Dependence on Initial Conditions of Attainable Sets of Control Systems with p-Integrable Controls*

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Abstract. Quasi-linear systems governed by p-integrable controls, for $1 < p < \infty$ with constraint $\|u(\cdot)\|_p \leq \mu_0$ are considered. Dependence on initial conditions of attainable sets are studied.

Keywords: control system, integral constraint, attainable set.

1 Introduction

In this paper quasi-linear control systems which are nonlinear with respect to phase state vector, linear with respect to control vector and where control inputs are constrained by an integral inequality are studied.

It is well known that attainable sets play an important role in control theory. Many problems of optimization, dynamics, game theory can be stated and solved in terms of attainable sets (see [1, 2]).

Many properties of attainable sets for linear and nonlinear systems without integral constraints is well known (see [3–5]). On the other hand attainable sets of control systems with p-integrable controls are still in interest. General properties and computability of attainable sets of latter completely differs from former (see [6–10]). Hence different techniques are required.

Consider a control system whose behavior is described by a differential equation

$$\dot{x}(t) = f\big(t, x(t)\big) + B\big(t, x(t)\big)u(t), \quad x(t_0) \in X_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the n-dimensional phase state vector of the system, $u \in \mathbb{R}^r$ is the r-dimensional control vector, $t \in [t_0, T]$ $(t_0 < T < \infty)$ is the time, $f(t, x)$ is n-dimensional vector function, $B(t, x)$ is an $(n \times r)$-dimensional matrix function and $X_0 \subset \mathbb{R}^n$.

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It is assumed that the realizations $u(t)$, $t \in [t_0, T]$, of the control u are restricted by the constraint

$$\int_{t_0}^{T} \|u(t)\|^p dt \le \mu_0^p, \quad \mu_0 > 0,\ \ 1 < p < \infty, \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm. Inequality (2) describes the constraint on the control pulse. This constraint is used for controls which have limited resources such as fuel reserve for jet engines, or capital for economical systems, etc. It is also assumed that the functions $(t,x) \to f(t,x)$, $(t,x) \to B(t,x)$ and the set X_0 satisfy the following conditions:

1. The set $X_0 \subset \mathbb{R}^n$ is compact.
2. The functions $(t,x) \to f(t,x)$ and $(t,x) \to B(t,x)$ are continuous with respect to (t,x) and locally Lipschitz with respect to x, that is for any bounded set $D \subset [t_0, T] \times \mathbb{R}^n$ there exist Lipschitz constants $L_i = L_i(D) \in (0,\infty)$ $(i = 1,2)$ such that

$$\|f(t,x^*) - f(t,x_*)\| \le L_1 \|x^* - x_*\|,$$
$$\|B(t,x^*) - B(t,x_*)\| \le L_2 \|x^* - x_*\|$$

   for any $(t,x^*) \in D$, $(t,x_*) \in D$.
3. There exist constants $\gamma_i \in (0,\infty)$ $(i = 1,2)$ such that

$$\|f(t,x)\| \le \gamma_1(1 + \|x\|), \quad \|B(t,x)\| \le \gamma_2(1 + \|x\|)$$

   for every $(t,x) \in [t_0,T] \times \mathbb{R}^n$.

Every function $u(\cdot) \in L_p([t_0,T], \mathbb{R}^r)$, ($1 < p < \infty$), satisfying the inequality (2) is said to be an *admissible control*, where $L_p([t_0,T], \mathbb{R}^r)$ denotes the space of p-power integrable functions. By the symbol $\mathcal{U}$ we denote the set of all admissible control functions $u(\cdot)$.

Let $u_*(\cdot) \in \mathcal{U}$. The absolutely continuous function $x_*(\cdot)\colon [t_0,T] \to \mathbb{R}^n$ which satisfies the equation $\dot{x}_*(t) = f(t, x_*(t)) + B(t, x_*(t))u_*(t)$ a.e. in $[t_0,T]$ and the initial condition $x_*(t_0) = x_0 \in X_0$ is said to be a *solution* of the system (1) with initial condition $x_*(t_0) = x_0$, generated by the admissible control function $u_*(\cdot)$. By the symbol $X(t_0,x_0)$ we denote the set of all solutions of the system (1) with initial condition $x(t_0) = x_0$, generated by all admissible control functions $u(\cdot) \in \mathcal{U}$ and we set

$$X(t_0, X_0) = \big\{x(\cdot) \in X(t_0,x_0)\colon x_0 \in X_0\big\},$$
$$X(t; t_0, X_0) = \big\{x(t) \in \mathbb{R}^n\colon x(\cdot) \in X(t_0, X_0)\big\}.$$

The set $X(t;t_0,X_0)$ is called the *attainable set* of the system (1) with constraint (2) at the instant of time t. It is obvious that the set $X(t;t_0,X_0)$ consists of all $x \in \mathbb{R}^n$, at which the solutions of the system (1) which are generated by all possible controls $u(\cdot) \in \mathcal{U}$ arrive at the instant of time $t \in [t_0, T]$.

The calculation of attainable sets can be a tedious task and it is generally treated numerically with the use of a computer. Therefore it is very important to determine how attainable set changes when the initial conditions change. This is studied in Propositions 1–6.

The Hausdorff distance between the nonempty sets $E, F \subset \mathbb{R}^n$ is defined as

$$\alpha(E, F) = \inf\{r > 0 \colon\ E \subset F + rB,\ F \subset E + rB\}, \tag{3}$$

where B is unit ball in $\mathbb{R}^n$.

2 Preliminaries

First, let us give a useful inequality:

$$\int_{t_0}^{t} \big(K_1 + K_2\|u(\tau)\|\big)d\tau \le K_1(T - t_0) + K_2(T - t_0)^{\frac{p-1}{p}}\mu_0 \tag{4}$$

for every $u(\cdot) \in \mathcal{U}$ and all $t \in [t_0, T]$, where K_1 and K_2 are positive constants. Inequality (4) will be used frequently in the following sections and it can be easily obtained via Hölder's integral inequality (see [11, pp. 122]).

The following proposition states that the graphs of all solutions of the system (1) with constraint (2) is bounded.

Proposition 1. *The inequality*

$$\|x(t)\| \le r$$

is fulfilled for all $x(\cdot) \in X(t_0, X_0)$ *and* $t \in [t_0, T]$, *where*

$$q = \gamma_1(T - t_0) + \gamma_2\mu_0(T - t_0)^{\frac{p-1}{p}},$$
$$d_* = \max\big\{\|x\| \colon x \in X_0\big\}$$

and

$$r = (d_* + q)\exp(q). \tag{5}$$

Proof. Let $x(\cdot) \in X(t_0, X_0)$ be any solution of the system (1). Then there exist $x_0 \in X_0$ and $u(\cdot) \in \mathcal{U}$ such that

$$x(t) = x_0 + \int_{t_0}^{t} \big[f\big(\tau, x(\tau)\big) + B\big(\tau, x(\tau)\big)u(\tau)\big]d\tau, \quad t \in [t_0, T]$$

holds. After taking the norm of both sides, on using Condition 3 and recalling that $d_* = \max\{\|x\| : x \in X_0\}$, we obtain,

$$\|x(t)\| \le d_* + \gamma_1(T - t_0) + \gamma_1 \int_{t_0}^{t} \|x(\tau)\| d\tau + \gamma_2 \int_{t_0}^{t} \|u(\tau)\| d\tau$$
$$+ \gamma_2 \int_{t_0}^{t} \|x(\tau)\|\|u(\tau)\| d\tau.$$

In view of Hölder's integral inequality we have

$$\int_{t_0}^{t} \|u(\tau)\| d\tau \le \left(\int_{t_0}^{t} 1^{\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \left(\int_{t_0}^{t} \|u(\tau)\|^p d\tau \right)^{\frac{1}{p}} \le \mu_0 (T - t_0)^{\frac{p-1}{p}}. \tag{6}$$

By virtue of (6) and since $q = \gamma_1(T - t_0) + \gamma_2 \mu_0 (T - t_0)^{\frac{p-1}{p}}$ we obtain

$$\|x(t)\| \le d_* + q + \int_{t_0}^{t} \big(\gamma_1 + \gamma_2 \|u(\tau)\|\big) \|x(\tau)\| d\tau.$$

It follows from Gronwall's inequality that

$$\|x(t)\| \le (d_* + q) \exp\left(\int_{t_0}^{t} \big(\gamma_1 + \gamma_2 \|u(\tau)\|\big) d\tau \right).$$

From inequality (4) it follows that

$$\|x(t)\| \le (d_* + q) \exp(q).$$

The right hand side of this last inequality is exactly the number r (see (5)). Thus the inequality

$$\|x(t)\| \le r$$

holds for all $x(\cdot) \in X(t_0, X_0)$ and all $t \in [t_0, T]$. □

The set

$$Z(t_0, X_0) = \big\{ \big(t, x(t)\big) \in [t_0, T] \times \mathbb{R}^n \colon x(\cdot) \in X(t_0, X_0) \big\}$$

is called the *integral funnel* of the system (1) with constraint (2).

A corollary of the previous proposition is that the graphs of all solutions of the system (1) is bounded by the cylinder

$$D = \big\{ (t, x) \in [t_0, T] \times \mathbb{R}^n \colon \|x\| \le r \big\}. \tag{7}$$

That is, the inclusion $Z(t_0, X_0) \subset D$ holds. Here, $r > 0$ is defined by (5). From now on D will denote the cylinder (7).

3 Dependence on initial conditions

The following proposition determines the dependence of attainable sets on the initial set X_0.

Proposition 2. *Let X_0 and X_1 be compact subsets of $\mathbb{R}^n$. Then the inequality*

$$\alpha\big(X(t;t_0,X_0),X(t;t_0,X_1)\big)\le K\alpha(X_0,X_1)$$

is valid for all $t\in[t_0,T]$. Here, K is positive constant.

Proof. Let $x_0(\cdot)\in X(t_0,X_0)$ be arbitrary. Then there exist $x_0\in X_0$ and $u(\cdot)\in\mathcal{U}$ such that

$$x_0(t)=x_0+\int\limits_{t_0}^{t}\big[f\big(\tau,x_0(\tau)\big)+B\big(\tau,x_0(\tau)\big)u(\tau)\big]d\tau$$

holds for all $t\in[0,T]$. Since X_0 and X_1 are compact subsets of $\mathbb{R}^n$, by the definition of Hausdorff distance there exists $x_1\in X_1$ such that

$$\|x_1-x_0\|\le\alpha(X_0,X_1)<+\infty$$

holds.

Therefore we obtain a new trajectory $x_1(\cdot)\in X(t_0,X_1)$ for the system (1) which is generated by the same control $u(\cdot)\in\mathcal{U}$ that satisfies the initial condition $x_1(t_0)=x_1$. Thus we can write

$$x_1(t)=x_1+\int\limits_{t_0}^{t}\big[f\big(\tau,x_1(\tau)\big)+B\big(\tau,x_1(\tau)\big)u(\tau)\big]d\tau$$

for all $t\in[t_0,T]$.

By Condition 1 we have

$$\|x_0(t)-x_1(t)\|\le\alpha(X_0,X_1)+\int\limits_{t_0}^{t}\big(L_1+L_2\|u(\tau)\|\big)\|x_0(\tau)-x_1(\tau)\|d\tau$$

for all $t\in[t_0,T]$.

It follows from Gronwall's inequality (see [11, pp. 189]) that

$$\|x_0(t)-x_1(t)\|\le\alpha(X_0,X_1)\exp\Bigg(\int\limits_{t_0}^{t}\big(L_1+L_2\|u(\tau)\|\big)d\tau\Bigg) \tag{8}$$

is valid for all $t\in[t_0,T]$. Taking (4) into account we obtain

$$\|x_0(t)-x_1(t)\|\le\alpha(X_0,X_1)\exp\big(L_1(T-t_0)+L_2(T-t_0)^{\frac{p-1}{p}}\mu_0\big)$$

for all $t \in [t_0, T]$. To shorten notation let us set

$$K = \exp\big(L_1(T - t_0) + L_2(T - t_0)^{\frac{p-1}{p}}\mu_0\big). \tag{9}$$

Thus we get

$$\|x_0(t) - x_1(t)\| \le K\alpha(X_0, X_1)$$

for all $t \in [t_0, T]$.

The inclusion

$$X(t; t_0, X_0) \subset X(t; t_0, X_1) + K\alpha(X_0, X_1)B, \quad t \in [t_0, T] \tag{10}$$

is then immediate.

Similar arguments yield the inclusion

$$X(t; t_0, X_1) \subset X(t; t_0, X_0) + K\alpha(X_0, X_1)B \tag{11}$$

for all $t \in [t_0, T]$.

Hence the desired inequality

$$\alpha\big(X(t; t_0, X_0), X(t; t_0, X_1)\big) \le K\alpha(X_0, X_1), \quad t \in [t_0, T]$$

is an immediate consequence of (3). □

Our next result, an easy corollary of the Proposition 2, tells us that the set valued map $X_0 \subset \mathbb{R}^n \to X(t; t_0, X_0) \subset \mathbb{R}^n$ is Lispschitz continuous with Lipschitz constant K which is defined by (9). It means that attainable set at any instant of time t continuously depends on the initial set X_0.

Proposition 3. *Let $T > t_1 > t_0$, $X_0, X_1 \subset \mathbb{R}^n$ be compact subsets,*

$$r_0 = \alpha(X_0, X_1) + d_1(t_1 - t_0) + d_2\mu_0(t_1 - t_0)^{\frac{p-1}{p}}, \tag{12}$$

and

$$r = r_0 \exp\big(L_1(T - t_1) + L_2\mu_0(T - t_1)^{\frac{p-1}{p}}\big).$$

Then the inequality

$$\alpha\big(X(t; t_0, X_0), X(t; t_1, X_1)\big) \le r, \quad t \in [t_1, T]$$

holds for the system (1) *with constraint* (2). *Here d_1 and d_2 are positive constants.*

Proof. Let $t \in [t_1, T]$ and $y_0 \in X(t; t_0, X_0)$ be arbitrary, then there exist $x_0 \in X_0$, $x_0(\cdot) \in X(t_0, x_0)$ and $u(\cdot) \in \mathcal{U}$ such that

$$y_0 = x_0(t) = x_0 + \int_{t_0}^{t} \big[f\big(\tau, x_0(\tau)\big) + B\big(\tau, x_0(\tau)\big)u(\tau)\big]d\tau$$

holds. By the definition of Hausdorff distance there exists $x_1 \in X_1$ such that

$$\|x_0 - x_1\| \leq \alpha(X_0, X_1). \tag{13}$$

Let $x_1(\cdot) \in X(t_1, x_1)$ be solution of the system (1) starting from the initial point $x_1 \in X_1$ and generated by the same control $u(\cdot) \in \mathcal{U}$ as $x_0(\cdot)$, then

$$x_1(t) = x_1 + \int\limits_{t_1}^{t} \left[f\big(\tau, x_1(\tau)\big) + B\big(\tau, x_1(\tau)\big)u(\tau)\right] d\tau$$

is fulfilled for all $t \in [t_1, T]$. Therefore we obtain the inequality,

$$\begin{aligned} \|x_0(t) - x_1(t)\| \leq \|x_0 - x_1\| &+ \int\limits_{t_1}^{t} \big\|f\big(\tau, x_0(\tau)\big) - f\big(\tau, x_1(\tau)\big)\big\| d\tau \\ &+ \int\limits_{t_1}^{t} \big\|\big[B\big(\tau, x_0(\tau)\big) - B\big(\tau, x_1(\tau)\big)\big]u(\tau)\big\| d\tau \\ &+ \int\limits_{t_0}^{t_1} \big\|f\big(\tau, x(\tau)\big) + B\big(\tau, x_0(\tau)\big)u(\tau)\big\| d\tau \end{aligned} \tag{14}$$

for all $t \in [t_1, T]$.

From Proposition 1 there exists a cylinder D_* such that the inclusions $Z(t_0, X_0) \subset D_*$ and $Z(t_1, X_1) \subset D_*$ holds.

Let

$$d_1 = \max_{(t,x) \in D_*} \|f(t, x)\| \text{ and } d_2 = \max_{(t,x) \in D_*} \|B(t, x)\|,$$

then it follows from (14) and Condition 1 that

$$\begin{aligned} \|x_0(t) - x_1(t)\| \leq \|x_0 - x_1\| &+ \int\limits_{t_1}^{t} \big(L_1 + L_2\|u(\tau)\|\big)\big(\|x_0(\tau) - x_1(\tau)\|\big) d\tau \\ &+ \int\limits_{t_0}^{t_1} \big(d_1 + d_2\|u(\tau)\|\big) d\tau \end{aligned} \tag{15}$$

for all $t \in [t_1, T]$.

In view of (13) and (4) the inequality

$$\|x_0(t) - x_1(t)\| \leq r_0 + \int\limits_{t_1}^{t} \big(L_1 + L_2\|u(\tau)\|\big)\|x_0(\tau) - x_1(\tau)\| d\tau$$

is valid, where r_0 is defined by (12).

By virtue of Gronwall's inequality and (4) we find

$$\begin{aligned}\|x_0(t) - x_1(t)\| &\le r_0 \exp\left(\int_{t_1}^{t} \big(L_1 + L_2\|u(\tau)\|\big) d\tau\right)\\ &\le r_0 \exp\big(L_1(T - t_1) + L_2\mu_0(T - t_1)^{\frac{p-1}{p}}\big)\end{aligned}$$

for all $t \in [t_0, T]$.

Hence the inclusion

$$X(t; t_0, X_0) \subset X(t; t_1, X_1) + rB$$

holds for all $t \in [t_1, T]$. Here, $r = r_0 \exp(L_1(T - t_1) + L_2\mu_0(T - t_1)^{\frac{p-1}{p}})$.

Similarly choosing an arbitrary element from $X(t; t_1, X_1)$ one can prove that the inclusion

$$X(t; t_1, X_1) \subset X(t; t_0, X_0) + rB$$

also holds for all $t \in [t_1, T]$.

Thus the desired inequality

$$\alpha\big(X(t; t_0, X_0), X(t; t_1; X_1)\big) \le r, \quad t \in [t_1, T]$$

follows from (3). □

An immediate corollary of Proposition 3 is the following.

Let $X_0 \subset \mathbb{R}^n$ and $X_n \subset \mathbb{R}^n$ ($n = 1, 2, \ldots$) be compact subsets, $\alpha(X_n, X_0) \to 0$ and $t_n \to t_0$ as $n \to \infty$. Then the inequality

$$\alpha\big(X(t; t_n, X_n), X(t; t_0, X_0)\big) \to 0, \quad t \in [t_0, T]$$

holds as $n \to \infty$.

Let μ_0 and μ_1 be positive,

$$\mathcal{U}_0 = \big\{u(\cdot) \in L_p([t_0, T], \mathbb{R}^m)\colon \|u(\cdot)\|_p \le \mu_0\big\}$$

and

$$\mathcal{U}_1 = \big\{u(\cdot) \in L_p([t_0, T], \mathbb{R}^m)\colon \|u(\cdot)\|_p \le \mu_1\big\}.$$

The set of all solutions and attainable set at instant of time t of the system (1) from the initial set (t_0, X_0) which are generated by all controls from $\mathcal{U}_0$ and $\mathcal{U}_1$ are denoted by $X_0(t_0, X_0)$, $X_0(t; t_0, X_0)$ and $X_1(t_0, X_0)$, $X_1(t; t_0, X_0)$ respectively.

The following proposition gives the dependence of attainable sets on the μ_0.

Proposition 4. *Let $K > 0$ be constant, $r_0 = K(T-t_0)^{\frac{p-1}{p}}|\mu_0 - \mu_1|$ and*

$$r = r_0\Big[1+\big(L_1(T-t_0)+L_2\mu_1(T-t_0)^{\frac{p-1}{p}}\big) \times \exp\big(L_1(T-t_0)+L_2\mu_1(T-t_0)^{\frac{p-1}{p}}\big)\Big],$$

then the inequality

$$\alpha\big(X_0(t;t_0,X_0), X_1(t;t_0,X_0)\big) \le r$$

is fulfilled for all $t \in [t_0,T]$.

Proof. Let $y_0 \in X_0(t;t_0,X_0)$ be arbitrary for $t \in [t_0,T]$, then there exist $x_0 \in X_0$, $x_0(\cdot) \in X_0(t_0,x_0)$ and $u_0(\cdot) \in \mathcal{U}_0$ such that

$$y_0 = x_0(t) = x_0 + \int_{t_0}^{t} \big[f\big(\tau, x_0(\tau)\big) + B\big(\tau, x_0(\tau)\big)u_0(\tau)\big]d\tau$$

holds.

Let us define a new control function $u_1(\cdot)$ via $u_0(\cdot) \in \mathcal{U}_0$ such that

$$u_1(t) = \frac{\mu_1}{\mu_0}u_0(t), \quad t \in [t_0,T].$$

Since

$$\|u_1(\cdot)\|_p = \Bigg(\int_{t_0}^{T} \|u_1(t)\|^p dt\Bigg)^{\frac{1}{p}} = \frac{\mu_1}{\mu_0}\Bigg(\int_{t_0}^{T} \|u_0(t)\|^p dt\Bigg)^{\frac{1}{p}} \le \mu_1,$$

we get $u_1(\cdot) \in \mathcal{U}_1$.

We denote the solution of the system (1) starting from the initial point (t_0, x_0) and generated by the control $u_1(\cdot) \in \mathcal{U}_1$, by $x_1(\cdot) \in X_1(t_0,x_0) \subset X_1(t_0,X_0)$.

Setting $x_1(t) = y_1$, we get

$$y_1 = x_1(t) = x_0 + \int_{t_0}^{t} \big[f\big(\tau, x(\tau)\big) + B\big(\tau, x(\tau)\big)u(\tau)\big]d\tau.$$

Hence we obtain the inequality

$$\|y_0 - y_1\| \le \int_{t_0}^{t} \big\|f\big(\tau, x_0(\tau)\big) - f\big(\tau, x_1(\tau)\big)\big\|d\tau + \int_{t_0}^{t} \big\|B\big(\tau, x_0(\tau)\big)u_0(\tau) - B\big(\tau, x_1(\tau)\big)u_1(\tau)\big\|d\tau.$$

It follows from Condition 1.

$$\|y_0 - y_1\| \le \int_{t_0}^{t} \big(L_1 + L_2\|u_0(\tau)\|\big)\|x_0(\tau) - x_1(\tau)\|d\tau$$
$$+ \int_{t_0}^{t} \big\|B\big(\tau, x_1(\tau)\big)\big\|\|u_0(\tau) - u_1(\tau)\|d\tau.$$

Taking $K = \max_{(t,x)\in D} \|B(t,x)\|$ and using the definition of the control $u_1(\cdot)$ we clearly have

$$\|y_0 - y_1\| \le \int_{t_0}^{t} \big(L_1 + L_2\|u_0(\tau)\|\big)\|x_0(\tau) - x_1(\tau)\|d\tau$$
$$+ K\Big|1 - \frac{\mu_1}{\mu_0}\Big| \int_{t_0}^{t} \|u_0(\tau)\|d\tau,$$

where D is defined by (7).

From the Hölder's integral inequality it follows that

$$\|y_0 - y_1\| \le \int_{t_0}^{t} \big(L_1 + L_2\|u_0(\tau)\|\big)\|x_0(\tau) - x_1(\tau)\|d\tau$$
$$+ K|\mu_0 - \mu_1|(T - t_0)^{\frac{p-1}{p}}.$$

Let us set

$$r_0 = K|\mu_0 - \mu_1|(T - t_0)^{\frac{p-1}{p}}.$$

Using Gronwall's inequality and (4) we find

$$\|x_0(t) - x_1(t)\| \le r_0 \exp\big(L_1(T - t_0) + L_2\mu_0(T - t_0)^{\frac{p-1}{p}}\big).$$

Define $r = r_0 \exp(L_1(T - t_0) + L_2\mu_0(T - t_0)^{\frac{p-1}{p}})$, then it follows that

$$\|x_0(t) - x_1(t)\| \le r.$$

Therefore the inclusion

$$X_0(t; t_0, X_0) \subset X_1(t; t_0, X_0) + rB \tag{16}$$

valid for $t \in [t_0, T]$.

Similarly, one can obtain the inclusion

$$X_1(t;t_0,X_0) \subset X_0(t;t_0,X_0) + rB \tag{17}$$

for $t \in [t_0, T]$.

According to inclusions (16) and (17) we obtain the validity of the inequality

$$\alpha\big(X_0(t;t_0,X_0), X_1(t;t_0,X_0)\big) \le r, \quad t \in [t_0,T]$$

as desired. □

Let us define

$$U_n = \big\{u(\cdot) \in L_p([t_0,T],\mathbb{R}^n) \colon \|u(\cdot)\|_p \le \mu_n\big\}$$

and denote the set of all solutions and attainable set at instant of time t of the system (1) with initial set (t_0, X_0) corresponding to control sets $\mathcal{U}_n$ by $X_n(t_0,X_0)$ and $X_n(t;t_0,X_0)$ respectively.

Proposition 4 implies that for $\mu_n \to \mu_0$ as $n \to \infty$, the inequality

$$\alpha\big(X_n(t;t_0,X_0), X_0(t;t_0,X_0)\big) \to 0$$

holds as $n \to \infty$ for all $t \in [t_0,T]$.

By the following proposition it is proved that the set valued map $t \to X(t;t_0,X_0)$ is Hölder continuous.

Proposition 5. *For the system* (1) *with constraint* (2) *the inequality*

$$\alpha\big(X(t_1;t_0,X_0), X(t_2;t_0,X_0)\big) \le M|t_1-t_2|^{\frac{p-1}{p}}$$

holds for every $t_1, t_2 \in [t_0,T]$. *Here,* $M > 0$ *is constant.*

Proof. Without loss of generality we can suppose $t_1 < t_2$. Let $y_1 \in X(t_1;t_0,X_0)$ be arbitrary, then there exist $x_0 \in X_0$, $x_*(\cdot) \in X(t_0,x_0)$ and $u_*(\cdot) \in \mathcal{U}$ such that

$$y_1 = x_*(t_1) = x_0 + \int_{t_0}^{t_1} \big[f\big(\tau, x_*(\tau)\big) + B\big(\tau, x_*(\tau)\big)u_*(\tau)\big]d\tau$$

holds.

If we take $y_2 = x_*(t_2) \in X(t_2;t_0,X_0)$

$$y_2 = x_*(t_2) = x_0 + \int_{t_0}^{t_2} \big[f\big(\tau, x_*(\tau)\big) + B\big(\tau, x_*(\tau)\big)u_*(\tau)\big]d\tau.$$

is obtained. Therefore we clearly have

$$\|y_1 - y_2\| \le \int_{t_1}^{t_2} \big\|f\big(\tau, x_*(\tau)\big)\big\| d\tau + \int_{t_1}^{t_2} \big\|B\big(\tau, x_*(\tau)\big)u_*(\tau)\big\| d\tau.$$

Let $K_1 = \max\{\|f(t,x)\| \colon (t,x) \in D\}$ and $K_2 = \max\{\|B(t,x)\| \colon (t,x) \in D\}$, then we find

$$\|y_1 - y_2\| \le \int_{t_1}^{t_2} (K_1 + K_2)\|u_*(\tau)\| d\tau$$

Finally, applying Hölder's integral inequality we obtain

$$\|y_1 - y_2\| \le (K_1 + K_2)\mu_0 |t_1 - t_2|^{\frac{p-1}{p}}.$$

If we set $M = (K_1 + K_2)\mu_0$, then we get

$$\|y_1 - y_2\| \le M|t_1 - t_2|^{\frac{p-1}{p}}.$$

Therefore the inclusion

$$X(t_1; t_0, X_0) \subset X(t_2; t_0, X_0) + M|t_1 - t_2|^{\frac{p-1}{p}} B \tag{18}$$

is valid for all $t_1, t_2 \in [t_0, T]$.

Similarly, choosing an arbitrary element y_2 from $X(t_2; t_0, X_0)$ the inclusion

$$X(t_2; t_0, X_0) \subset X(t_1; t_0, X_0) + M|t_1 - t_2|^{\frac{p-1}{p}} B \tag{19}$$

can be obtained. Combining inclusions (18) and (19) we obtain the desired result. □

Let $E \subset \mathbb{R}^n$. Then diameter of E is denoted by

$$\operatorname{diam} E = \sup_{x,y \in E} \|x - y\|.$$

The following proposition gives an upper bound for the diameter of the attainable sets.

Proposition 6. *Let*

$$K = \max_{(t,x)\in D} \|B(t,x)\| \text{ and } d = \operatorname{diam} X_0, \tag{20}$$

then the inequality

$$\operatorname{diam} X(t; t_0, X_0) \le \big(d + 2K\mu_0 (t - t_0)^{\frac{p-1}{p}}\big) \exp\big(L_1(T - t_0)\big)$$

holds for all $t \in [t_0, T]$.

Proof. Let $t \in [t_0, T]$ and $y_1, y_2 \in X(t; t_0, X_0)$ be arbitrary, then there exist $x_1 \in X_0$, $x_1(\cdot) \in X(t_0, X_0)$, $u_1(\cdot) \in \mathcal{U}$ such that

$$y_1 = x_1(t) = x_1 + \int_{t_0}^{t} \left[f\big(\tau, x_1(\tau)\big) + B\big(\tau, x_1(\tau)\big) u_1(\tau) \right] d\tau$$

holds and there exist $x_2 \in X_0$, $x_2(\cdot) \in X(t_0, X_0)$, $u_2(\cdot) \in \mathcal{U}$ such that

$$y_2 = x_2(t) = x_2 + \int_{t_0}^{t} \left[f\big(\tau, x_2(\tau)\big) + B\big(\tau, x_2(\tau)\big) u_2(\tau) \right] d\tau$$

is valid. It follows from Condition 1 and (20) that

$$\begin{aligned} \|y_1 - y_2\| &\le \|x_1 - x_2\| + L_1 \int_{t_0}^{t} \|x_1(\tau) - x_2(\tau)\| d\tau \\ &\quad + \int_{t_0}^{t} \big\|B\big(\tau, x_1(\tau)\big)\big\| \|u_1(\tau)\| d\tau + \int_{t_0}^{t} \big\|B\big(\tau, x_2(\tau)\big)\big\| \|u_2(\tau)\| d\tau \\ &\le d + L_1 \int_{t_0}^{t} \|x_1(\tau) - x_2(\tau)\| d\tau + K \left[\int_{t_0}^{t} \|u_1(\tau)\| d\tau + \int_{t_0}^{t} \|u_2(\tau)\| d\tau \right]. \end{aligned}$$

In accordance with Hölder's integral inequality we obtain

$$\|y_1 - y_2\| \le d + L_1 \int_{t_0}^{t} \|x_1(\tau) - x_2(\tau)\| d\tau + 2K\mu_0 (t - t_0)^{\frac{p-1}{p}}$$

Therefore utilizing the Gronwall's inequality (see [11, pp. 189]) we find

$$\|y_1 - y_2\| \le \left(d + 2K\mu_0 (t - t_0)^{\frac{p-1}{p}} \right) \exp\left(L_1 (T - t_0) \right).$$

Since $t \in [t_0, T]$ and $y_1, y_2 \in X(t; t_0, X_0)$ arbitrary, we find

$$\operatorname{diam} X(t; t_0, X_0) \le \left(d + 2K\mu_0 (t - t_0)^{\frac{p-1}{p}} \right) \exp\left(L_1 (T - t_0) \right)$$

for all $t \in [t_0, T]$. □

It is clear from Proposition 6 that $\operatorname{diam} X(t; t_0, X_0) \to \operatorname{diam} X_0$ as $t \to t_0$.

We conclude from Propositions 1–6 that attainable set of the system (1) with constraint (2) at the instant of time $t \in [t_0, T]$ continuously depends on initial set X_0 and μ_0. Besides, the set valued maps $X_0 \to X(t; t_0, X_0)$ and $t \to X(t; t_0, X_0)$ are Lipschitz and Hölder continuous respectively.

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