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# Discrete Multistage Optimization and Hierarchical Market

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**Abstract.** New simple form of mixed solutions is described by bilinear continuous optimization processes. It enables investigate an analytic solutions and the connection between discrete and continuous optimization processes. Connection between discrete and continuous processes is stochastic. Discrete optimization processes are used for the control works in levels and groups of the hierarchical market. Equilibrium between local and global levels of works is investigated in hierarchical market.

**Keywords:** connection between discrete and continuous optimization processes, equilibrium control between local and global levels of the works in hierarchical market.

## 1 Introduction

Our first region of the works includes mathematical models and their mixed solutions. Models are described by discrite multistage optimization problems and mixed solutions are described in the continuous optimal control processes forms. This region arose in the investigation the gold mining problems stated by R. Bellman and other authors [1]. Simple forms of mixed solutions are described in the bilinear dfferential trajectories and bilinear integral criterions. Different forms of mixed solutions are used in the investigation pure solutions. These forms of mixed solutions are used in the approximation different optimization processes and enable investigate the wider class of solutions than ordinary. We investigate a simple bilinear form of mixed solutions. Analogous continuous form of mixed solutions is investigated by the author [2] for the capital investment and the gold mining problems and generalized optimization processes problems. Discrete bilinear form of mixed solutions is described by the author [3] for the mentioned problems. These forms of mixed solutions are investigated for the inventory control problems by the authors [4, 5]. Analogous continuous approximations are investigated by P. Rutkauskas and the author [6, 7]. Continuous approximation models are surveyed by K. A. Smith, I. V. Potvin and T. Kwok [8] for neutral network methods when their length $\Delta \to 0$. Analogous continuous approximation methods are compared with discrete by S. E. Lyshevski [9] and by I.N. Sigal and A. P. Ivanova [10].

Second region of our works includes the applications of multistage optimization processes in hierarchical market. We use hierarchical market in the estimations resources, production and other valuable things. These values are met in payments, prices and other estimations. Hierarchical market forms the levels and groups for workers. Every group of corresponding kind and level includes corresponding workers which belong the next higher level and lower levels of different kinds. Representatives from the adjacent lower level are chosen for the declaration of estimations for the service in higher level. Estimatios of the service are not higher than declarations.

Groups of lower levels are preparing workers for higher levels. Connection between levels and groups controls works in groups and coordinates local and global market. We describe mathematical model for control the works in levels and groups and use dynamic programming equations in the investigation control in hierarchical market. Several levels are used in the agriculture market and in the invertory control by the authors [11, 12].

## 2 Bilinear mixed solutions

We investigate mixed solutions described in the bilinear continuous optimization processes form. This form of mixed solutions enables the analytic investigation of problems. Our discrete optimization problem is described by dynamic programming equations as follows

$$f_{N-k}\big(x(t_k)\big) = \max_{u_k} \Big[g_0\big(t_k, x(t_k), u_k\big) + f_{N-k-1}\big(x(t_{k+1})\big)\Big] \\ \big(k \in N^0 = \{0, \dots, N-1\}\big), \quad f_0(\dots) \equiv 0; \tag{1}$$

$$x(t_{k+1}) = g\big(t_k, x(t_k), u_k\big) \in R^n \quad (k \in N^0), \quad x(t_0) = \bar{x}_0; \tag{2}$$

$$u_k \in U = \big\{a(1), \ldots, a(m)\big\} \in R^r, \tag{3}$$

$g_0(t_k, \ldots)$, $g(t_k, \ldots) = (g_1(t_k, \ldots), \ldots, g_n(t_k, \ldots))$ ($k \in N^0$) are given general functions. Only their discrete values $g_j(t_k, x(t_k), a(i))$ $(i = 0, \ldots, n,\ j = 0, \ldots, n)$ are used in the investigations. $U$ is discrete set for control $u_k,\ x(t_k)$ ($k \in N^0$) are phase variables, $\bar{x}_0$ is given vector.

Simple form of bilinear mixed solutions is described as follows

**Theorem 1.** *Mixed solutions of the problem* (1)–(3) *are described in continuous optimization processes form with bilinear differential trajectories and bilinear integral criterion as follows*

$$y_0(t_N) \Rightarrow \max_{\nu_i(\cdot)},$$

$$\dot{y}_j(t) = \frac{y_j(t)}{\Delta(k)} \sum_{i=1}^{m} \nu_i(t) \ln \big|\psi_j\big(t_k, y(t_k), i\big)\big|$$

$$\big(j \in n^* = \{0, \ldots, n\}, \quad t \in T_k = (t_k, t_{k+1}], \quad \Delta(k) = t_{k+1} - t_k, \quad k \in N^0\big),$$

$$y_j(t_k + 0) = y_j(t_k)\ \mathrm{sign} \sum_{i=1}^{m} \psi_j\big(t_k, y(t_k), i\big)\nu_i(t_k + 0) \quad (\mathrm{sign}\, 0 = 1),$$

$$y_j(t_0) = x_j(t_0)\ (j \in n^*), \quad \big(x_1(t_0), \ldots, x_n(t_0)\big) = \bar{x}_0,$$

$y_0(t_0)$ *is chosen positive number*;

$$\nu(t) = \big(\nu_1(t), \ldots, \nu_m(t)\big) \in \vartheta,$$

$$\vartheta = \big\{(\nu_1, \ldots, \nu_m) \geq 0\colon \nu_i + \nu_j = 1,\ \nu_s = 0;\ i \neq j \neq s;\ i, j, s = 1, \ldots, m\big\},$$

$$t \in T_k, \quad k \in N^0,$$

*where* $\nu(t)$ $(t \in T_k)$ *is piecewise continuous function on left;*

$$\psi_j(t_k, \ldots, i) = \bar{\psi}_j(t_k, y(t_k), i) \quad (j = 1, \ldots, n),$$

$$\bar{\psi}_j(t_k, \ldots, i) = \frac{g_j\big(t_k, y(t_k), a(i)\big)}{y_j(t_k)} \quad (j = 0, \ldots, n),$$

$$y(t_k) = \big(y_1(t_k), \ldots, y_n(t_k)\big), \quad y_j(t_k) \neq 0,$$

$$\psi_0(t_k, \ldots, i) = \bar{\psi}_j(t_k, \ldots, i) + 1.$$

**Remark 1.** *We illiustrate the* $\nu$ *for* $m = 2, 3$*. If* $m = 2$ *then*

$$\nu = \big\{(\nu_1, \nu_2) \geqslant 0\colon\ \nu_1 + \nu_2 = 1\big\}.$$

*If* $m = 3$ *then*

$$\begin{aligned}\nu = \big\{(\nu_1, \nu_2, \nu_3) \geqslant 0\colon\ &\nu_1 + \nu_2 = 1, \quad \nu_3 = 0, \quad \nu_2 + \nu_3 = 1,\\ &\nu_1 = 0, \quad \nu_1 + \nu_3 = 1, \quad \nu_2 = 0\big\}.\end{aligned}$$

*Proof.* Proof of theorem is concluded from limit transition to bilinear discrete mixed solutions described by the author [3] for our problem. Using these results we describe discrete bilinear mixed solutions of problem (1)–(3) as follows

$$\begin{aligned}
&z_0(t_N) \to \max_{u_s(k_e)},\\
&z_j(k_{l+1}) = z_j(k_l) \sum_{s=1}^{m} u_s(k_l) \big|\psi_j\big(t_k, z(t_k), s\big)\big|^{\delta(k_l)}\\
&\quad \big(j \in n^* = \{0, \ldots, n\},\ l \in n^0(k) = \big\{0, \ldots, n(k) - 1\big\}\big),\\
&k_0 = t_k,\ k_{n(k)} = t_{k+1},\ z(t_k) = \big(z_1(t_k), \ldots, z_n(\ldots)\big),\\
&z_j(t_{k+0)} = z_j(t_{k-0}) \operatorname{sign} \sum_s u_s(t_k + 0)\psi_j(t_k, \ldots, s)\\
&\quad \big(k \in N^0,\ z_j(t_0) = y_j(t_0)\big);\\
&\delta(k_l) = \frac{k_{l+1} - k_l}{\Delta(k)} > 0 \quad \big(\Delta(k) = t_{k+1} - t_k\big);\\
&u(k_l) = \big(u_n(k_l), \ldots, u_m(k_l)\big) \in \nu\ (l \in n^0(k),\ k \in N^0).
\end{aligned}$$

Using Taylor formula we describe our optimization process as follows

$$\begin{aligned}
&z_0(t_N) \to \max_{u_s(k_l)},\\
&\frac{z_j(k_{l+1}) - z_j(k_l)}{k_{l+1} - k_l} = \frac{z_j(k_l)}{\Delta(k)} \sum_{s=1}^{m} u_s(k_l) \cdot \ln |\psi_j(t_k, \ldots, s)| + \mathrm{O}\big(\delta(k_l)\big)\\
&\quad \big(j \in n^*,\ l \in n^0(k),\ k \in N^0\big);\\
&u(k_l) \in v\big(l \in n^0(k),\ k \in N^0\big).
\end{aligned}$$

Limit transition when $\delta(k_l) \to 0$ to last formulas gives the statement of theorem. □

**Remark 2.** *Theorem 1 gives new simple form of bilinear continuous mixed solutions investigated by author* [7]. *Proof of the theorem is simpler than earlier by the author* [7].

## 3 Hierarchical market

We investigate the connection between global and local markets in the estimation of works in the hierarchical market. Estimations of the works in the first levels and the workers in these levels compose the local market and estimations in higher levels and corresponding levels of workers compose the global market in the hierarchical market. We define the hierarchical market as folows

$n(j)$ $(j \in J' = \{1, \ldots, J\},\ n \in N' = \{1, \ldots, N\})$ is $n$ group of $j$ level works and corresponding workers $i(n(j))$ $(i \in I' = \{1, \ldots, I\}),\ i(\ldots)$ is worker including in $n(j)$ group $(i(n(j)) \in n(j))$;

$n(j, j+1)$ is subgroup of the group $n(j)$ including in the group $n(j+1)$ $(n(j, j+1) \subset n(j),\ n(j, j+1) \subset n(j+1)),\ j \neq J$;

$N(j+1)$ $(j \in J^0 = \{1, \ldots, J-1\})$ is subgroup of the group $n(j, j+1)$ which workers $i(N(j+1))$ declarate the corresponding estimations $a(i(N(j+1)))$ for their works of the unit production in the group $n(j+1)$ $(N(j+1) \subset n(j, j+1) \subset n(j+1)$. Workers in the group $N(j)(j > 1)$ are chosen by workers of the group $n(j),\ j = 2, \ldots$.

$b(n(j))$ $(j > 1)$ is estimation of the unit works production in the group $n(j)$ for the level $j$ after declarations. It is not higher than declarations. Using this principle estimations $b(n(j)$ may be defined as follows

$$b\big(n(j)\big) = \min_{i(\ldots)} a\Big(i\big(N(j)\big)\Big).$$

Control of the works for the workers in groups and the levels is described by dynamic programming equations.

**Definitions.** $p_g(i(n(j)))$ are probabilities for the corresponding demands $u_g(i(\ldots))$ of production in the groups $n(j)$ for the woker $i(n(j))$ $(n \in N',\ j \in J'$, $i \in I',\ g \in G' = \{1, \ldots, G\})$.

$q(i(n(j)), m(r))$ are probabilities of the corresponding expenses $c(i(n(j)),\ m(r))$ for the preparation the worker $i(n(j))$ to work in the second period in group $m(r)$ $(m, n \in N',\ j,\ r \in J',\ i \in I';\ n(j) \neq m(r))$;

$f_M(i(n(j))$ is maximum expected value of the payment for the worker $i(n(j))$ in the $M$ period of hierarchical market control process. From definition we have following dynamic programming equations

$$f_M\big(i\big(n(j)\big)\big) = \max_{u_g(\ldots),l(\ldots)} \Big[\sum_m \sum_s p_g\big(i\big(m(s)\big)\big)u_g\big(i(\ldots)\big)b\big(m(s)\big) - \sum_l \sum_r q\big(i\big(n(j)\big), l(r)\big)c\big(i(\ldots), l(r)\big) + f_{M-1}\big(i\big(n(j)\big)\big)\Big] \quad (M \geqslant 1),$$

$$f_0(\ldots) = 0, \quad (l, m, n \in N';\ s, j, r \in J').$$

Optimal control for the estimations of the works in the groups and the levels convergences to the equilibrium by Nikaido [13]. This result is stated for the global market as follows

**Theorem 2.** *If the estimation of the works in the levels corresponds the estimation of the works production after declaration then they are equilibrium in the global market for the higher levels.*

*Proof.* Let $d(i(n(j)))$ is an estimation of the production for $i(n(j))$ worker. From definition and assumption of the theorem we have

$$d\big(i\big(n(j)\big)\big) = a\big(s\big(N(j)\big)\big),$$

where $s(\ldots)$ is chosen a representative by worker $i(\ldots)$. After declaration estimations we conclude following inequalities

$$d\big(i\big(n(j)\big)\big) = d^*\big(i(\ldots)\big) = b\big(n(j)\big) \leqslant a\big(s(N(j))\big) \quad \text{for all } i(\ldots).$$

These inequalities prove statement of theorem. □

Equilibrium for the local market is investigated analogously.

**Remark 3.** *Division into global and local markets and equilibrium in hierarchical market is investigated for the first time.*

## 4 Applications

We illiustrate mentioned market principle for problem of gathering the milk and problem for evaluations in research works.

## 4.1 Problem for gathering the milk

We have $n$ points of milk production. Let $m$ $(m < n)$ points of them may to collect the milk for the deliver to the center post and choose their representatives $i$ $(i = 1, \ldots, m)$ for declarations the corresponding prices $c(i)$. Along the market the price of the service $c$ may be calculated as follows

$$c = \frac{c\big(i(1)\big) + c\big(i(2)\big)}{2},$$

where $c(i(1)) = \min_i c(i),\ c(i(2)) = \min_{i \neq i(1)} c(i)$.

## 4.2 Problem for the evaluations in the research works

Let the scientific workers are formed $n$ directions in research works. Every direction of the research includes the levels. Let the first level in the research works includes the scientific workers from pedagogical region or applications. They participate in the research of region connected with the local resources. Second level of groups includes applied research works for the first level research problems and corresponding scientific workers. Third level of the group includes the fundamental research works for the lower levels problems and the corresponding scientific workers. Connection between the levels and the groups controls by subgroups of scientific workers which are chosen along the mentioned hierarchical market.

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