Process Distribution in the Network Systems

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Received: 22.09.2006 **Revised:** 19.03.2007 **Published online:** 05.05.2007

Abstract. Performing risk analysis of systems, evaluating reliability of technological objects, hazard of technological processes, we usually have to systems of network type and distribution of various processes in such systems. A well-known mathematical apparatus of diffusive processes example is dispersion in continuum medium (air, water, etc.). Process distribution in network systems is simpler, however, it much depends on network features. In this article theory of Markov chains is selected, distributions of different processes in transitional regimes are analysed as well as issues of their stability. Created models may be used in many different ways, for example, for the analysis or viruses in computer networks, hazard distribution in transport systems regarding transportation of hazardous materials, etc.

Keywords: distribution, process, network system.

1 Introduction

Development and expansion of various networks and network structures create favourable conditions for the spread of process through network channels. One of the most visible examples of such phenomenon is the spread of computer viruses in the internet network. It seems that a similar situation is developing in the networks of mobile connection as well. It is also obvious that together with the improvement of means of transportation and the increase in the quantity and size of the loads, the assessment of process distribution becomes more prominent in the systems of transportation. It has to be noted that the majority of the scientific research articles and works on the process distribution assessment in the network systems has been made during the last several decades [1–4] and this topic is still under active investigation. Here mathematical modelling and process analysis of process distribution through network channels at both transition periods and steady modes are presented. The main aim of the article is the analysis of process distribution in the network systems. Process can be distributed through the channels of various networks and concentrated in the nodes of the networks. Process transmission through the channels that connect network nodes can take place in many ways: for example, process can be transmitted to a single or to several nodes, as an undivided value or divided into parts.

2 Distribution of non-additive process in the network systems

In this section non-additive process that has a feature $H_1 + H_2 = \max\{H_1; H_2\}$ will be analysed. Let's suppose that our network has N nodes. The network can be depicted with an oriented graph and let's also say that the degrees of all its nodes are $S(i) \geq 2$, $i = 1, 2, \ldots, N$. In other words, process can access every node and spread further.

Let's begin investigating a case, in which one network node (e.g., the first) is a point source with process $f_1(0) = f$.

Network flow matrix will be marked as follows:

$$Q = \lfloor q_{ij} \rfloor, \quad \text{where } \ i, j = 1, 2, \ldots, N.$$

Let's hold that

$$\breve{q}_j = \sum_{i=1, i\neq j}^{N} q_{ij} > 0 \ \text{ and } \ 0 < \widehat{q}_j = \sum_{k=1, k\neq j}^{N} q_{jk} < 1, \quad \text{where } \ j = 1, 2, \ldots, N.$$

Zero step process distribution in the network nodes is as follows:

$$\vec{f}(0) = \left[f_1(0), f_2(0), \ldots, f_N(0)\right] = [f, 0, \ldots, 0].$$

In the first cycle it will become:

$$\vec{f}(1) = [f, fq_{12}, fq_{13}, \ldots, fq_{1N}] = \left[f_1(1), f_2(1), \ldots, f_N(1)\right].$$

In the second cycle process in each node already has to be calculated separately:

$$f_1(2) = \max\left\{f; f_2(1)q_{21}; \ldots; f_N(1)q_{N1}\right\},$$
$$f_N(2) = \max\left\{f_N(1); f_2(1)q_{2N}; \ldots; f_{N-1}(1)q_{N-1N}\right\}.$$

Such expression for process determination in the nodes can be used after each cycle:

$$f_i(n+1) = \max\left\{f_i(n); f_1(n)q_{1i}; \ldots; f_N(n)q_{Ni}\right\}, \quad \text{where } \ i = 1, 2, \ldots, N.$$

Now we shall prove that after N cycle's process in all the nodes stabilizes and does not fluctuate, if the flow intensity of all the nodes of the network satisfies the following inequalities:

$$0 < \breve{q}_i < 1 \ \text{ and } \ 0 < q_{ij} < 1, \quad \text{where } \ i, j = 1, 2, \ldots, N,$$

and one node has a point source of process, then

$$f_i(N) = f_i(N+1), \quad \text{where } \ i = 1, 2, \ldots, N,$$

here N – a number of network nodes.

At first, we can note that process during every cycle varies according to the formula:

$$f_i(n+1) = \max\left\{f_i(n); f_1(n)q_{1i}; \ldots; f_N(n)q_{Ni}\right\}, \quad \text{where } i = 1, 2, \ldots, N.$$

As the process is non-additive, i.e. $f_1 + f_2 = \max\{f_1; f_2\}$, then

$$f_i(n) \geq f_i(n-1), \quad \text{for each } i = 1, 2, \ldots, N.$$

It is also obvious that if during the n cycle process is brought back to the node from which it has been transferred in the previous cycle, this process $f_i(n)$ does not change and in this case

$$f_i(n) = f_i(n-1) = \max\left\{f_i(n-1); f_i(n-1)q_{ij}\right\}.$$

It is clear that process $f_i(n)$ will not increase, if it passes a closed way, i.e. a cycle with k nodes and comes back because in this case:

$$f_i(n+k) = \max\left\{f_i(n); f_i(n)q_{ij_1}; f_i(n)q_{ij_1}q_{j_1j_2}; \ldots; f_i(n)q_{ij_1}\ldots q_{j_{k-1}i}\right\} = f_i(n),$$

because $q_{ij} < 1$, for all $i, j = 1, 2, \ldots, N$.

Therefore, all the network lines links from the source that can increase process in the node have to be irreversible, i.e. not to produce cycles. However, in the graph that has N nodes the longest way that has no cycles is made up from $N-1$ links, and the process from the source reaches it within steps (starting with zero). Thus, in the $N+1$ step the processes in all the nodes will not increase and since they cannot be decreased, from $N+1$ cycle processes do not vary.

3 Distribution of additive process in networks and networks systems

In this section the distribution of process that can be divided or added in the network nodes will be analysed. Two process distribution methods in the network will be analysed separately. In the first case it will be assumed that process can be transferred from every node only to one of the possible nodes, while in the second case, let's allow the process spreading through the entire network.

The analysis will be started with process distribution in Markov chain. Let's suppose that we have a network with N nodes. Process from the node i can be transferred only to one node j, which is selected according to transfer probability P_{ij}. Thus, during each cycle, process can occur in only one network node. In the paper an assumption will be made that transfer probabilities have Markov feature. Thus, if the hazard that exists in node i after n cycles will be marked as $X(n)$, so

$$\begin{aligned} P_{ij} &= P\big(X(n) = j \mid X(n-1) = i\big) \\ &= P\big(X(n) = j \mid X(1) = i_1;\ X(2) = i_2; \ldots; X(n-1) = i_{n-1}\big). \end{aligned}$$

This way the process $X(n)$ will be Markov chain with finite set of the states $\{1; 2; \ldots; N\}$. The homogeneous Markov chain should also be discussed since P_{ij} is not dependent on n. Let's mark hazard occurrence probability in the i node after n cycles $\pi_i(n)$. It is clear that

$$\sum_{i=1}^{N} \pi_i(n) = 1.$$

Now it can be returned to the process calculation in each node after n cycles. Naturally, it is possible to determine only average process $\overline{f}_i(n)$ in each node since process after n steps is a random value. If we made an assumption that all the network line flows are equal to 1, the following would be obtained:

$$\overline{f}_i(n) = \frac{1}{n} \sum_{i=1}^{n} f\pi_i(k) = \frac{f}{n} \sum_{i=1}^{n} \pi_i(k),$$

here f – the process that has occurred in one of the network nodes during zero step, i.e., we hold that this node is a point source of the process.

From the theory of the Markov chains [5, 6] we know that state probabilities after n cycles are described using recursive formulas

$$\big[\pi_1(1), \pi_2(1), \ldots, \pi_N(1)\big] = \big[\pi_1(0), \pi_2(0), \ldots, \pi_N(0)\big] \lfloor P_{ij} \rfloor,$$

or in matrix form

$$\vec{\pi}(1) = \vec{\pi}(0)\, P,$$

here $P = [P_{ij}]$ – transfer probability matrix and $\vec{\pi}(0) = [1, 0, \ldots, 0]$, if we make an assumption that the point source of the process is located in the first node.

Then it follows:

$$\vec{\pi}(2) = \vec{\pi}(1)P = \big(\vec{\pi}(0)\, P\big)P = \vec{\pi}(0)\, P^2, \quad i, j = 1, 2, \ldots, N.$$

Given that $\vec{\pi}(0) = [1, 0, \ldots, 0]$, we receive:

$$\vec{\pi}(n) = \vec{\pi}(n-1)\, P = \vec{\pi}(0)\, P^n = \lfloor P_{11}^{(n)}, P_{12}^{(n)}, \ldots, P_{1N}^{(n)} \rfloor.$$

Thus, we can calculate the average hazard in the node i after n cycles $\overline{f}_i(n)$ recursively, using the following formula:

$$\overline{f}_i(n) = \frac{f}{n} \sum_{k=1}^{n} P_{1i}^{(k)}, \tag{1}$$

here $P_{1i}^{(k)}$ is the i element of the first line of matrix.

According to the expression of $\pi_1(n)$ and [5] it is not difficult to prove the theorem of the marginal distribution of the process average, when n converge to infinity.

If Markov chain with N states and transfer probability matrix $P = \lfloor P_{ij} \rfloor$ is ergodic, i.e., $\lim_{n\to\infty} \pi_i(n) = \pi_i$, $i = 1, 2, \ldots, N$, so marginal process average values in all the nodes of the network exist as well.

When, in the equation (1) we reach the limit when n converge to infinity we get:

$$\lim_{k\to\infty} \overline{f}_i(n) = \lim_{k\to\infty} \frac{f}{n} \sum_{k=1}^{n} \pi_i(k).$$

As $\lim_{k\to\infty} \pi_i(k) = \pi_i$, so there is a vanishing function $\varepsilon(k)$ which is $\pi_i(k) = \pi_i + \varepsilon_k(i)$, where $\lim_{k\to\infty} \varepsilon_k(i) = 0,\ k = 1, 2, \ldots$. Then

$$\lim_{k\to\infty} \overline{f}_i(n) = \lim_{k\to\infty} \frac{f}{n} \sum_{k=1}^{n} \bigl(\pi_i + \varepsilon_k(i)\bigr) = \lim_{n\to\infty} \Bigl(\frac{f}{n}\pi_i\Bigr) n + \lim_{n\to\infty} \frac{f}{n} \sum_{k=1}^{n} \varepsilon_k(i).$$

Let's select $\varepsilon^{(n)}(i) = \max_{1\le i\le n}\{\varepsilon_1(i); \varepsilon_2(i); \ldots; \varepsilon_n(i)\}$. Then:

$$0 \le \lim_{n\to\infty} \frac{f}{n} \sum_{k=1}^{n} \bigl|\varepsilon_k(i)\bigr| \le \lim_{n\to\infty} \frac{f}{n} \cdot n\varepsilon^{(n)}(i) = 0.$$

Let $\lim_{n\to\infty} \frac{f}{n} \sum_{k=1}^{n} \varepsilon_k(i) = 0$, and, therefore, $\overline{f}_i = f\pi_i$.

Thus, the marginal average process exists in every node, besides, it is equal to the product of the initial process f and the marginal node probability.

4 The distribution of the additive process in the network nodes the transitional period

We analyse a network system in which process from each node can be transferred to other nodes during one cycle, by dividing process $f_i(n)$ of the node in proportion to the flows q_{ij}, when $i = 1, 2, \ldots, N$ and $\sum_{j=1}^{N} q_{ij} \le 1$.

First of all, let's assume that one network node, for example, the first one, is a point source of the additive process, in which process $f_1(0)$ occurs. Thus, at the zero step we have the following process distribution in the nodes:

$$\vec{f}(0) = \bigl[f_1(0), 0, \ldots, 0\bigr].$$

During the following cycles, process modification will occur in each node. From that node process will be transferred to other nodes by flows q_{ij}. The total transfer will be:

$$f_i(n)(q_{i1} + q_{i2} + \ldots + q_{ii-1} + q_{ii+1} + \ldots + q_{iN}) = f_i(n)\widehat{q}_i.$$

In the node i it will remain

$$f_i(n) - f_i\widehat{q}_i = f_i(n)q_{ii}$$

part of process. The process $\sum_{j=1}^{N} f_j q_{ij}$ will be respectively transferred from other nodes to the node i. Thus, after n cycles, we will have the following process in the node i:

$$f_i(n+1) = f_1(n)q_{i1} + f_2(n)q_{i2} + \ldots + f_N(n)q_{iN}, \quad \text{where } i = 1, 2, \ldots, N. \tag{2}$$

After defining network flow $Q = \lfloor q_{ij} \rfloor$, we can write the system of equations in the form of matrix:

$$\vec{f}(n+1) = \vec{f}(n)\, Q.$$

Thus, we have received process distribution in the iterative process. As the process is stationary, i.e. matrix Q is not dependent on the number of cycle's n, so irrespective of the initial process distribution, this process converges only when all matrixes Q own values will be less than one. This is as well the obligatory and sufficient condition for the marginal distribution of the additive process in the network systems [7].

The iterative process of process distribution $\vec{f}(n)$ converges when the number of cycles is $n \to \infty$ and it is not dependent on the initial process distribution, if all the flows $0 < q_{ij} < 1$. It is important to analyse risk distribution after certain number of iterations.

From the expression (2) we get:

$$\vec{f}(n+1) = \vec{f}(n)\, Q.$$

What follows is

$$\vec{f}(n+1) = \vec{f}\,(n-1)\, Q\, Q = \vec{f}(n-1)\, Q^2 = \ldots = \vec{f}(0)\, Q^{n+1}.$$

Therefore

$$\vec{f}(n+1) = \vec{f}(0)\, Q^{(n+1)}.$$

This equality allows employing the ideas that are used when proving the ergodic theorems of Markov's chain states [8].

We shall mark the elements of matrix Q^n this way: $q_{ij}^{(n)}$, and $q_{ij} = q_{ij}^{(1)},\ i, j = 1, 2, \ldots, N$. When we come to the limit of $n \to \infty$, we mark $\lim_{n\to\infty} \vec{f}(n+1) = \overline{f}$, and get:

$$\lim_{n\to\infty} \vec{f}(n+1) = \lim_{n\to\infty} \vec{f}(0)\, Q^{n+1} = f(0)\, \overline{Q}.$$

Or

$$\vec{f} = \vec{f}(0)\, \overline{Q} = [f, 0, \ldots, 0] \begin{bmatrix} q_1 & q_2 & \cdots & q_N \\ q_1 & q_2 & \cdots & q_N \\ \cdots & \cdots & \cdots & \cdots \\ q_1 & q_2 & \cdots & q_N \end{bmatrix}.$$

Thus

$$\vec{f} = [f_1, fq_2, \ldots, fq_N].$$

This distribution does not depend on initial conditions.

5 Marginal distribution of the process in the network nodes with immunity

In this section we will analyse only the marginal processes in different network systems with the infinitive sources and node immunities. For example, in the computer networks, antivirus systems that destroy the majority of viruses are implemented, human immune system destroys disease viruses, customs does not allow free movement of prohibited goods, etc. The immunity of the network node i is considered the number I_i $(0 \leq I_i \leq 1)$, by which the process H that occurs in this node is multiplied. Thus, when $I_i = 1$, process is fully transmitted, and when $I_i = 0$, all of the process, that makes way to the node i, is destroyed. Let's assume that the first node of the network is the infinitive source of process that increases process by the value H during each new cycle. First we shall make an assumption that a marginal process distribution exists under such conditions, i.e. $\lim_{n\to\infty} H_i(n) = H_i$, where $i = 1, 2, \ldots, N$.

Then, if at the end of each cycle the process that is located in the node is multiplied by the immunity $0 < I_i \leq 1$, and the node in which the source of process is located, does not have immunity (let's say it is the first one), in this case process in the first node will not change, if it is reduced by the value H after each cycle, i.e.

$$H_2 q_{21} + H_3 q_{31} + \ldots + H_N q_{N1} = H_1(q_{12} + q_{13} + \ldots + q_{1N}) - H.$$

In the second node, the incoming process multiplied by the immunity I_2 has to be equal to the outgoing process. Thus,

$$I_2(H_2 q_{12} + H_3 q_{32} + \ldots + H_N q_{N2}) = H_2(q_{21} + q_{23} + \ldots + q_{2N}).$$

The case is analogous with the other nodes:

$$I_j \sum_{i=1, i\neq j}^{N} H_i q_{ij} = H_j \sum_{i=1, i\neq j}^{N} q_{ji}, \quad j = 2, 3, \ldots, N.$$

Joining those equations we get a system of linear equations which can be written in the form of matrix:

$$\widetilde{Q}\,\overline{H} = \overline{B}. \tag{3}$$

The main matrix of this system is:

$$\widetilde{Q} = \begin{bmatrix} -\widehat{q}_1 & q_{21} & q_{31} & \cdots & q_{N1} \\ q_{12} & -\widehat{q}_2/I_2 & q_{32} & \cdots & q_{N2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{1N} & q_{2N} & q_{3N} & \cdots & -\widehat{q}_N/I_N \end{bmatrix},$$

constants vector $\overline{B} = [H, 0, \ldots, 0]^T$, and unknowns vector $\overline{H} = [H_1, H_2, \ldots, H_N]^T$.

To investigate the existence of the marginal distribution, hazard distribution $\vec{H}(n)$ will be expressed as iterative equation. As there is the immunity, we shall introduce a diagonal matrix

$$I = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & I_N \end{bmatrix}.$$

Now we can express the system of equations this way:

$$\begin{aligned} &\big[H_1(n+1), H_2(n+1), \dots, H_N(n+1)\big] \\ &\quad = \big[H_1(n), H_2(n), \dots, H_N(n)\big] \times I \times Q + [H, 0, \dots, 0], \end{aligned}$$

here $Q = [q_{ij}],\ i, j = 1, 2, \dots, N.$

We mark $\vec{H}(n) = \big[H_1(n), H_2(n), \dots, H_N(n)\big]^T$, $\vec{H}(0) = [H, 0, \dots, 0]^T$ and obtain that

$$\vec{H}(n+1) = \vec{H}(n)\, IQ + \vec{H}(0). \tag{4}$$

In the paper the convergence conditions of the pro cess (4) were not fully set; however, for this process, general condition that is necessary and sufficient for the convergence of iterative processes applies: for the process (4) to converge, it is necessary and sufficient that all the absolute values of the IQ matrix are not less than 1 [9].

6 Conclusions

Distributions of processes of different types in network systems are analyzed in the paper. In case of non-additive processes an algorithm is created for the evaluation of process after each step. It is revealed that after infinite number of steps (coinciding with the number of network nodes) the process settles.

Process stability conditions are determined in case of additive processes, when the process, while spreading, during each step transfers only to one node and when the process transfers to more than one node.

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