists a unique solution of Problem* (66)–(69)*. Moreover, the solution satisfies*

$$u \in W^{1,2}(0, T; V) \cap C^1([0, T]; H), \quad \ddot{u} \in L^2(0, T; V'), \tag{70}$$

$$\varphi \in W^{1,2}(0, T; W), \tag{71}$$

$$\beta \in W^{1,\infty}\big(0, T; L^2(\Gamma_3)\big) \cap \mathcal{Z}. \tag{72}$$

We conclude that, under the stated assumptions, Problem P has a unique *weak solution* which satisfies (70)–(72).

4 Proof of Theorem 1

The proof of Theorem 1 will be carried out in several steps. We assume in the following that (42)–(54) hold and below in this section c will denote a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, θ, μ, e, α p, L and T, but does not depend on t, nor on the rest of the input data, and whose value may change from place to place. Let $\eta \in L^2(0, T; V')$ be given. In the first step we consider the following variational problem.

Problem PV^1_η**.** *Find a displacement field* $u_\eta\colon [0,T]\to V$ *such that*

$$\langle \ddot{u}_\eta(t), w\rangle_{V'\times V} + a_\theta\big(\dot{u}_\eta(t), w\big) + \langle \eta(t), w\rangle_{V'\times V} = \langle f(t), w\rangle_{V'\times V}$$
$$\forall\, w\in V,\ \textit{a.e. } t\in(0,T), \tag{73}$$
$$u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = v_0. \tag{74}$$

We have the following result.

Lemma 1. *There exists a unique solution of Problem* PV^1_η *and it has the regularity expressed in* (70).

Proof. We define the operator $A_\theta\colon V\to V'$ by

$$\langle A_\theta v, w\rangle_{V'\times V} = a_\theta(v,w) \quad \forall\, v,w\in V. \tag{75}$$

It follows from (75), the continuity of the bilinear form a_θ and (59) that the linear operator A_θ is continuous and positively definite, i.e.

$$\langle A_\theta w, w\rangle_{V'\times V} \geq \theta^*\,\|w\|_V^2 \quad \text{for all } w\in V.$$

Recall also that $f-\eta\in L^2(0,T;V')$ and $v_0\in H$. Then, from a classical result on ordinary differential equations in abstract spaces (see, e.g. [24, p. 140]), it follows that there exists a unique function v_η which satisfies

$$v_\eta\in L^2(0,T;V)\cap C([0,T];H), \quad \dot{v}_\eta\in L^2(0,T;V'), \tag{76}$$
$$\dot{v}_\eta(t) + A_\theta v_\eta(t) + \eta(t) = f(t) \quad \text{a.e. } t\in(0,T), \tag{77}$$
$$v_\eta(0) = v_0. \tag{78}$$

Let $u_\eta\colon [0,T]\to V$ be the function defined by

$$u_\eta(t) = \int_0^t v_\eta(s)\,\mathrm{d}s + u_0 \quad \forall\, t\in[0,T]. \tag{79}$$

It follows from (75) and (76)–(79) that u_η is a solution of the variational problem PV^1_η and it satisfies the regularity expressed in (70). This concludes the existence part of Lemma 1. The uniqueness of the solution follows from the uniqueness of the solution of problem (76)–(78). $\square$

In the next step, we use the displacement field u_η obtained in Lemma 1 to define the following variational problem for the electrical potential field.

Problem PV^2_η**.** *Find an electrical potential field* $\varphi_\eta\colon [0,T]\to W$ *such that*

$$a_\alpha\big(\varphi_\eta(t),\psi\big) - a_e\big(u_\eta(t),\psi\big) = \big(q(t),\psi\big)_W \quad \forall\,\psi\in W,\ t\in[0,T], \tag{80}$$

The well-posedness of Problem PV^2_η follows.

Lemma 2. *There exists a unique solution $\varphi_\eta \in W^{1,2}(0,T;W)$ of Problem PV_η^2.*

Proof. Let $t \in [0,T]$. We use the properties of the bilinear form a_α and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta(t) \in W$ which solves (80) at any moment $t \in [0,T]$. Consider now $t_1, t_2 \in [0,T]$; using (80) and (60) we find that

$$\begin{aligned}\alpha^* \|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W^2 \le \|e\|_{L^\infty(\Omega)} \|u_\eta(t_1) - u_\eta(t_2)\|_V \|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \\ + \|q(t_1) - q(t_2)\|_W \|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W,\end{aligned}$$

which implies that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \le c\,(\|u_\eta(t_1) - u_\eta(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \tag{81}$$

We note that regularity $u_\eta \in W^{1,2}(0,T;V)$ combined with (65) and (81) imply that $\varphi_\eta \in W^{1,2}(0,T;W)$ which concludes the proof. ☐

We use again the solution u_η obtained in Lemma 1 to construct the following Cauchy problem for the bonding field.

Problem PV_η^3. *Find a bonding field $\beta_\eta \colon [0,T] \to L^2(\Gamma_3)$ such that*

$$\dot{\beta}_\eta(t) = -\big(\gamma\,\beta_\eta(t) R\big(u_\eta(t)\big)^2 - \epsilon_a\big)_+, \tag{82}$$

$$\beta_\eta(0) = \beta_0. \tag{83}$$

We have the following existence and uniqueness result.

Lemma 3. *There exists a unique solution to Problem PV_η^3. Moreover, the solution satisfies $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$.*

Proof. For the sake of simplicity, we omit the explicit display of of the dependence of various functions on $\mathbf{x} \in \Gamma_3$. Consider the mapping $F \colon [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\big(\gamma\beta R\big(u_\eta(t)\big)^2 - \epsilon_a\big)_+,$$

for $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows that F_η is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\beta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus, using a version of Cauchy-Lipschitz theorem (see, e.g. [23, p. 48]), we obtain that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ which satisfies (82)–(83). The regularity $\beta_\eta \in \mathcal{Z}$ follows from (82)–(83) and the assumption (54). Indeed, equation (82) implies that for a.e. $\mathbf{x} \in \Gamma_3$ the function $t \longmapsto \beta_\eta(\mathbf{x},t)$ is decreasing and its derivative vanishes when $\gamma\,\beta_\eta(t) R(u_\eta(t))^2 \le \epsilon_a$. Combining these properties with the inequality $0 \le \beta(0) \le 1$ we deduce that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3, which shows that $\beta_\eta \in \mathcal{Z}$. ☐

Now, for $\eta \in L^2(0,T;V')$ we denote by u_η the solution of problem PV_η^1 obtained in Lemma 1, by φ_η the solution of problem PV_η^2 obtained in Lemma 2 and by β_η the solution of Problem PV_η^3 given by Lemma 3. Let $\Lambda\eta(t)$ denote the element of V' defined by

$$\langle \Lambda\eta(t), w\rangle_{V'\times V} = a_\mu\big(u_\eta(t), w\big) + a_e^*\big(\varphi_\eta(t), w\big) + j\big(\beta_\eta(t), u_\eta(t), w\big), \tag{84}$$

for all $w \in V$ and $t \in [0,T]$. We have the following result.

Lemma 4. *For all $\eta \in L^2(0,T;V')$ the element $\Lambda\eta$ belongs to $C([0,T];V')$. Moreover, the operator $\Lambda\colon L^2(0,T;V') \to L^2(0,T;V')$ has a unique fixed point η^*.*

Proof. Let $\eta \in L^2(0,T;V')$ and let $t_1,\, t_2 \in [0,T]$. Using (84), the continuity of the bilinear forms a_μ and a_e^* and (64), we obtain

$$\begin{aligned}\|\Lambda\eta(t_1)-\Lambda\eta(t_2)\|_{V'} \le c\big(&\|u_\eta(t_1)-u_\eta(t_2)\|_V + \|\varphi_\eta(t_1)-\varphi_\eta(t_2)\|_W \\ &+ \big\|p\big(\beta_\eta(t_1)\big)R\big(u_\eta(t_1)\big) - p\big(\beta_\eta(t_2)\big)R\big(u_\eta(t_2)\big)\big\|_{L^2(\Gamma_3)}\big).\end{aligned}$$

Now, keeping in mind (37), assumptions on the function p, the inequality $0 \le \beta_\eta \le 1$ and the properties of the operator R we find

$$\begin{aligned}\|\Lambda\eta(t_1)-\Lambda\eta(t_2)\|_{V'} \le c\,\big(&\|u_\eta(t_1)-u_\eta(t_2)\|_V + \|\varphi_\eta(t_1)-\varphi_\eta(t_2)\|_W \\ &+ \|\beta_\eta(t_1)-\beta_\eta(t_2)\|_{L^2(\Gamma_3)}\big).\end{aligned} \tag{85}$$

Since $u_\eta \in W^{1,2}(0,T;V)$, $\varphi_\eta \in W^{1,2}(0,T;W)$ and $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ we deduce from inequality (85) that $\Lambda\eta \in C([0,T];V')$.

Let now $\eta_1,\, \eta_2 \in L^2(0,T;V')$ and let $t \in [0,T]$. In what follows we use the notation $u_i = u_{\eta_i}$, $v_i = v_{\eta_i} = \dot{u}_{\eta_i}$, $\varphi_i = \varphi_{\eta_i}$ and $\beta_i = \beta_{\eta_i}$ for $i = 1,2$. Using arguments similar to those in the proof of (85) we find that

$$\begin{aligned}\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_{V'} \le c\big(&\|u_1(t)-u_2(t)\|_V + \|\varphi_1(t)-\varphi_2(t)\|_W \\ &+ \|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\big).\end{aligned} \tag{86}$$

On the other hand, (80) and arguments similar as those used in the proof of (81) yield

$$\|\varphi_1(t)-\varphi_2(t)\|_W \le c\,\|u_1(t)-u_2(t)\|_V. \tag{87}$$

Moreover, using (82), (83) and the properties of the function R it follows that

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)} \le c\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}\,\mathrm{d}s + c\int_0^t \|u_1(s)-u_2(s)\|_V\,\mathrm{d}s$$

and, by using Gronwall's inequality, we find

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)} \le c\int_0^t \|u_1(s)-u_2(s)\|_V\,\mathrm{d}s. \tag{88}$$

We combine now the inequalities (86), (87) and (88) to obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V'} \le c\,\|u_1(t) - u_2(t)\|_V + c\int_0^t \|u_1(s) - u_2(s)\|_V \,\mathrm{d}s.$$

Also, since u_1 and u_2 have the same initial value it follows that

$$\|u_1(t) - u_2(t)\|_V \le \int_0^t \|v_1(s) - v_2(s)\|_V \,\mathrm{d}s.$$

We use now the last two inequalities to obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V'} \le c\int_0^t \|v_1(s) - v_2(s)\|_V \,\mathrm{d}s$$

which implies

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V'}^2 \le c\int_0^t \|v_1(s) - v_2(s)\|_V^2 \,\mathrm{d}s. \tag{89}$$

Next, we obtain from (73)

$$\langle \dot{v}_1 - \dot{v}_2, v_1 - v_2\rangle_{V'\times V} + a_\theta(v_1 - v_2, v_1 - v_2) + \langle \eta_1 - \eta_2, v_1 - v_2\rangle_{V'\times V} = 0$$

a.e. on $(0,T)$. We integrate this relation with respect to the time and use the initial conditions $v_1(0) = v_2(0) = v_0$ and (59) to find

$$\begin{aligned}\theta^*\int_0^t \|v_1(s) - v_2(s)\|_V^2 \,\mathrm{d}s &\le -\int_0^t \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s)\rangle_{V'\times V} \,\mathrm{d}s\\ &\le \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}\|v_1(s) - v_2(s)\|_V \,\mathrm{d}s\\ &\le \frac{1}{\theta^*}\int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 \,\mathrm{d}s + \frac{\theta^*}{4}\int_0^t \|v_1(s) - v_2(s)\|_V^2 \,\mathrm{d}s.\end{aligned}$$

Therefore, from the previous inequality we obtain

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2 \,\mathrm{d}s \le c\int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 \,\mathrm{d}s, \tag{90}$$

and from (89), (90) we deduce that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V'}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 \, \mathrm{d}s.$$

Reiterating this inequality m times yields

$$\|\Lambda^m\eta_1 - \Lambda^m\eta_2\|_{L^2(0,T;V')}^2 \leq \frac{c^m}{m!} \|\eta_1 - \eta_2\|_{L^2(0,T;V')}^2,$$

which implies that for m sufficiently large a power Λ^m of Λ is a contraction in the Banach space $L^2(0,T;V')$; therefore there exists a unique element $\eta^* \in L^2(0,T;V')$ such that $\Lambda\eta^* = \eta^*$. □

Proof of Theorem 1. *Existence.* Let $\eta^* \in L^2(0,T;V')$ be the fixed point of the operator Λ and let u, φ, β be the solutions of Problems PV_η^1, PV_η^2 and PV_η^3 respectively with $\eta = \eta^*$, i.e. $u = u_{\eta^*}$, $\varphi = \varphi_{\eta^*}$, $\beta = \beta_{\eta^*}$. Clearly, equalities (67)–(69) hold from PV_η^1, PV_η^2 and PV_η^3. Moreover, since $\eta^* = \Lambda\eta^*$ it follows from (73) and (84) that (66) holds too. The regularity of the solution expressed in (70)–(72) follows from Lemmas 1–3. We conclude that (u, φ, β) is a solution of Problem PV and it satisfies (70)–(72).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of Λ combined with the unique solvability of Problems PV_η^1, PV_η^2 and PV_η^3, guaranteed by Lemmas 1–3. □

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