Existence Theorems for Some Classes of Boundary Value Problems Involving the $P(X)$-Laplacian

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Abstract. We prove an alternative for a nonlinear eigenvalue problem involving the $p(x)$-Laplacian and study a subcritical boundary value problem for the same operator. The theoretical approach is the Mountain Pass Lemma and one of its variants, which is very useful in the study of eigenvalue problems.

Keywords: $p(x)$-Laplacian, nonlinear eigenvalue problem, critical point theory.

1 Introduction

For any fixed real number $p \in (1, +\infty)$ the p-Laplacian is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

This operator appears in a variety of physical fields. For example, applications of Δ_p have been seen in Fluid Dynamics. The equation governing the motion of a fluid involves the p-Laplacian. More exactly the shear stress $\vec{\tau}$ and the velocity gradient ∇u of the fluid are related in the manner that

$$\vec{\tau}(x) = r(x)|\nabla u|^{p-2} \nabla u,$$

where $p = 2$ (resp., $p < 2$ or $p > 2$) if the fluid is Newtonian (resp., pseudoplastic or dilatant). Other applications of the p-Laplacian also appear in the study of flow through porous media ($p = \frac{3}{2}$), Nonlinear Elasticity ($p \geq 2$), or Glaciology ($1 < p \leq \frac{4}{3}$).

This paper is motivated by recent advances in elastic mechanics and electrorheological fluids (sometimes referred to as "smart fluids") where some processes are modeled by nonhomogeneous quasilinear operators.

We refer mainly to the $p(x)$-Laplace operator $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, where $p > 1$ is a real constant.

However, the $p(x)$-Laplace operator possesses more complicated nonlinearities that the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous.

Throughout this paper, Ω stands for a bounded domain in $\mathbb{R}^N$. In the first section we are concerned with the following nonlinear eigenvalue problem with Dirichlet boundary condition and constraints on eigenvalues:

$$\begin{cases} -\Delta_{p(x)}u = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ 0 < \lambda \le a, \end{cases} \tag{1}$$

where $a > 0$ is a given constant. The function f is supposed to satisfy

(H$_1$) f is a Carathéodory function, i.e., measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$, with $f(x,0) \neq 0$ on a subset of Ω of positive measure;

(H$_2$) $|f(x,u)| \le C_1 + C_2|u|^{q(x)-1}$, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, with constants $C_1 \ge 0$, $C_2 \ge 0$ and $1 < p(x) \le q(x) < p^\star(x)$, where

$$p^\star(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \ge N; \end{cases}$$

(H$_3$) there are constants $b_1 \ge 0,\ b_2 \ge 0,\ 1 \le \gamma < p(x) < \nu$ such that, for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$,

$$f(x,u)u - \nu \int_0^u f(s,\tau)\,\mathrm{d}\tau \ge -b_1 - b_2|u|^\gamma.$$

By the Sobolev embedding Theorem, there exists a constant $C > 0$ such that, for every $u \in W_0^{1,p(x)}(\Omega)$,

$$\|u\|_{L^q}^{q(x)} \le C\|u\|_{W_0^{1,p(x)}}^{q(x)}. \tag{2}$$

For $p \in L^\infty(\Omega)$, let

$$p^-(\Omega) = \operatorname*{ess\,inf}_\Omega p(x), \quad p^+(\Omega) = \operatorname*{ess\,sup}_\Omega p(x).$$

For a later use we denote

$$a_1 = c_1|\Omega|^{(q^+-1)/q^+} \quad \text{and} \quad a_2 = C\big(c_1|\Omega|^{(q^+-1)/q^+} + c_2(q^+)^{-1}\big). \tag{3}$$

Our approach relies on the following version of the celebrated Mountain Pass Theorem of Ambrosetti-Rabinowitz (see [1, 2]):

Lemma 1 ([3]). *Let X be a Banach space and let $F\colon X \times \mathbb{R} \to \mathbb{R}$ be a C^1 functional verifying the hypotheses*

(i) *there exist constants $\rho > 0$ and $\alpha > 0$ provided $F(v, \rho) \geq \alpha$, for every $v \in X$;*

(ii) *there is some $r > \rho$ with $F(0,0) = F(0,r) = 0$.*

Then the number

$$c := \inf_{g\in\Gamma} \max_{0\leq\tau\leq 1} F\big(g(\tau)\big),$$

where

$$\Gamma = \{g \in C([0,1], X \times \mathbb{R});\;\; g(0) = (0,0),\; g(1) = (0,r)\},$$

is a critical value of F.

Let us now state our main result concerning the eigenvalue problem (1). We shall keep the notations given in (2), (3) and, for simplicity, we use in the sequel $\|\cdot\|$ in place of $\|\cdot\|_{W_0^{1,p(x)}}$.

Theorem 1. *Assume that the function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (H_1)–(H_3). *Let $\beta \in C^1(\mathbb{R}, \mathbb{R})$ be a function such that, for some constants $0 < \rho < r$, $\sigma > 0$, the following properties hold:*

(β_1) $\beta(0) = \beta(r) = 0$;

(β_2) $\rho^{\sigma+1} \geq q(x) a_2 \frac{||u||^{q^+}}{||u||^{q(x)}}$ *and* $\frac{\sigma+1}{q(x)}\beta(\rho) = a_1$;

(β_3) $\lim_{|t|\to\infty} \beta(t) = +\infty$;

(β_4) $\beta'(t) < 0$ *if and only if* $t < 0$ *or* $\rho < t < r$.

Then, for each $a > 0$, the following alternative holds:

either

(i) *$a > 0$ is an eigenvalue in problem* (1) *with a corresponding eigenfunction $u \in W_0^{1,p(x)}(\Omega)$ located by*

$$\alpha \leq -\int\limits_{\Omega}\int\limits_{0}^{u(x)} f(x,t)\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{ap(x)}\,\|u\|^{p(x)} \leq a_1 + \alpha$$

or

(ii) *one can find a positive number s with*

$$\rho < s < r \tag{4}$$

which determines an eigensolution $(u,\lambda)\in W_0^{1,p(x)}(\Omega)\times(0,a]$ *of the problem* (1) *by the relations*

$$\|u\| = s^{-\sigma/q(x)}(-\beta'(s))^{1/q(x)}, \tag{5}$$

$$\lambda^{-1} = a^{-1} + s^{(q(x)+\sigma p(x))/q(x)}\big(-\beta'(s)\big)^{(q(x)-p(x))/q(x)}, \tag{6}$$

$$\begin{aligned}\alpha &\le \frac{s^{q(x)+1}}{q(x)}\,\|u\|^{q(x)} + \frac{\sigma+1}{q(x)}\,\beta(s) - \int_\Omega\int_0^{u(x)} f(x,t)\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{ap(x)}\,\|u\|^{p(x)}\\ &\le a_1+\alpha.\end{aligned} \tag{7}$$

In the second section of this paper we consider another problem related to the $p(x)$-Laplacian operator:

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u + |u|^{q(x)-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega\,,\\ u\not\equiv 0, & \text{in } \Omega. \end{cases} \tag{8}$$

Our result on this problem is

Theorem 2. *If* $\lambda < \lambda_1(-\Delta_{p(x)}) := \inf\{\int_\Omega|\nabla u|^{p(x)};\ u\in W_0^{1,p(x)}(\Omega),\ u\neq 0,\ \|u\|_{L^{p(x)}}=1\}$ *and* $1<p(x)<q(x)<p^\star(x)$*, then the problem* (8) *has a weak solution.*

The key argument in the proof is the Mountain-Pass Theorem in the following variant:

Ambrosetti-Rabinowitz Theorem. *Let* X *be a real Banach space and* $F\colon X\to\mathbb{R}$ *be a* C^1*-functional. Suppose that* F *satisfies the Palais-Smale condition and the following geometric assumptions:*

$$\begin{aligned}&\textit{there exist positive constants } R \textit{ and } c_0 \textit{ such that}\\ &F(u)\ge c_0,\ \textit{for all } u\in X \textit{ with } \|u\|=R;\end{aligned} \tag{9}$$

$$\begin{aligned}&F(0)<c_0 \textit{ and there exists } v\in X \textit{ such that}\\ &\|v\|>R \textit{ and } F(v)<c_0.\end{aligned} \tag{10}$$

Then the functional F *possesses at least a critical point.*

We refer to [4] and [5] for related bifurcation results in the semi-linear case and to the works [6–9], and [10] for recent qualitative results both in the semi-linear and in the quasi-linear case.

2 Proof of Theorem 1

In order to set problem (1) in terms of Lemma 1 we introduce the functional $F \in C^1(W_0^{1,p(x)}(\Omega) \times \mathbb{R})$ by

$$F(v,t) = \frac{|t|^{\sigma+1}}{q(x)}\,\|v\|^{q(x)} + \frac{\sigma+1}{q(x)}\,\beta(t) - \int\limits_{\Omega}\int\limits_{0}^{v(x)} f(x,t)\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{ap(x)}\,\|v\|^{p(x)}. \tag{11}$$

From (β_1) and (11) we derive that condition (ii) of Lemma 1 is valid.

From (H_2), (2) and (3) we see that, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned}
\int\limits_{\Omega}\int\limits_{0}^{v(x)} f(x,t)\,\mathrm{d}t\,\mathrm{d}x &\le c_1\|v\|_{L^1} + c_2(q^+)^{-1}\,\|v\|_{L^{q^+}}^{q^+} \\
&\le c_1|\Omega|^{(q^+-1)/q^+}\,\|v\|_{L^{q^+}} + c_2(q^+)^{-1}\,\|v\|_{L^{q^+}}^{q^+} \\
&\le c_1|\Omega|^{(q^+-1)/q^+} + \big(c_1|\Omega|^{(q^+-1)/q^+} + c_2(q^+)^{-1}\big)\,\|v\|_{L^{q^+}}^{q^+} \\
&\le c_1|\Omega|^{(q^+-1)/q^+} + C\big(c_1|\Omega|^{(q^+-1)/q^+} + c_2(q^+)^{-1}\big)\,\|v\|^{q^+} \\
&= a_1 + a_2\,\|v\|^{q^+}
\end{aligned} \tag{12}$$

Relations (11), (12) and (β_2) yield

$$F(v,\rho) \ge \left(\frac{\rho^{\sigma+1}}{q(x)} - a_2\frac{||v||^{q^+}}{||v||^{q(x)}}\right)\|v\|^{q(x)} + \frac{\sigma+1}{q(x)}\,\beta(\rho) - a_1 \ge \alpha,$$

for every $v \in W_0^{1,p(x)}(\Omega)$. This shows that the requirement (i) of Lemma 1 is fulfilled.

We check now that F verifies the Palais-Smale condition. To this end, let (v_n,t_n) be a sequence in $W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ such that $F(v_n,t_n)$ is bounded and

$$F'(v_n,t_n) = \big(F_v(v_n,t_n), F_t(v_n,t_n)\big) \to 0, \quad \text{in } W^{-1,p'(x)}(\Omega) \times \mathbb{R},$$

where $p'(x) = \frac{p(x)}{p(x)-1}$. Therefore

$$|F(v_n,t_n)| \le M \tag{13}$$

$$\begin{aligned}
-F_v(v_n,t_n) &= |t_n|^{\sigma+1}\,\|v_n\|^{q(x)-p(x)}\,\Delta_{p(x)}v_n \\
&\quad + f(\cdot,v_n) + a^{-1}\,\Delta_{p(x)}v_n \to 0 \quad \text{in } W^{-1,p'(x)}(\Omega)
\end{aligned} \tag{14}$$

$$F_t(v_n,t_n) = |t_n|^{\sigma}(\operatorname{sgn} t_n)\,\|v_n\|^{q(x)} + \beta'(t_n) \to 0, \quad \text{in } \mathbb{R}. \tag{15}$$

From (11), (12) and (13) we infer that

$$M \ge \left(\frac{|t_n|^{\sigma+1}}{q^+} - a_2\right)\frac{||v||^{q^+}}{||v||^{q(x)}} + \frac{\sigma+1}{q(x)}\,\beta(t_n) - a_1.$$

But, by condition (β_3), this shows that (t_n) is bounded in $\mathbb{R}$.

Without loss of generality we may assume that (v_n) is bounded away from 0. We treat separately two cases.

Firstly, assume that along a subsequence one has $t_n \to 0$. Then, by (β_4), it follows that $\beta'(t_n) \to \beta'(0) = 0$. So, by (15),

$$|t_n|^\sigma \|v_n\|^{q(x)} \to 0, \quad \text{as } n \to \infty. \tag{16}$$

From (11), (13) and (16) we see that

$$\int_\Omega \int_0^{v_n(x)} f(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x - \frac{1}{ap(x)}\,\|v_n\|^{p(x)} \quad \text{is bounded in } \mathbb{R}. \tag{17}$$

Since $t_n \to 0$ and (v_n) is bounded away from zero it is clear from (16) that

$$\begin{aligned}
&|t_n|^{\sigma+1}\|v_n\|^{q(x)-p(x)}\|\,\|\Delta_{p(x)}v_n\|_{W_0^{-1,p'(x)}} \\
&\quad = |t_n|\,|t_n|^\sigma\|v_n\|^{q(x)-p(x)}\,\|v_n\|^{p(x)-1} \\
&\quad = |t_n|\,|t_n|^\sigma\|v_n\|^{q(x)}\,\|v_n\|^{-1} \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, (14) implies

$$f(\cdot, v_n) + a^{-1}\,\Delta_{p(x)}v_n \to 0, \quad \text{as } n \to \infty. \tag{18}$$

From (17) and (18) we find that, for some constant $M > 0$ and with $\nu > 2$ in (H_3),

$$\begin{aligned}
M + \nu^{-1}\|v_n\| &\geq \frac{1}{ap(x)}\,\|v_n\|^{p(x)} - \int_\Omega \int_0^{v_n(x)} f(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x \\
&\quad + \frac{1}{\nu}\left(\int_\Omega f(x,v_n)v_n\,\mathrm{d}x + a^{-1}\int_\Omega (\Delta_{p(x)}v_n)v_n\,\mathrm{d}x\right) \\
&= \frac{1}{a}\left(\frac{1}{p(x)} - \frac{1}{\nu}\right)\|v_n\|^{p(x)} + \frac{1}{\nu}\int_\Omega \left(f(x,v_n)v_n - \nu\int_0^{v_n(x)} f(x,\tau)\,\mathrm{d}\tau\right)\mathrm{d}x,
\end{aligned}$$

if n is sufficiently large. Then hypothesis (H_3) and inequality (2) ensure us that some new constants $d_1 \geq 0$ and $d_2 \geq 0$ exist such that

$$\begin{aligned}
M + \nu^{-1}\|v_n\| &\geq \frac{1}{a}\left(\frac{1}{p(x)} - \frac{1}{\nu}\right)\|v_n\|^{p(x)} - \frac{1}{\nu}\big(b_1|\Omega| + b_2\|v_n\|^\gamma_{L^\gamma}\big) \\
&\geq \frac{1}{a}\left(\frac{1}{p(x)} - \frac{1}{\nu}\right)\|v_n\|^{p(x)} - d_1 - d_2\,\|v_n\|^\gamma.
\end{aligned}$$

Recalling that $1 \leq \gamma < p(x) < \nu$, the last estimate shows that (v_n) is bounded in $W_0^{1,p(x)}(\Omega)$. On the other hand, the growth condition in (H_2) ensures that the restriction of Nemytskii's operator to $W_0^{1,p(x)}(\Omega)$, namely,

$$v \in W_0^{1,p(x)}(\Omega) \longmapsto f\big(\cdot, v(\cdot)\big) \in W^{-1,p'(x)}(\Omega),$$

is a compact mapping, in the sense that it maps any bounded set onto a relatively compact one (see, for details, de Figueiredo [11] or Rabinowitz [2]). Thus, passing eventually to a subsequence,

$$f\big(\cdot, v_n(\cdot)\big) \quad \text{converges in } W_0^{-1,p'(x)}(\Omega). \tag{19}$$

By (18) and (19) we conclude that (v_n) possesses a convergent subsequence in $W_0^{1,p(x)}(\Omega)$.

Assume now that (t_n) is bounded away from 0. Then, by (15), we see that (v_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Hence (19) holds. From (14) it follows that

$$\big(1 + a|t_n|^{\sigma+1}\,\|v_n\|^{q(x)-p(x)}\big)\,\Delta_{p(x)}v_n \quad \text{is convergent in } W^{-1,p'(x)}(\Omega),$$

which shows that $(\Delta_{p(x)}v_n)$ converges in $W^{-1,p'(x)}(\Omega)$. Finally, we obtain that, up to a subsequence, (v_n) converges in $W_0^{1,p(x)}(\Omega)$. This concludes the verification of the Palais-Smale condition for the functional F.

The hypotheses of Lemma 1 are now verified. Thus, there exists a point $(u,s) \in W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ satisfying

$$-\Delta_{p(x)}u = \frac{1}{a^{-1} + |s|^{\sigma+1}\|u\|^{q(x)-p(x)}}\, f(\cdot, u); \tag{20}$$

$$|s|^{\sigma}(\operatorname{sgn} s)\,\|u\|^{q(x)} + \beta'(s) = 0; \tag{21}$$

$$\frac{|s|^{\sigma+1}}{q(x)}\,\|u\|^{q(x)} + \frac{\sigma+1}{q(x)}\,\beta(\mathrm{s}) - \int\limits_{\Omega}\int\limits_{0}^{u(x)} f(x,t)\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{ap(x)}\,\|u\|^{p(x)} \geq \alpha. \tag{22}$$

From (21) we observe that

$$s\beta'(s) \leq 0. \tag{23}$$

There are two cases:

Case 1. $s = 0$. Then the assertion (i) in the alternative of Theorem 1 is deduced from (20) and (22). The last inequality of (i) is obtained from the definition of c and Γ in Lemma 1, making use of the path $g \in \Gamma$ given by $g(t) = (0, tr)$, for $0 \leq t \leq 1$.

Case 2. $s \neq 0$. We argue by contradiction. If $s < 0$ then, by (β_4), it follows that $\beta'(s) < 0$, which contradicts (23). So, the only possibility is $s > 0$. Using (β_4) again it turns out

$$\rho \leq t \leq r. \tag{24}$$

If $t = \rho$ or $t = r$, relation (21) and assumption (β_4) imply $u = 0$. This leads to a contradiction between (20) and our hypothesis (H_1). We proved that (24) reduces to (4). Since $s > 0$, (21) gives rise to (5). From (20) it is clear that $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ is an eigensolution of (1), where

$$\lambda = \frac{1}{a^{-1} + s^{\sigma+1}\|u\|^{q(x)-p(x)}}. \tag{25}$$

Substituting $\|u\|$ as determined by (5) in (25) we arrive at (6). The first inequality of (7) is just (22). The second inequality of (7) follows from Lemma 1, by choosing the path $g(t) = (0, tr)$, $0 \le t \le 1$.

Corollary 1. *Assume that the function* $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies hypotheses* (H_1)*–*(H_3) *and let* $a > 0$ *be a number which is not an eigenvalue of the problem* (1)*. Then there exists a sequence* $(u_n, \lambda_n) \in W_0^{1,p(x)}(\Omega) \times (0, a)$ *of eigensolutions of* (1) *with the properties*

$$u_n \to 0 \ \textit{ in } W_0^{1,p(x)}(\Omega), \ \lambda_n \to 0 \ \textit{ and } \ \lambda_n^{-1}\|u_n\|^{p(x)} \to 0, \quad \textit{as } n \to \infty.$$

Proof. For every $\varepsilon > 0$ one can find $\beta_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ satisfying (β_1)–(β_4) with $\rho = \rho_\varepsilon < r = r_\varepsilon$, which depends on ε, and $\sigma > 0$, $\alpha > 0$ independent of ε such that

$$|\beta_\varepsilon'(t)| \le \varepsilon^{q(x)} t^{-1}, \quad \text{for every } \ t \ge (q(x)a_2)^{1/(\sigma+1)}. \tag{26}$$

Applying Theorem 1, one obtains the number $s = s_\varepsilon \in (\rho_\varepsilon, r_\varepsilon)$ that describes an eigensolution $(u_\varepsilon, \lambda_\varepsilon)$ of (1) by equalities (5) and (6) with $u = u_\varepsilon$ and $\lambda = \lambda_\varepsilon$. Clearly, we can assume

$$s_\varepsilon \to +\infty, \quad \text{as } \ \varepsilon \to 0. \tag{27}$$

Hence, by (5), (26) and (27), we infer that

$$\|u_\varepsilon\| = s_\varepsilon^{-\sigma/q(x)}\big(-\beta'(s_\varepsilon)\big)^{1/q(x)} \le \varepsilon s_\varepsilon^{-(\sigma+1)/q(x)} \to 0, \quad \text{as } \ \varepsilon \to 0. \tag{28}$$

We know that the following equality holds

$$-\frac{1}{\lambda_\varepsilon}\,\Delta_{p(x)} u_\varepsilon = f(x, u_\varepsilon).$$

Letting $\varepsilon \to 0$ we notice that, in view of (H_1) and $u_\varepsilon \to 0$ in $W_0^{1,p(x)}(\Omega)$, it follows that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. In addition, we get from (6) that

$$(\lambda_\varepsilon^{-1} - a^{-1})^{q(x)} = s_\varepsilon^{q(x)+\sigma p(x)}\big(-\beta'(s_\varepsilon)\big)^{q(x)-p(x)} \le \varepsilon^{q(x)(q(x)-p(x))} s_\varepsilon^{(\sigma+1)p(x)}. \tag{29}$$

By (28) and (29) we observe that

$$\|u_\varepsilon\|^{p(x)}\big(\lambda_\varepsilon^{-1} - a^{-1}\big) \le \varepsilon^{q(x)},$$

which implies, taking into account (28), that

$$\lambda_\varepsilon^{-1}\,\|u_\varepsilon\|^{p(x)} \to 0, \quad \text{as } \ \varepsilon \to 0.$$

This completes our proof. □

Corollary 2. *Under the hypotheses of Corollary* 1*, for every function* $\beta \in C^1(\mathbb{R}, \mathbb{R})$ *satisfying conditions* (β_1)–(β_4) *with fixed constants* ρ, r, σ, α*, there is a one-to-one mapping from* $[1, +\infty)$ *into the set of eigensolutions* (u, λ) *of the problem* (1)*. In particular, there exist uncountable many solutions* (u, λ) *of* (1)*.*

Proof. Notice that if $\beta \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the requirements (β_1)–(β_4) for given numbers ρ, r, σ, α, then this is true for each function $\delta\beta$, with an arbitrary number $\delta \geq 1$. We may suppose that there is some $a > 0$ which is not an eigenvalue of (1). Applying Theorem 1 with $\delta\beta$, for $\delta \geq 1$, in place of δ, one finds an eigensolution $(u_\delta, \lambda_\delta) \in W_0^{1,p(x)}(\Omega) \times (0, a)$ and a number $s_\delta \in (\rho, r)$ such that

$$\|u_\delta\| = s_\delta^{-\sigma/q(x)} \big(- \beta'(s_\delta)\big)^{1/q(x)} \, \delta^{1/q(x)} \tag{30}$$

and, by (25),

$$\lambda_\delta^{-1} = a^{-1} + s_\delta^{\sigma+1} \, \|u_\delta\|^{q(x)-p(x)}. \tag{31}$$

Let $\delta_1, \delta_2 \geq 1$ with $\delta_1 \neq \delta_2$. Then (31) shows that $s_{\delta_1} = s_{\delta_2}$. Thus (30) yields $\delta_1 = \delta_2$. This contradiction completes the proof. $\square$

In some situations the qualitative informations provided by Theorem 1 and Corollaries 1 and 2 can be improved by direct methods in studying the eigenvalue problem (1).

Example. Assume that the Carathéodory function $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (H_1) and the growth condition

$$\left| \int_0^t f(x, \tau) \,\mathrm{d}\tau \right| \leq C_1 + C_2 |t|^{p(x)}, \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{32}$$

with constants $C_1 \geq 0$ and $C_2 \geq 0$. Using the constant $C > 0$ entering in (2), with $q(x) = p(x)$, we check that every number $\lambda > 0$ which satisfies

$$\lambda < \lambda^\star := \frac{1}{pCC_2} \tag{33}$$

is an eigenvalue of the boundary value problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x, u), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases}$$

In order to justify this, corresponding to each λ in (33) we introduce the functional $I_\lambda \colon W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$I_\lambda(v) = - \int_\Omega \int_0^{v(x)} f(x, t) \,\mathrm{d}t \,\mathrm{d}x + \frac{1}{\lambda p(x)} \, \|v\|^{p(x)}.$$

The assumption (32) allows us to write

$$\begin{aligned} I_\lambda(v) &\geq \frac{1}{\lambda p(x)} \|v\|^{p(x)} - C_1|\Omega| - C_2 \|v\|^{p(x)}_{L^{p(x)}} \\ &\geq \left(\frac{1}{\lambda p(x)} - CC_2 \right) \|v\|^{p(x)} - C_1 |\Omega|, \end{aligned} \tag{34}$$

for every $v \in W_0^{1,p(x)}(\Omega)$. From (33) and (34) it follows that the functional I_λ is bounded from below, coercive and (sequentially) weakly lower semicontinuous on $W_0^{1,p(x)}(\Omega)$. Therefore the infimum of I_λ is achieved at some $u \in W_0^{1,p(x)}(\Omega)$ which solves the above boundary value problem corresponding to any λ in (33).

3 Proof of Theorem 2

Our hypothesis

$$\lambda < \lambda_1(-\Delta_{p(x)}) := \inf_{W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\int_\Omega |u|^{p(x)}\,\mathrm{d}x}$$

implies the existence of some $C_0 > 0$ such that, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$\int_\Omega \left(|\nabla v|^{p(x)} - \lambda |v|^{p(x)} \right) \mathrm{d}x \geq C_0 \int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x. \tag{35}$$

Set

$$g(u) = \begin{cases} u^{q(x)-1}, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0 \end{cases}$$

and $G(u) = \int_0^u g(t)dt$. Denote

$$F(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} - \lambda |u|^{p(x)} \right) \mathrm{d}x - \int_\Omega G(u)\,\mathrm{d}x.$$

Observe that

$$|G(u)| \leq C\,|u|^{q(x)}$$

and, by our hypothesis $1 < p(x) < q(x) < p^\star(x)$, $W_0^{1,p(x)}(\Omega) \subset L^{q(x)}(\Omega)$, which implies that F is well defined on $W_0^{1,p(x)}(\Omega)$.

A straightforward computation shows that F is a C^1 function and, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$F'(u)(v) = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v - \lambda\,|u|^{p(x)-2} uv \right) \mathrm{d}x - \int_\Omega g(u) v\,\mathrm{d}x.$$

We prove in what follows that F satisfied the hypotheses of the Mountain-Pass Theorem.

Verification of (9): We may write, for every $u \in \mathbb{R}$,

$$|g(u)| \leq |u|^{q(x)-1}.$$

Thus, for every $u \in \mathbb{R}$,

$$|G(u)| \leq \frac{1}{q(x)} |u|^{q(x)}. \tag{36}$$

Now, by (36) and the Sobolev embedding Theorem,

$$F(u) \geq C_0 \|u\|^{p(x)} - \frac{1}{q(x)} \|u\|^{q(x)}, \tag{37}$$

for every $u \in W_0^{1,p(x)}(\Omega)$.

For $\varepsilon > 0$ and $R > 0$ small enough, we deduce by (36) that, for every $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = R$,

$$F(u) \geq c_0 > 0.$$

Verification of (10): Choose $u_0 \in W_0^{1,p(x)}(\Omega)$, $u_0 > 0$ in Ω. Then, by $1 < p(x) < q(x) < p^\star(x)$, it follows that if $t > 0$ is large enough,

$$F(tu_0) = \int_\Omega \frac{t^{p(x)}}{p(x)} \left(|\nabla u_0|^{p(x)} - \lambda |u_0|^{p(x)}\right) \mathrm{d}x - \int_\Omega t^{q(x)}\, u_0^{q(x)} \,\mathrm{d}x < 0.$$

Verification of the Palais-Smale condition: Let (u_n) be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\sup_n |F(u_n)| < +\infty, \tag{38}$$

$$\|F'(u_n)\|_{W^{-1,p'(x)}} \to 0, \quad \text{as } n \to \infty. \tag{39}$$

We prove firstly that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Remark that (39) implies that, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla v - \lambda |u_n|^{p(x)-2} u_n v\right) \mathrm{d}x \\ &\quad = \int_\Omega g(u_n) v \,\mathrm{d}x + o(1)\|v\|, \quad \text{as } n \to \infty. \end{aligned} \tag{40}$$

Choosing $v = u_n$ in (40) we find

$$\int_\Omega \left(|\nabla u_n|^{p(x)} - \lambda\, |u_n|^{p(x)}\right) \mathrm{d}x = \int_\Omega g(u_n) u_n \,\mathrm{d}x + o(1)\|u_n\|. \tag{41}$$

Remark that (38) means that there exists $M > 0$ such that, for any $n \geq 1$,

$$\left| \int_\Omega \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} - \lambda \, |u_n|^{p(x)} \right) \mathrm{d}x - \int_\Omega G(u_n) \, \mathrm{d}x \right| \leq M. \tag{42}$$

But a simple computation yields

$$\int_\Omega g(u_n) u_n \, \mathrm{d}x = q(x) \int_\Omega G(u_n) \, \mathrm{d}x. \tag{43}$$

Combining (41), (42) and (43) we find

$$\alpha \int_\Omega G(u_n) \, \mathrm{d}x = O(1) + o(1) \|u_n\|, \tag{44}$$

where $\alpha = q(x) - p(x) > 0$. Thus, by (41) and (44),

$$\|u_n\|^{p(x)} \leq O(1) + o(1) \, \|u_n\|,$$

which means that $\|u_n\|$ is bounded.

It remains to prove that (u_n) is relatively compact. We consider the case $p(x) < N$. First of all we remark that (40) may be written

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla v \, \mathrm{d}x = \int_\Omega h(u_n) v \, \mathrm{d}x + o(1) \|v\|, \tag{45}$$

for every $v \in W_0^{1,p(x)}(\Omega)$, where

$$h(u) = g(u) + \lambda \, |u|^{p(x)-2} u.$$

Obviously, h is continuous and there exists $C > 0$ such that

$$|h(u)| \leq C \left(1 + |u|^{(Np(x)-N+p(x))/(N-p(x))} \right). \tag{46}$$

Moreover

$$h(u) = o\left(|u|^{Np(x)/(N-p(x))} \right), \quad \text{as } |u| \to \infty. \tag{47}$$

Observing that $(-\Delta_{p(x)})^{-1} \colon W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is a continuous operator, it follows by (45) that it suffices to show that $h(u_n)$ is relatively compact in $W^{-1,p'(x)}(\Omega)$. By Sobolev's Theorem, this will be achieved by proving that a subsequence of $h(u_n)$ is convergent in $\left(L^{(Np(x))/(N-p(x))}(\Omega) \right)^\star = L^{(Np(x))/(Np(x)-N+p(x))}(\Omega)$.

Since (u_n) is bounded in $W_0^{1,p(x)}(\Omega) \subset L^{(Np(x))/(N-p(x))}(\Omega)$ we can suppose that, up to a subsequence,

$$u_n \to u \in L^{Np(x)/(N-p(x))}(\Omega), \quad \text{a.e. in } \Omega.$$

Moreover, by Egorov's Theorem, for each $\delta > 0$, there exists a subset A of Ω with $|A| < \delta$ and such that

$$u_n \to u, \quad \text{uniformly in } \ \Omega \setminus A.$$

So, it is sufficient to show that

$$\int\limits_A |h(u_n) - h(u)|^{Np(x)/(Np(x)-N+p(x))}\,\mathrm{d}x \le \eta,$$

for any fixed $\eta > 0$. But, by (46),

$$\int\limits_A |h(u)|^{Np(x)/(Np(x)-N+p(x))}\,\mathrm{d}x \le C \int\limits_A \left(1 + |u|^{Np(x)/(N-p(x))}\right)\mathrm{d}x,$$

which can be made arbitrarily small if we choose a sufficiently small $\delta > 0$.

We have, by (47),

$$\int\limits_A |h(u_n)-h(u)|^{Np(x)/(Np(x)-N+p(x))}\,\mathrm{d}x \le \varepsilon \int\limits_A |u_n-u|^{Np(x)/(N-p(x))}\,\mathrm{d}x + C_\varepsilon|A|,$$

which can be also made arbitrarily small, by Sobolev's Theorem and by the boundedness of (u_n) in $W_0^{1,p(x)}(\Omega)$.

Hence, F satisfies Palais-Smale Condition and, by Ambrosetti-Rabinowitz Theorem, the problem (8) has a weak solution.

Remark. *We are not able to decide at this stage what happens if* $\lambda > \lambda_1(-\Delta_{p(x)})$. *The main difficulty consists in the impossibility of defining in a suitable manner the orthogonal of a set, so to split the Banach space* $W_0^{1,p(x)}(\Omega)$, $p \neq 2$, *as a direct sum of its first eigenspace and the corresponding orthogonal. A more general version of Theorem* 2 *can be obtained by replacing the term* $|u|^{q(x)-2}u$ *in* (8) *by a function f(x,u) whose behaviour at* $u = 0$ *and for* $|u| \to +\infty$ *is similar to the one of* $|u|^{q(x)-2}u$. *The final part of the proof of Theorem* 2, *that is, the deduction of the relative compactness of* u_n *from its boundedness, can also be derived using the continuity of Nemytskii's operator* $u \longmapsto h(u)$ *on* $L^{p^\star(x)}(\Omega)$.

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