Stabilizing Unstable Periodic Orbits of the Multi-Scroll Chua's Attractor

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Abstract. This paper addresses the control of the n-scroll Chua's circuit. It will be shown how chaotic systems with multiple unstable periodic orbits (UPOs) detected in the Poincaré section can be stabilized as well as taking the system dynamics from one UPO to another.

Keywords: chaos, Chua's circuit, multi-scroll chaotic attractor, nonlinear control.

1 Introduction

Controlling chaos has become a challenging topic in nonlinear dynamics. It has been studied in many scientific and engineering fields such as physics, chemistry, electrical circuit, etc., and several extension and applications of the original OGY control method [1] have been reported [2–6].

Chua's circuit is known as an electrical circuit and having the ability of generating chaos [7]. Recent research results include modifying its nonlinear characteristics by using a generalized piecewise linear function (PWL) with multiple breakpoints to generate the so-called multi-scroll chaotic attractor.

Suykens and Vandewalle [8] designed a simple recurrent neural network model that can produce a chaotic attractor like the double-scroll attractor of Chua's circuit. Later on, Suykens et al. [9] proposed a method for generating a more complete family of multi-scroll instead of n-double scroll chaotic attractors. In [10], the author presented a PWL function approach for creating multi-spiral chaotic attractors from both autonomous and non autonomous differential equations. Yalcin et al. [11,12] also proposed a simple circuit model for generating n-scroll chaotic attractors. The main design idea of most of the aforementioned methodologies is the same – to add some additional breakpoints into the PWL function of the nonlinear resistor in Chua's circuit, or other nonlinear circuits [13].

Most, if not all, of the aforementioned multi-scroll chaotic attractors were verified only by numerical simulations. However, known to electronic engineers, it is much more difficult to physically realize these multi-scroll chaotic attractors by analogue circuits. But great efforts have been made by many. In this endeavour, Arena et al. [14] experimentally verified some n-double scroll chaotic attractors by using a state-controlled CNN-based circuit. Eguchi et al. [15] also constructed an FPGA chaotic circuit for creating n-scroll chaotic attractors. Tang et al. [16] proposed a secure digital communication system using digitized n-scroll chaotic attractors. Yu et al. [17] produced a unidirectional coupled synchronization scheme for n-scroll chaotic attractors from the modified Chua's circuit.

It can be foreseen that multi-scroll chaotic attractors will have many unusual practical applications in such fields as digital and secure communications, synchronous prediction, random bit generation, information systems, and so on.

Controlling such systems was first reported first in [18] in which unstable fixed points were well stabilized.

It is therefore interesting to ask if the multi-scroll Chua's circuit can be stabilized on one of its multiple unstable periodic orbits (UPOs) as well as taking the system dynamics from one UPO to another. Moreover, there may be errors present in the measurements of the system states used in identifying the system. The location of the coordinates of the unstable periodic orbit we wish to control may thus differ from its true coordinates. Similarly, in real systems there is often noise present. This paper provides a positive answer to these problems.

2 Chaos control method

The chaos control algorithm that we introduce in the following uses, in a large sense, the Poincaré section properties. Since chaos is the superposition of a number of periodic motions, it is represented in the Poincaré section by a number of fixed points, called the system chaotic attractor. The chaos control algorithm developed here relies on the knowledge of the chaotic attractor and its response to small perturbations of the system. It is based on the analysis of the Poincaré section to determine how the system approaches the desired orbit or fixed point. The analysis is carried out in three steps [6]:

1. Among the unstable periodic orbits (UPO) of the attractor, choose the one that represents the desired performances.

2. Determine the influence of control parameter on the chosen UPO. For this, we vary the control parameter around the value for which we want to control the system and each time to generate the associated Poincaré section.

3. Determine the variation that should be applied to the control parameter in order to force the system to rejoin the desired UPO or fixed point.

After information about this Poincaré section has been gathered, the system is kept to remain on the desired orbit by perturbing the appropriate parameter. Similar to the original OGY control method [1], we wish to make only small controlling perturbations to the system. We do not envision creating new orbits with different properties from the already existing orbits.

The basic idea of our control algorithm is as follows. Given a periodic orbit represented by a fixed point at the Poincaré section, we wait for the system trajectory to come close to the control region of the desired UPO to bring the system trajectory near the control region. When the system state is in the control region, we will try to use a small parametric perturbation to control the unstable directions of the chaotic state variables.

Let us consider a three dimensional continuous-time system of nonlinear autonomous differential equations described by:

$$\dot{x}(t) = f\big(x(t), p\big), \tag{1}$$

where x is the state vector, f is a smooth function of its variables, and p is an externally accessible control parameter.

We can reduce the three-dimensional phase space flow that results from integrating equation (1) to a two-dimensional map:

$$x_{k+1} = f_M(x_k, p), \tag{2}$$

by intercepting the flow with the Poincaré section.

Let x_f be one of the fixed points of the system (1) at the nominal parameter value p_0 that we wish to stabilize. In other words, we attempt to bring the deviation $\delta x_k = (x_k - x_f)$ to lie on the linearized stable direction.

The control law (3) below is directly derived from the Poincaré section:

$$\delta p_k = \frac{\partial p}{\partial x_f}\delta x_k, \tag{3}$$

where $\delta p_k = (p_k - p_0)$ determine the parameter perturbations and $\frac{\partial p}{\partial x_f}\delta x_k$ determine the influence of small parametric variation on fixed points variation. We restrict parameter perturbations to be small.

This perturbation control law acts instantaneously on the system. However, in real cases, the future system state of a chaotic system depends on the current parametric variation as well as the previous parametric variations, so the system must take sometime to react to the correction. It seems more sensitive, from a practical point of view, to introduce some delay between the computation of the control law and the effective modification of the control parameter. This is realized by adding to the computed law a term depending on the previous value of the control parameter weighted with a parameter γ, which is determined by trial and error.

Thus, equation (3) becomes:

$$\delta p_k = \frac{\partial p}{\partial x_f}\delta x_k + \gamma\delta p_{k-1}. \tag{4}$$

In terms of the quality of control performance, once the control is activated, the controlled system must be maintained at its new trajectory along its evolution.

We expect that, under forward applications of the control law (4), points in the local neighbourhood of the fixed point will eventually fall into the local neighbourhood and then be controlled.

The activating region of control is limited to the following set of points:

$$\{\delta:\ \delta x_1^2 + \delta x_2^2 < 1\}. \tag{5}$$

A general objective of control is to force a given chaotic system into a desired behaviour. This often means driving the trajectory from the chaotic attractors to an equilibrium point or an unstable periodic orbit. Another objective can be to change from one UPO to another.

As example for a system possessing multiples UPOs, we choose the n-scroll Chua's circuit.

3 n-scroll Chua's circuit

In this section, a generalized Chua's circuit is introduced for generating n-scroll chaotic attractors.

The generalized Chua's circuit, which exhibits n number of scrolls, is given by the equations:

$$\begin{aligned} \dot{x} &= \alpha\big(y - f(x)\big), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned} \tag{6}$$

where

$$f(x) = \begin{cases} \dfrac{b\pi}{2a}(x - 2ac), & \text{if } x \geq 2ac, \\ -b\sin\left(\dfrac{\pi x}{2a} + d\right), & \text{if } -2ac < x < 2ac, \\ \dfrac{b\pi}{2a}(x + 2ac), & \text{if } x \leq -2ac. \end{cases} \tag{7}$$

System (6) can generate n-double scroll chaotic attractors for the following relationship:

$$n = c + 1 \tag{8}$$

and

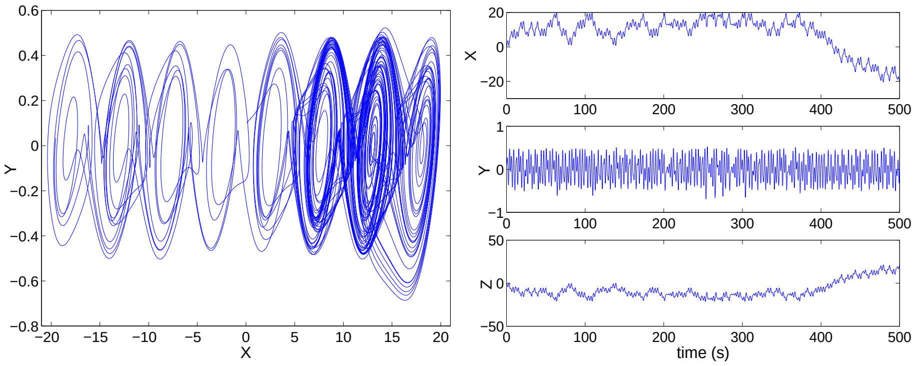
$$d = \begin{cases} \pi, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases} \tag{9}$$

With equations (6) and (7), when $\alpha = 10.814$, $\beta = 14$, $a = 1.3$, $b = 0.11$, $c = 7$, $d = 0$ and starting from the initial state $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$, a 8-scroll attractor is generated as depicted in Fig. 1.

Poincaré section is chosen by plotting the current maxima against the pervious maxima of the x state variable.

The firs state of the fixed point is determined by

$$x_{f1} = 4.507. \tag{10}$$



(a) (b)

Fig. 1. The 8-scroll Chua's attractor: (a) phase space; (b) time response.

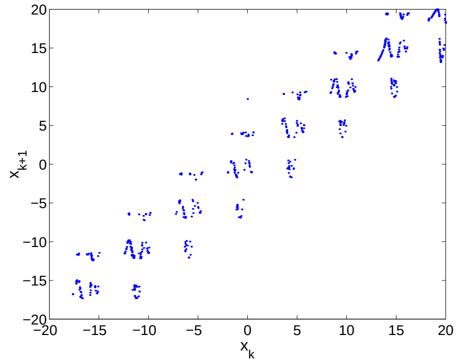


Fig. 2. Return map of the x state variable.

Then, we generate a Poincaré section, at a value near the desired one, for example, we choose $\alpha = 10.914$. In this case,

$$x'_{f1} = 4.810. \tag{11}$$

The control law is defined by:

$$\begin{aligned}\delta\alpha_k &= \frac{\partial\alpha}{\partial x_f}\delta x_k + \gamma\delta\alpha_{k-1} \\ &= \frac{10.914 - 10.814}{4.810 - 4.507}(x_k - x_{f1}) + \gamma\delta\alpha_{k-1} \\ &= 0.33(x_k - 4.507) + \gamma\delta\alpha_{k-1}.\end{aligned} \tag{12}$$

The question now is how to guarantee that the dynamical system (6) is asymptotically stable toward the desired unstable periodic orbit, i.e., the resulting dynamics is asymptotically stable. This can be ensured by choosing suitable value of the parameter γ in the above formula. After many numerical tests, this parameter must be small and it is chosen from the interval $[0.01, 0.5]$.

The n-scroll Chua's circuit is under control of the form:

$$\delta\alpha_k = 0.33(x_k - 4.507) + 0.1\delta\alpha_{k-1}. \tag{13}$$

The perturbation is activated only when the state variables x and y are located in the neighbourhood of the appropriates fixed points x_f and y_f respectively. The activation region of the control is:

$$(x_k - x_f)^2 + (y_k - y_f)^2 < 1, \tag{14}$$

where $y_f = 0.278$.

Computer results of applying control are shown in Fig. 3.

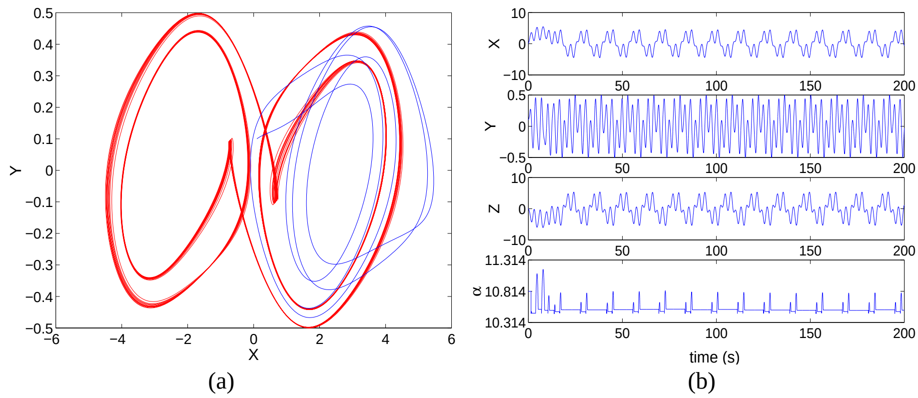


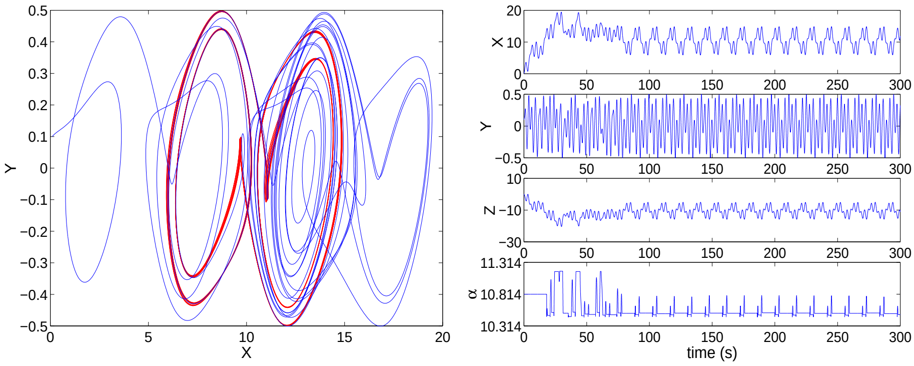
Fig. 3. Stabilizing the first UPO of the 8-scroll Chua's chaotic attractor: (a) phase space; (b) time response.

To stabilize the chaos on its real unstable periodic orbits, one can see that control generate a pulse train; each pulse is activated automatically so that, at sufficient amplitude, determined by the Poincaré section at each travelling from the fixed point, eventually the system orbits converges to the desired unstable periodic orbits. We also tested our chaos control strategy with different initial conditions and it was found to be robust.

In the same way, and by exploring the Poincaré section again, we found $x_{f2} = 14.93$.

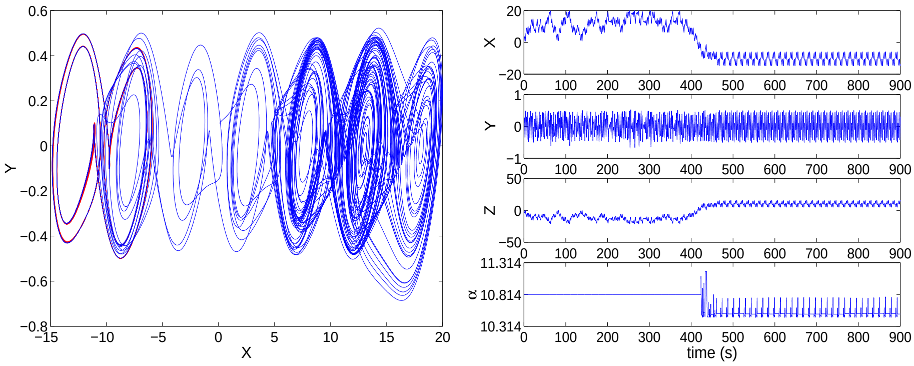
In the last case, for $x_{f3} = -5.80$; control results are shown in Fig. 5.

We now present numerical results about the changing dynamics from one UPO to another one. Starting from the same initial state, control switch on and stabilize the first UPO (appropriate unstable fixed points x_{f1}). It is switched off, the system returns to a chaotic state. Once the control test (14) is verified again, this time with the second unstable fixed points x_{f2} as goal, the control leads the trajectory to the desired UPO. It remains switched on for a same time as in the preceding case. The same control action can be carried out for the last unstable fixed point x_{f3}. Stabilization is reached quickly. Control results are depicted in Fig. 6.



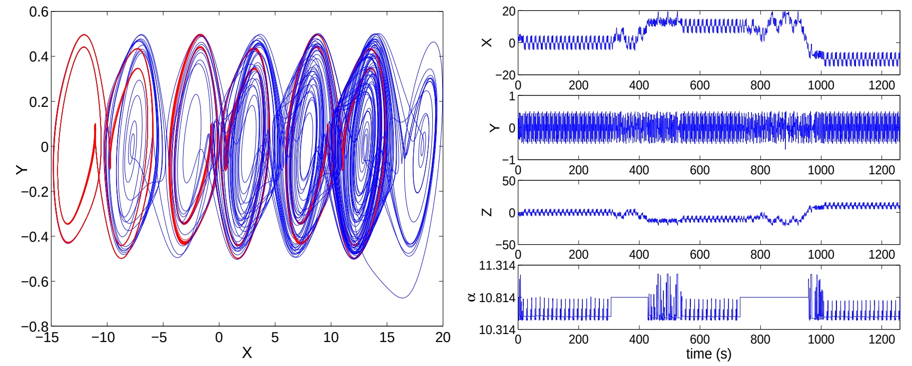
(a) (b)

Fig. 4. Stabilizing the second UPO: (a) phase space; (b) time response.



(a) (b)

Fig. 5. Stabilizing the last UPO: (a) phase space; (b) time response.



(a) (b)

Fig. 6. The transition from one UPO to another one.

To study the robustness of our chaos control method against noise, we add a term $\varepsilon\xi(t)$ to the right-hand side of the linearized equations (13), where $\xi(t)$ is a random variable and ε is a small parameter specifying the intensity of the noise. If the noise is bounded, i.e., $|\varepsilon\xi| < \delta\alpha_{\min}$, then the control will hardly be affected by noise as shown in Fig. 7.

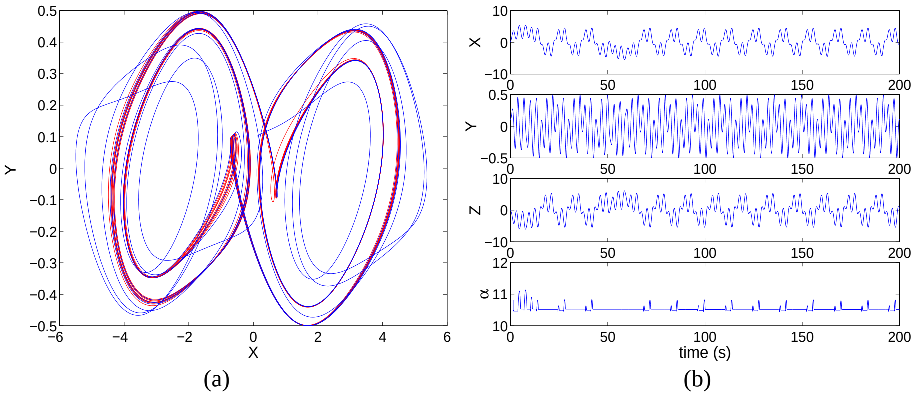


Fig. 7. The effect of noise on the controlled 8-scroll Chua's attractor.

4 Conclusion

This paper has developed a simple control method for stabilizing multiples unstable periodic orbits in the chaotic n-scroll Chua's circuit. From the control point of view, the analysis has shown that this control method is easy to implement, has a fair degree of robustness and can stabilize several high order chaotic systems. The most difficult task is to determine the unstable fixed point that corresponds to the originally targeted unstable periodic orbit (UPO).

Satisfactory control performances are demonstrated for both stabilizing one UPO of the n-scroll Chua's circuit as well as taking the dynamics from one UPO to another. In addition, numerical results showed the robustness of the controller against external noise.

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