Chaotic Dynamics in a Three Species Aquatic Population Model with Holling Type II Functional Response

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Abstract. A three-trophic model for marine community is proposed and investigated by means of numerical bifurcation analysis. The proposed model based on a modified version of the Leslie-Gower scheme, incorporates mutual interference in all the three populations and generalizes several other known models in the ecological literature. We investigate the dynamical behavior of the model system by considering the Holling type II functional response of toxin liberation process. Bifurcation diagram and two-dimensional parameter scan suggest that chaotic dynamics is robust to variations in toxin production by phytoplankton. Our study suggests that toxic substances released by TPP population may act as bio-control by changing the state of chaos to order. The mutual interference also induces chaos and acts as both stabilizing and destabilizing factors.

Keywords: chaotic dynamics, toxin producing phytoplankton, aquatic system, functional response.

1 Introduction

Ecological systems have all the necessary characteristics (nonlinearity, high-dimensions, etc.) to support chaotic dynamics [1]. Chaotic dynamics and limit cycles are common in tri-trophic food chain model and are of common interest to both the theoretical and experimental population biologists/ecologists. To assess the ecological implications of chaotic dynamics in different natural system, it is important to explore changes in the dynamics when structural assumptions of the system are varied. One such approach to the study of the dynamics of marine ecological community is its food web and the coupling of interacting species with each other [2]. Upadhyay and Rai [3] provided new examples of a chaotic population system in a simple tri-trophic food chain with Holling type II functional response. Aziz- Alaoui [4] revisited the Upadhyay and Rai model and found

that the chaotic dynamics is observed via sequences of period-doubling bifurcation of limit cycles which however suddenly break down and reverse giving rise to a sequence of period-halving bifurcation leading to limit cycles. Upadhyay and Chattopadhyay [5] modified the model of Upadhyay and Rai [3], by introducing an extra mortality term in middle predator and interpreted the system for aquatic environment consisting of TPP-Zooplankton-Molluscs food chain model. They observed that an increase in the strength of toxic substance released by toxin producing phytoplankton population reduces the propensity of chaotic dynamics and changes the state of chaos to limit cycle and finally settles down to stable focus. Ruxton [6] also showed that the system of linked populations has a stabilizing effect on tri-trophic food chain model. Further study [7] reveals that the rate of toxin production by TPP plays an important role for controlling oscillations in the plankton system.

Many studies investigated the effect of mutual interference on the population dynamics. DeAngelis et al. [8] studied the dynamical properties of a continuous-time autonomous model system incorporating their interference model. This model was studied by Hwang [9] to establish that the periodic orbits, if they exist, are unique. The models considered for interference have different mathematical expressions and different conceptual foundations [10]. From theoretical studies and empirical evidences, a consensus has been reached to conclude that interference has a stabilizing influence on population dynamics [11], although Hassell and May [12] pointed out that there was an upper limit on the interference constant beyond which the dynamics becomes unstable. Predator-prey models incorporating mutual interference were first proposed in Hassell [13] and Rogers and Hassell [14]. A model incorporating density-dependent death rates was considered by Levin [15]. Freedman and Rao [16] considered the Gause-type model incorporating mutual interference among predators and a density-dependent predator death rate. Erbe and Freedman [17] applied it to the simple food chain, modeled by Lotka -Volterra dynamics. Here, I have used the concept for modelling it with modified version of Leslie-Gower scheme in a simple food-chain model modelling marine ecosystems. Motivated by the above studies, we show that the chaotic behaviour as described by Upadhyay and Rai [3, 18] could be controlled by an auto-control mechanism.

In this paper, we propose a generalized model of aquatic ecological system by introducing mutual interference in all the three populations, an extra mortality term in zooplankton population and also taking into account the toxin liberation process of TPP population. This model generalizes several other known models in the literature like Upadhyay and Rai model [3, 18] and Hastings and Powell model [19]. One of the main objectives of this study is to examine the roles of mutual interference parameters and the parameter θ, the rate of toxin release by TPP population on the dynamics of the model system. Different types of toxin release function $f_1(x_1)$, which represents the toxin liberation process of TPP population is considered. The results reported in this paper are only for Holling type II functional response.

This paper is organized as follows: in Section 2, we present the details of the model system. The methodology used is presented in Section 3, to help us in selecting the parameter values to perform simulation experiments. Numerical results are summarized in Section 4 and conclusions are presented in Section 5.

2   The mathematical model

Consider a situation where a prey population x_1 is predated by individuals of a population x_2. The population x_2, in turn serves as a favourite food for individuals of a population x_3. This interaction is represented by the following system (prey – specialist predator – generalist predator interaction) of ordinary differential equations:

$$\frac{dx_1}{dt} = g_1(x_1, x_2, x_3) \equiv a_1 x_1 - b_1 x_1^2 - w_0 \left(\frac{x_1}{x_1 + D_0} \right)^{m_1} x_2^{m_2}, \tag{1a}$$

$$\frac{dx_2}{dt} = g_2(x_1, x_2, x_3) \equiv -a_2 x_2 + w_1 \left(\frac{x_1}{x_1 + D_1} \right)^{m_1} x_2^{m_2}$$

$$- w_2 \left(\frac{x_2}{x_2 + D_2} \right)^{m_2} x_3^{m_2} - \theta f_1(x_1) x_2, \tag{1b}$$

$$\frac{dx_3}{dt} = g_3(x_1, x_2, x_3) \equiv c x_3^{m_3} - w_3 f_2(x_2) x_3^{m_3}, \tag{1c}$$

where $m_i > 0$ for $i = 1, 2, 3$, $a_1, a_2, b_1, \theta, w_0, w_1, w_2, w_3, c$ and D_0, D_1, D_2, D_3, D_4 are the positive constants and $f_i \in C^0(R_+)$ for $i = 1, 2$. The parameters m_i for $i = 1, 2, 3$ are mutual interference parameters that model the intraspecific competition among predators when hunting for prey [13, 16, 17, 20, 21].

In this model, TPP population (prey) of size x_1 serves as the only food for the specialist predator (zooplankton) population of size x_2. This zooplankton population, in turn, serves as a favorite food for the generalist predator (molluscs) population of size x_3. The equations for rate of change of population size for prey and specialist predator have been written following the Volterra scheme that is, predator population dies out exponentially in the absence of its lone prey. The interaction between this predator x_2 and the generalist predator x_3 is modeled by the modified version of the Leslie-Gower scheme, where the loss in a predator population is proportional to the reciprocal of per capita availability of its most favorite food. a_1 is the intrinsic growth rate of the prey population x_1, a_2 is the intrinsic death rate of the predator population x_2 in the absence of the only food x_1, c measures the rate of self-reproduction of generalist predator x_3, w_0, w_1, w_2, w_3 are the maximum values which per capita growth rate can attain. b_1 measures the strength of intra-specific competition among the individuals of the prey species x_1. D_0 and D_1 quantify the extent to which environment provides protection to the prey x_1 and may be thought of as a refuge or a measure of the effectiveness of the prey in evading a predator's attack. D_2 is the value of x_2 at which per capita removal rate of x_2 becomes $w_2/2$. For $m_1 = 1$ the coefficient $w_0/(x_1 + D_0)$, of the third term on the right hand side of (1a) is obtained by considering the probable effect of the density of the prey's population on predators attack rate. If this coefficient is multiplied by x_1 (the prey population at any instant of time), it gives the attack rate on the prey per predator. Denote $p(x_1) = w_0 x_1/(x_1 + D_0)$, when $x_1 \to \infty$, $p(x_1) \to w_0$ which is the maximum value that it can reach. Some aquatic organisms condition their medium by releasing substances that stimulate growth of species, which have similar genetic make-up. Sparse populations rarely provide sufficient opportunities for social interaction necessary for reproduction.

Here, $f_1(x_1)$ represents the toxin liberation process of TPP population for which the mortality of zooplankton increases and as a result, the grazing pressure of zooplankton on TPP population decreases. The parameter θ is the rate of toxin release by TPP population. w_3 measures the limitation on the growth of the generalist predator x_3 by its dependence on per capita availability of its most favorite prey x_2 represented by the function $f_2(x_2)$. $f_2(x_2) = \frac{1}{x_2 + D_3}$ where D_3 represents the residual loss in x_3 population due to severe scarcity of its favourite food x_2. Equations (1a)–(1c) describe the proposed model system.

A model system could be more realistic from ecological point of view and interesting from mathematical point of view if one considers different predator's functional response and compares the dynamic effects of these functional responses. Since functional response encapsulates attributes of both prey and predator biology, so handling time, search efficiency, encounter rate, prey escape ability, etc. should alter, in general, the functional responses [22]. Therefore, predator's functional response may be different when a particular predator preys different prey having different escape ability and if a particular prey is predated by different predators having different hunting ability. The structure of prey habitat is also responsible to alter the functional response. Thus, a predator which follows type II functional response in homogeneous habitat may follow type III in a heterogeneous medium. Anderson [23] experimentally observed in a kelp bass-kelp parch predator-prey interaction for none and medium amounts of habitat structure, the type II functional response had a better fit than linear models. However, for the highest amount of habitat structure a type III functional response had a better fit. In reality, the raptorial behaviour of copepods is highly complex and exhibits a hunting behaviour [24] and hence type II or type III is an appropriate choice. To characterize interface between phytoplankton and zooplankton populations in the presence of toxic chemical, Holling type II functional responses for $f_1(x_1)$ is considered to study the dynamical behaviour of the model system.

It is easy to see that the functions g_i, $i = 1, 2, 3$ in (1a)–(1c) are continuous on R_+^3, where $R_+ = [0, \infty)$. When $m_i \geq 1$, the functions $\frac{\partial g_i}{\partial x_k}$ are continuous on R_+^3. Following Erbe et al. [17], we determine the conditions under which the solutions of (1a)–(1c) form a dynamical system.

A separate investigation is required when the parameters m_i are sub-linear $(0 < m_i < 1)$. In this case, we make the following assumptions:

Assumption 1. There exist functions h_j continuous on R_+^3, where

$$h_j(x_1, x_2, x_3) = x_j^{-m_i} g_j(x_1, x_2, x_3) \quad \text{with} \quad 0 < m_j < 1, \ j = 1, 2, 3.$$

Assumption 2. $x_k^{m_k} \frac{\partial}{\partial x_k} h_j(x_1, x_2, x_3)$ are continuous on R_+^3 for $j \neq k = 1, 2, 3$.

Assumption 3. All solutions of the system $\frac{du_i}{dt} = h_i(u_1, u_2, u_3)$ for $i = 1, 2, 3$ are continuous on R_+^3.

As in Erbe et al. [17], we consider the following change of variables for (1a)–(1c)

$$u_1 = x_1^{1-m_1}, \quad u_2 = x_2^{1-m_2}, \quad u_3 = x_3^{1-m_3}. \tag{2}$$

The system (1a)–(1c) transforms as

$$u_1' = (1-m_1)\left[a_1 u_1 - b_1 u_1^{\frac{2-m_1}{1-m_1}} - w_0 \frac{u_2^{\frac{m_2}{1-m_2}}}{\left(u_1^{\frac{1}{1-m_1}} + D_0\right)^{m_1}}\right] \equiv h_1(u_1, u_2, u_3), \quad (3a)$$

$$u_2' = (1-m_2)\left[-a_2 u_2 + w_1 \frac{u_1^{\frac{m_1}{1-m_1}}}{\left(u_1^{\frac{1}{1-m_1}} + D_1\right)^{m_1}} - w_2 \frac{u_3^{\frac{m_2}{1-m_3}}}{\left(u_2^{\frac{1}{1-m_2}} + D_2\right)^{m_2}}\right.$$

$$\left. -\theta f_1\left(u_1^{\frac{1}{1-m_1}}\right) u_2^{\frac{1}{1-m_2}}\right] \equiv h_2(u_1, u_2, u_3), \quad (3b)$$

$$u_3' = (1-m_3)\left[c - w_3 f_2\left(u_2^{\frac{1}{1-m_2}}\right)\right] \equiv h_3(u_1, u_2, u_3). \quad (3c)$$

The change of variables given in (2) transforms the sublinear system (3c) into (3b) in which no sublinearities are present. Biologically, this amounts to requiring that the mutual interferences are not too strong.

The above discussions may be summarized as follows.

Theorem 1. *Consider the system of equations* (1a)–(1c) *in which* $x_i(0) \geq 0$, $0 < m_i < 1$ *for* $i = 1, 2, 3$. *Assume that the Assumptions 1–3 hold. Then, the solutions of the system equations* (1a)–(1c) *form a dynamical system in the sense of Nemytskii and Stepanov* [25], *provided the mutual interference parameters satisfy the following inequalities:*

$$m_1 \geq \frac{1}{2}, \quad m_2 \geq \frac{1}{2} \quad and \quad m_2 + m_3 \geq 1.$$

3 Methods of investigation

The model system presented above is a multi-parameter system. Model parameters are selected in accordance with a method given in upadhyay et al. [3, 18]. A few hundred parameter combinations (choosing two at a time) are possible. This is simply not feasible for any one to scan the system in all the parameter spaces. Application of non-linear dynamics is in unison with the knowledge of biology of the system and enables one to choose parameter combinations for simulation experiments. The most crucial part of the present methodology is the following conjecture:

Two coupled Kolmogorov systems in oscillatory mode would yield either cyclic (stable limit cycles and quasi-periodic) or chaotic solutions depending on the strength of coupling between the two.

In the present case, the set of parameter values for which the system admits a limit

cycle solution is found to be

$$a_1 = 2.0, \ b_1 = 0.05, \ w_0 = 1.0, \ a_2 = 1.0, \ w_1 = 2.0, \ D_1 = 10,$$
$$w_2 = 0.55, \ D_2 = 10, \ D_4 = 10, \ \theta = 0.003, \ c = 0.03, \tag{4}$$
$$w_3 = 1.0, \ D_3 = 20, \ m_1 = 0.95, \ m_2 = 0.95, \ m_3 = 2.0.$$

There is one more important aspect of these simulation experiments i.e., choosing the step size for the variation of a system parameter from a parameter combination within the chosen range. It depends on the nature of the parameter concerned: whether it is a slow varying or fast varying one.

The most useful way to study such a dynamical system is to monitor the amplitude (maxima) of the subsequent oscillations as the control parameter of the system is varied. A small change in parameter values may lead to a bifurcation: an abrupt, qualitative change in the dynamics. There are number of ways to detect chaotic dynamics in dynamical systems. We have used in our study the phase space representation, bifurcation diagram and two dimensional scan.

4 Numerical results

Model system is integrated numerically using six-order Runge-Kutta method along with predictor corrector method. It is observed that the model system (1a)–(1c) has a chaotic solution at the following set of parameter values (see Fig. 1).

$$a_1 = 1.93, \ b_1 = 0.06, \ w_0 = 1.0, \ D_0 = 10.0, \ a_2 = 1.0, \ w_1 = 2.0,$$
$$D_1 = 10.0, \ w_2 = 0.405, \ D_2 = 10.0, \ c = 0.03, \ w_3 = 1, \ D_3 = 20.0, \tag{5}$$
$$D_4 = 10.0, \ m_1 = 1.0, \ m_2 = 1.0, \ m_3 = 2.0, \ \theta = 0.0.$$

The parameter values are selected on the basis of previous studies [4, 7] and correspond to quantitative measures of attributes of the TPP-Zooplankton-Molluscs food chain.
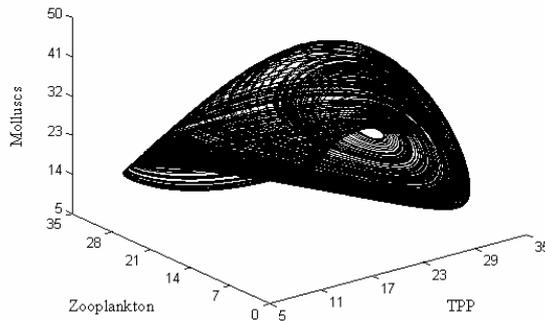

Fig. 1. Phase plane diagram for model system (1a)–(1c) depicting chaotic attractor for $\theta = 0$, other parameter are same as given in (5) [7].

To confirm the existence of chaos, the dynamics of the model system is studied by constructing bifurcation diagram. For Holling type II functional response form for toxin liberation process, we have plotted the successive maxima of top predator x_3 as a function of the parameter θ (rate of toxin substances release by TPP population) keeping other parameters fixed as given in equation (5) for model system (1a)–(1c). The Fig. 2 represents the bifurcation diagrams of model system (1a)–(1c) with Holling type II functional response. This figure shows the transition from chaos to order through sequences of period-halving bifurcation. From this bifurcation diagram, it is observed that an increase in the value of toxic substances released by TPP population has a stabilizing effect. The blow-up bifurcation diagram (see Fig. 2(b)) shows that the model system possesses rich variety of dynamical behaviour for bifurcation parameter θ in the range $[0, 0.06]$. A period – doubling cascade is observed. After the accumulation point, the behaviour settles down onto a chaotic attractor. When θ, the bifurcation parameter is decreased, new periodic orbits are created. The chaotic attractor emanating from the main one is destroyed by a boundary crisis with the unstable periodic orbit created by the saddle-node bifurcation. A saddle-node bifurcation is merge and disappearance of two steady states one of them is saddle and other is node. Two co-existing period – doubling cascades are then observed.
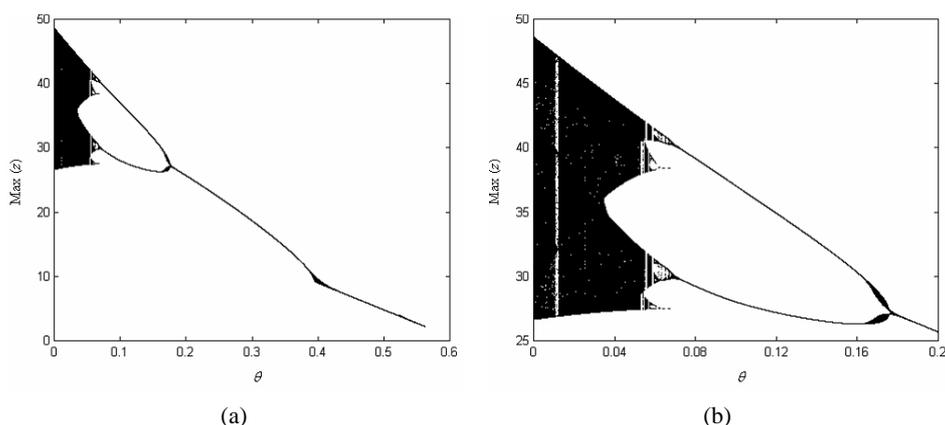

Fig. 2. (a) Bifurcation diagram as a function of θ for model system with $f_1(x_1)$ of Holling type II; (b) blown up bifurcation diagram of (a) in the range $0 \leq \theta \leq 0.2$. Here z stands for x_3 in model system (1a)–(1c) [7].

Dynamical behavior of model system (1a)–(1c) depending on the results of bifurcation diagrams given in Fig. 2 is presented in Table 1. From this result, we observe stable focus, different order limit cycles and strange chaotic attractor in different ranges of θ, the rate of toxic substance released by TPP. Also, we conclude that for the model system, the increase in the value of toxic substances released by TPP has a stabilizing effect. These observations indicate that to maintain the order of an ecosystem functioning, Holling type II functional form for toxin liberation process is more appropriate.

Table 1. Dynamical behavior of model system (1a)–(1c) depending on the results of bifurcation diagrams given in Fig. 2. Pi – limit cycle of period i for (i $= 2, 4, 5, 6$), SF – stable focus, LC – limit cycle, LP – long period, SCA – strange chaotic attractor

Results of model (1a)–(1c) for Holling type II functional response $f(x) = x/(x + D_4)$, where $D_4 = 10$			
θ	Dynamical behavior	θ	Dynamical behavior
0.001–0.0111	SCA	0.061	P6
0.0112	P6	0.062–0.068	P4
0.0113–0.0115	P5	0.07–0.16	P2
0.0116–0.0123	P4	0.17	LP
0.0124–0.059	SCA	0.18–0.39	LC
0.06	LP	0.4–0.6	SF

We have also investigated the role of mutual interference parameters on the dynamics of trophic system in detail. The values of mutual interference parameters were chosen on the basis of the values reported in Katz [26]. We have observed stable focus, limit cycles and chaotic dynamics phenomena in the model system by changing the mutual interference parameters m_i, $i = 1, 2, 3$ and the rate of toxin release by TPP population θ, in the fixed range. We have also reported the function error or argument domain error, the region in the parameter space, where no dynamics is observed. In this domain, the values of mutual interference parameters are not conducive for simulation experiment i.e., in real situation, no species can attain these values of mutual interference. Our approach is to fix m_1 and m_2 then vary m_3 in the interval $[1, 3]$ and θ in the interval $[0, 1)$ and then observe the exchange of states (stability – limit cycle – period doubling – chaos) in the model system for three different cases of m_i ($m_i >, =, < 1$).

The results for model system (1a-1c) are summarized below:

Case I. When $m_i > 1$.

(A) For $f_1(x_1) = \frac{x_1}{x_1 + D_4}$, $f_2(x_2) = \frac{1}{x_2 + D_3}$ (see Table 2).

(i) For $m_1 = m_2 = 1.05$ and $1.5 \leq m_3 \leq 3.0$, $0 \leq \theta \leq 1$.
Chaos exists at some discrete points. For example, chaos exists for $(m_3, \theta) = (1.75, 0.4)$, $(2.0, 0.45)$, $(2.0, 0.5)$, $(2.0, 0.55)$, $(2.25, 0.5)$, $(2.25, 0.55)$. Rest of the points it shows the limit cycle attractor.

(ii) For $m_1 = m_2 = 2.0$ and $1 \leq m_3 \leq 3$, $0 \leq \theta \leq 1$.
It is found that in most of the cases, x_2 becomes extinct and (x_1, x_3) rests on stable focus for higher values of θ. For lower values of θ, all the three populations rest on stable focus and limit cycle attractor in the phase plane. It is also observed that for $m_1 = m_2 = 1.25, 1.5, 1.75$ and for whole range of the parameter space (m_3, θ) (i.e., $1 \leq m_3 \leq 3$, $0 \leq \theta \leq 1$), the model system (1a)–(1c) predicts no dynamics. The simulation results show function error or argument domain error.

Case II. When $m_i = 1$ (i.e., $m_1 = m_2 = m_3 = 1$).

(A) For $f_1(x_1) = \frac{x_1}{x_1 + D_4}$, $f_2(x_2) = \frac{1}{x_2 + D_3}$, $0 \leq \theta \leq 1$.

Chaos exists in the interval $0 \leq \theta \leq 0.25$. For $\theta \in [0.3, 0.4]$ and $\theta \in [0.45, 0.7]$, we obtain the limit cycle and stable focus behaviour respectively. For the values of $\theta \in [0.8, 1.0]$, (x_2, x_3) becomes extinct and x_1 rests on a stable focus but at $\theta = 0.75$, only x_3 becomes extinct and other species rests on stable focus. Fig. 3 shows the chaotic behaviour of the model system (1a)–(1c) observed in the domain $0.75 \leq m_3 \leq 2.25$, $0 \leq \theta \leq 0.35$.

Table 2. Simulation experiments of model system (1a)–(1c) with Holling type II functional response. The values of the common parameters used in the model system are same as given in (4) with $D_4 = 10.0$. The mutual interference parameters $m_i > 1$ $(i = 1, 2, 3)$, and θ varies in the range $[0, 1]$

Values of m_1, m_2	m_3 (in [1;3])	θ (in [0;1])	Dynamical behaviour
$m_1 = m_2 = 1.05$	1.5	0.0002–0.001	Limit cycle
	1.75	0.0001–0.0003; 0.002–0.0095	Limit cycle
		0.09; 0.5–0.6	Limit cycle
		0.4	Chaos
	2.0	0–0.0002; 0.0005–0.0008	Limit cycle
		0.0075–0.02	Limit cycle
		0.3–0.4; 0.6–0.65	Limit cycle
		0.45–0.55	Chaos
	2.25	0–0.002	Limit cycle
		0.0035–0.015;0.09; 0.3–0.45	Limit cycle
		0.5–0.55	Chaos
		0.6–0.7	Limit cycle
	2.5	0–0.02; 0.3–0.75	Limit cycle
	2.75	0.09, 0.3–0.75	Limit cycle
	3.0	0.35–0.42; 0.7– 0.75	Limit cycle
$m_1 = m_2 = 2.0$	1.0	0–1.0	x_1 SF; (x_2, x_3) extinct
	1.25–2.0	0–0.1	(x_1, x_3) SF; x_2 extinct
	2.25	0–1.0	(x_1, x_2, x_3) SF
		0.15–1.0	(x_1, x_3) SF; x_2 extinct
	2.5	0–0.4	(x_1, x_2, x_3) SF
		0.45–1.0	(x_1, x_3) SF; x_2 extinct
	2.75	0–0.5	(x_1, x_2, x_3) SF
		0.55–0.6	(x_1, x_2, x_3) Limit cycle
		0.65–1.0	(x_1, x_3) SF; x_2 extinct
	3.0	0–0.5	(x_1, x_2, x_3) SF
		0.55–0.85	(x_1, x_2, x_3) Limit cycle
		0.86–1.0	(x_1, x_3) SF; x_2 extinct

Case III. When $m_i < 1$.

In this case, chaos does not exist at all. The domain in which we perform the two dimensional scans is

$$m_1 = m_2 = 0.25, \ 0.5, \ 0.75, \ 0.95, \quad m_3 = 0.25, \ 0.5, \ 0.75 \quad \text{and} \quad 0 \leq \theta \leq 1.$$

We obtain only function error in this domain except for $m_1 = m_2 = 0.95$. For $m_1 = m_2 = 0.95$ and in the whole range of m_3 and θ, stable focus and limit cycles are observed. Re-

sults are presented in Table 3, from which it is observed that for $m_1 = m_2 = 0.25, 0.5, 0.75$ and $m_3 \in [0.25, 3], \theta \in [0, 1]$ the dynamics is settles down to stable focus.
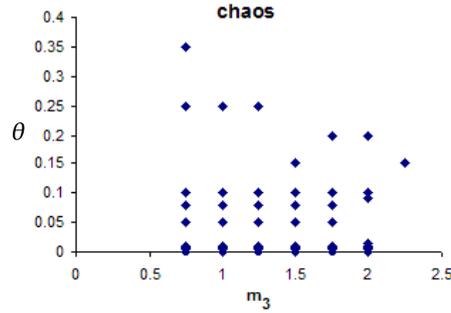

Fig. 3. 2D scan diagram between (m_3, θ) parameter space for Holling type II functional responses with the parameter values $a_1 = 2$, $b_1 = 0.05$, $w_2 = 0.55$, $D_4 = 10.0$, $\theta = 0.003$ other parameters are same as given in (5).

Table 3. Simulation experiments of model system (1a)–(1c) with Holling type II functional response. The values of the common parameters used in the model system are same as given in (4) with $D_4 = 10.0$. The mutual interference parameters m_3 and θ varies in the ranges [0.25, 3] and [0, 1] respectively

Values of m_1, m_2	m_3 (in [0.25;3])	θ (in [0;1])	Dynamical behaviour
$m_1 = m_2 = 0.25$	0.25–0.75	0–1.0	Function Error
	1.00–3.0	0–1.0	Stable Focus
$m_1 = m_2 = 0.5$	0.25–0.75	0–1.0	Function Error
	1.0–3.0	0–1.0	Stable Focus
$m_1 = m_2 = 0.75$	0.25–0.75	0–1.0	Function Error
	1.0–3.0	0–1.0	Stable Focus
$m_1 = m_2 = 0.95$	0.25	0–0.004	Limit cycle
		0.01–0.1	Limit cycle
		0.2–0.4	Stable Focus
	0.5–0.75	0–0.15	Limit cycle
		0.2–0.4	Stable Focus
	1.0	0–0.15	Limit cycle
		0.2–0.7	Stable Focus
		0.75–1.0	(x_1, x_2) SF, x_3 extinct
	1.25–2.25	0–0.15	Limit cycle
		0.2–1.0	Stable Focus
	2.5	0–0.0002	Limit cycle
		0.0005–0.0006	Limit cycle
		0.001–0.006	Limit cycle
		0.03–0.15	Limit cycle
		0.2–1.0	Stable Focus
	2.75	0.2–1.0	Stable Focus
	3.0	0.25–1.0	Stable Focus

5   Conclusions

In this article, we have attempted to find whether mutual interference and toxic substances released by TPP always stabilize the predator – prey dynamics in aquatic environment? Our simulation experiments support the conclusion that TPP stabilizes the predator – prey dynamics in aquatic environment. From the tables, it is observed that for different values of mutual interference parameters in different ranges, dynamics of the model system is also influenced by the functional form of toxin liberation process. For $m_i < 1$, $i = 1, 2, 3$, no dynamics was observed in the range $0.25 \leq m_i \leq 0.75$, but if we take the values of any one of the interference parameters close to 1, the system dynamics converges to stable focus. In this case, the top predator becomes extinct as m_3 reaches 1. For $m_i > 1$, most often the dynamics rests on stable limit cycle or stable focus. From Tables 2, it is found that for $m_1 = m_2 = 1.05$ (i.e., close to 1) and m_3 in the range $[1, 3]$, system dynamics settles down to limit cycle attractor. In this case, model system also supports chaotic dynamics only at a few discrete points. But for $m_1 = m_2 = 2$ and m_3 in the range $[1, 3]$, the system dynamics mostly settled on stable focus and middle predator becomes extinct. These results show that the interaction between predators is a stabilizing factor. Chaotic dynamics/situation may arise from an equilibrium state for different reasons in any ecosystem. But to overcome this chaotic situation sometimes system itself has some mechanism and self-adaptability. There are many ways by which system can be self-adjusted and one of such ways is toxin production by phytoplankton, which reduces the zooplankton grazing, helps the system to recover chaotic situation. In aquatic system of such condition it is reported in Mandal et al. [2], that toxins are produces by many phytoplankton and these toxins may turn the ecosystem into ordered state from chaos by reducing the grazing pressure of zooplankton.

From the tables and 2D scan diagram, it was also observed that the model system supports chaotic dynamics for $m_i \geq 1$. We also observe from bifurcation diagram that chaotic dynamics is robust to changes in changes against rates in toxin production by phytoplankton as it exists for large range of θ value. Period doubling bifurcations seem to be responsible for this kind of dynamical behaviour.

In real life situations, it has been observed that increasing the strength of toxic substances and mutual interference parameters has a greater stabilizing effect. Here, we like to see whether this is true or not in our considered model system. Our simulation results show that interference might actually strongly destabilize the dynamics as well leading to chaotic dynamic behaviour. Further studies are needed to ascertain if this defense mechanism suppresses chaotic dynamics in model aquatic systems.

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