Stability of Nuclear Reactor: Point Model Analysis

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Abstract. A point model of a nuclear reactor with delay in feedback line "ower – reactivity" estimating the influence of six groups of delayed neutrons is presented and investigated by means of linear and nonlinear analysis methods. The results of numerical experiments and the comparison of them to the asymptotic solution of differential equations are presented as well.

Keywords: nuclear reactor, differential equation, delay, periodic solution.

1 Introduction

Let's take the point model of a nuclear reactor that was introduced in [1]. The dynamics equations are given by

$$\dot{N}(t) = r_N \cdot \left[1 + a \cdot \left(1 - \frac{C(t)}{C_0}\right) - \frac{N(t-h_N)}{N_0}\right] \cdot N(t), \tag{1}$$

$$\dot{C}(t) = r_C \cdot \left[\frac{N(t)}{N_0} - \frac{1}{C_0}\sum_{j=1}^{6} \alpha_j\, C(t-h_j)\right] \cdot C(t), \tag{2}$$

where $N(t)$ is the density of neutrons at the time moment t; N_0 is its steady value; r_N is the linear growth coefficient of the density of neutrons; $C(t)$ is the integral density of all delayed neutrons at the time t; C_0 is its steady state value; r_C is the coefficient of linear growth of the density of delayed neutrons; $h_N > 0$ is the delay in the feedback line "power – reactivity"; $j = \overline{1,6}$ is a number of delayed neutrons group; $h_j > 0$ generation time of delayed neutrons of group j; $\alpha_j = \frac{\beta_j}{\beta}$ ($\sum_{j=1}^{6} \alpha_j = 1$) is the relative yield of delayed neutrons; β_j is a part of delayed neutrons belonging to group j; β is a number of all delayed neutrons ($\beta = \sum_{j=1}^{6} \beta_j$); a ($-1 < a \leq 0$) is the feedback parameter regulating the power of the reactor.

As the delayed neutrons make up from $0.7\,\%$ to $1.5\,\%$ of the whole number of neutrons, so a will be considered as a small parameter. Let $a = 0$. Then the system (1), (2) is transformed into

$$\dot{N}(t) = r_N \cdot \left[1 - \frac{N(t-h_N)}{N_0}\right] \cdot N(t), \tag{3}$$

$$\dot{C}(t) = r_C \cdot \left[\frac{N(t)}{N_0} - \frac{1}{C_0}\sum_{j=1}^{6} \alpha_j C(t-h_j)\right] \cdot C(t). \tag{4}$$

2 Linear analysis

The system (3), (4) has equilibrium states

$$N(t) \equiv 0, \quad C(t) \equiv 0, \tag{5}$$

$$N(t) \equiv N_0, \quad C(t) = 0, \tag{6}$$

$$N(t) \equiv N_0, \quad C(t) = C_0. \tag{7}$$

As shown in [2] the equilibrium states (5), (6) are unstable. So, the further analysis of the system (3), (4) is needed in the neighbourhood of non-zero equilibrium state (7). After the substitution of

$$N(t) = N_0\big[1 + x(t)\big], \tag{8}$$

$$C(t) = C_0\big[1 + y(t)\big] \tag{9}$$

into equations (3), (4), we get the equations

$$\dot{x}(t) + r_N \cdot x(t-h_N)\big[1 + x(t)\big] = 0, \tag{10}$$

$$\dot{y}(t) - r_C \cdot \left[x(t) - \sum_{j=1}^{6} \alpha_j y(t-h_j)\right] \cdot \big[1 + y(t)\big] = 0. \tag{11}$$

The linear parts of (10), (11) are given by

$$\dot{x}(t) = -r_N \cdot x(t - h_N), \tag{12}$$

$$\dot{y}(t) = r_C \cdot \left[x(t) - \sum_{j=1}^{6} \alpha_j y(t - h_j)\right]. \tag{13}$$

The characteristic equation of the system (12), (13) is defined as

$$\left[\lambda + r_N \exp(-\lambda h_N)\right] \cdot \left[\lambda + r_C \sum_{j=1}^{6} \alpha_j \exp(-\lambda h_j)\right] = 0. \tag{14}$$

The analysis of the roots of (14) splits into the investigation of two quasi-polynomial roots. The disposition of the roots of the quasi-polynomial $P(\lambda) = \lambda + r_N \exp(-\lambda h_N)$ on the complex plane is well-known [2], but in order to determine the disposition of the roots of the quasi-polynomial

$$P(\lambda) = \lambda + r_C \sum_{j=1}^{6} \alpha_j \exp(-\lambda h_j) \tag{15}$$

on the complex plane the further analysis is performed below.

The roots of the quasi-polynomial

$$P(\lambda) = \lambda + p + r_C \sum_{j=1}^{6} \alpha_j \exp(-\lambda h_j) \tag{16}$$

are analysed using D-decomposition method [3]. If $\lambda = 0$, then

$$p + r_C = 0. \tag{17}$$

The line (17) becomes one of the D-decomposition curves on the plane pr_C. Let $\lambda = i\sigma$. Other curves are determined by the following parametric equations:

$$r_C = \frac{\sigma}{\sum_{j=1}^{6} \alpha_j \sin(\sigma h_j)}, \tag{18}$$

$$p = \frac{\sigma \sum_{j=1}^{6} \alpha_j \cos(\sigma h_j)}{\sum_{j=1}^{6} \alpha_j \sin(\sigma h_j)} = -r_C \sum_{j=1}^{6} \alpha_j \cos(\sigma h_j). \tag{19}$$

In the case of $\sigma \to 0$, the coordinates of recurrence point of the curves (17), (18) and (19) are calculated as follows:

$$(p_0; r_0) = \left(-\frac{1}{\sum_{j=1}^{6} \alpha_j h_j}; \frac{1}{\sum_{j=1}^{6} \alpha_j h_j} \right) \approx (-0.11; 0.11). \tag{20}$$

The results of D-decomposition of the parameters p and r_C are presented as Fig. 1. The values of h_j and α_j are taken from Table 1 [4]. Two roots of (16) with positive real parts appear in the D_2 domain.

Table 1. Numerical characteristics of delayed neutrons

Fuel	j	$T_{1/2} = h_j(s)$	α_j
	1	54.28	0.035
	2	23.04	0.298
^{239}PU	3	5.60	0.211
(Plutonium)	4	2.13	0.326
	5	0.618	0.086
	6	0.257	0.044

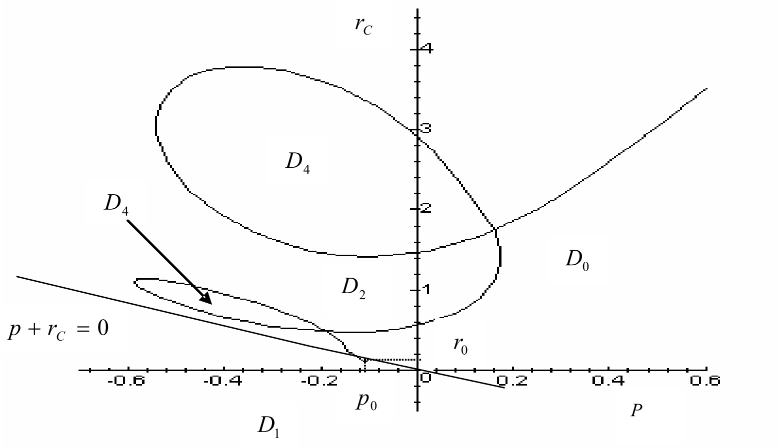

Fig. 1. The D-decomposition of quasi-polynomial (15).

3 Nonlinear analysis

3.1 The reason of oscillation – the feedback line

Let's take differential equations (10), (11). If the parameter of bifurcation $\varepsilon = r_N h_N - \frac{\pi}{2}$ has a sufficiently small and positive value, then as shown in [3] equation (10) has a stable periodic solution.

$$x(t) = \xi x_1(t) + \xi^2 x(t) + O(\xi^3), \tag{21}$$

where

$$x_1(t) = \cos\sigma t, \quad x_2(t) = \frac{1}{10}(\sin 2\sigma t + 2\cos 2\sigma t),$$
$$\sigma\Big(1 + \frac{c_2}{b_2}\varepsilon + O(\varepsilon^2)\Big) = \frac{\pi}{2h_N}, \quad \xi = \sqrt{\frac{\varepsilon}{b_2}},$$
$$c_2 = \frac{1}{10\pi}, \quad b_2 = \frac{3\pi - 2}{40}.$$

Theorem 1. *If* $r_C \cdot \sum_{j=1}^{6} \alpha_j h_j \leq 1$*, then the roots of quasi-polynomial* (15) *satisfy the inequality* $\operatorname{Re}\lambda < 0$ *and the differential equation* (11) *has a stable periodic solution* [3].

The periodic solution of equation (11) can be calculated using the formula

$$y(t) = \xi y_1(t) + \xi^2 y_2(t) + O(\xi^3), \tag{22}$$

where functions $y_j(t)$ are found from the linear differential equations.

$$\dot{y}_1(t) + r_C \sum_{j=1}^{6} \alpha_j y_1(t - h_j) = r_C x_1(t), \tag{23}$$

$$\dot{y}_2(t) + r_C \sum_{j=1}^{6} \alpha_j y_2(t - h_j) = r_C x_2(t) + y_1(t)\dot{y}_1(t). \tag{24}$$

Then from (23)

$$y_1(t) = \frac{r_C}{\big|P(i\sigma)\big|^2}\big[\mathrm{Im}P(i\sigma)\sin(\sigma t) + \mathrm{Re}P(i\sigma)\cos(\sigma t)\big], \tag{25}$$

and from (24)

$$y_2(t) = A\sin(2\sigma t) + B\cos(2\sigma t), \tag{26}$$

where

$$A = \frac{1}{\left|P(2i\sigma)\right|^2}\left[W_1 \mathrm{Re} P(2i\sigma) + W_2 \mathrm{Im} P(2i\sigma)\right], \tag{27}$$

$$B = \frac{1}{\left|P(2i\sigma)\right|^2}\left[W_2 \mathrm{Re} P(2i\sigma) - W_1 \mathrm{Im} P(2i\sigma)\right], \tag{28}$$

$$W_1 = \frac{r_C}{10} + \frac{\sigma r_C^2}{2\left|P(2i\sigma)\right|^2}\left[\mathrm{Im}^2 P(i\sigma) - \mathrm{Re}^2 P(i\sigma)\right], \tag{29}$$

$$W_2 = \frac{r_C}{5} + \frac{\sigma r_C^2}{2\left|P(2i\sigma)\right|^2}\left[\mathrm{Re} P(i\sigma) \cdot \mathrm{Im} P(i\sigma)\right], \tag{30}$$

and $P(\lambda)$ is the quasi-polynomial (15).

Therefore, differential equations (3) and (4) have the following stable periodic solutions

$$N(t) = N_0\Big[1 + \xi\cos\frac{\pi}{2h_N}\tau + \xi^2 x_2(\tau) + O(\xi^3)\Big], \tag{31}$$

$$C(t) = C_0\big[1 + \xi y_1(\tau) + \xi^2 y_2(\tau) + O(\xi^3)\big], \tag{32}$$

where functions $x_2(\tau)$, $y_1(\tau)$, $y_2(\tau)$ and variable ξ, b_2, c_2 are defined by formulas (21), (25)–(30), with $\sigma = \frac{\pi}{2h_N}$, $\tau = \frac{t}{1+c_2\xi^2}$.

When

$$h_N = 10^{-3}\,\mathrm{s}, \quad r_N h_N = 1.8, \quad r_N = 1800, \quad r_C = 0.1,$$
$$N_0 = 100\,\mathrm{kW}, \qquad C_0 = 1\,\mathrm{kW} \quad \varepsilon = 0.229,$$

then

$$b_2 \approx 0.1856194, \;\; c_2 \approx 0.0318309, \;\; \xi \approx 1.1107237,$$
$$\sigma \approx 1511.4419, \quad \tau \approx 0.9622138t \quad .$$

The stable periodic solution (31) of differential equation (3)

$$N(t) \approx 100 \cdot \big[1 + 1.1107237\cos 1511,442t$$
$$+ 0.1233707(\sin 3022,884t + 2\cos 3022.884t)\big]$$

is presented in Fig. 2.

The stable periodic solution (32) of differential equation (4)

$$C(t) \approx 1 + 4.86242\cdot 10^{-8}(1511.442 \sin 1511.442t - 0.025197 \cos 1511.442t) + 1.2337072(6.5151\cdot 10^{-6} \sin 3022.884t - 2.5033\cdot 10^{-3} \cos 3022.884t)$$

is presented in Fig. 3.

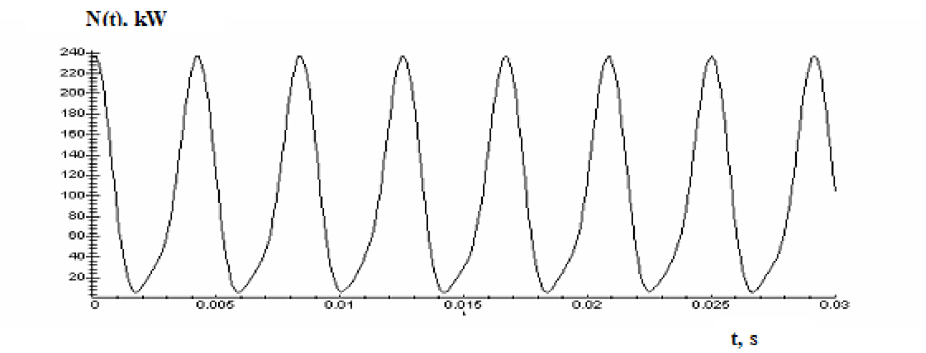

Fig. 2. The stable periodic solution of equation (3), when $r_N = 1800$, $N_0 = 100\,\text{kW}$.

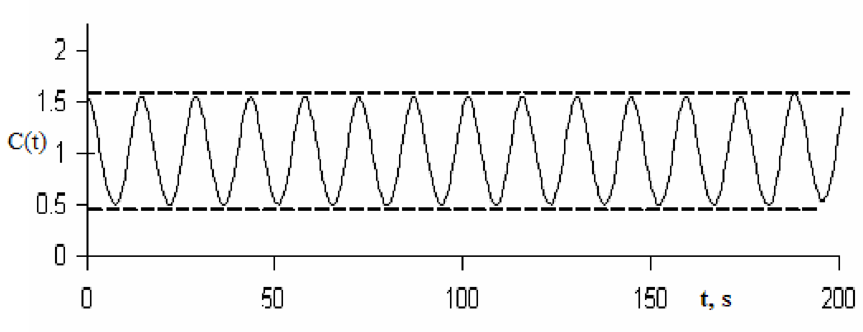

Fig. 3. The stable periodic solution of equation (4), when $r_C = 0.1$, $C_0 = 1\,\text{kW}$.

3.2 Oscillations of two frequencies

Let $r_N = \frac{\pi}{2h_N} + \varepsilon$, $r_C = r_C^* + \mu$, parameters $\varepsilon,\ \mu$ are assumed to be small, and $r_C^* = \frac{\sigma_*}{\sum_{j=1}^6 \alpha_j \sin \sigma_* h_j}$, where σ_* is the unique root of the equation $\sum_{j=1}^6 \alpha_j \sin \sigma_j = 0$ belonging to the interval $(0, \frac{\pi}{2h_N})$.

The substitution of (8), (9) into equations (3), (4) will produce

$$\dot{x}(t) + \left(\frac{\pi}{2h_N} + \varepsilon\right) \cdot x(t - h_N) \cdot \big[1 + x(t)\big] = 0, \tag{33}$$

$$\dot{y}(t) + (r_C + \mu) \cdot \left[\sum_{j=1}^{6} \alpha_j y(t - h_j) - x(t)\right] \cdot \big[1 + y(t)\big] = 0. \tag{34}$$

When $0 < r_N - \frac{\pi}{2h_N} = \varepsilon \leq 1$, the differential equation (33) has a stable periodic solution [3]

$$x(t) = \xi \cos \frac{\pi}{2h_N} \nu_1 t + O(\xi^2), \tag{35}$$

where

$$\xi = \sqrt{\frac{h_N \varepsilon}{b_2}}, \quad \nu_1 = 1 - c_2 \xi^2, \tag{36}$$

$$b_2 = \frac{3\pi - 2}{40}, \quad c_2 = \frac{1}{10\pi}. \tag{37}$$

When $h_N = 0.001\,\mathrm{s}$; $\varepsilon = 0.3$, then $b_2 = 0.1856$; $c_2 = 0.0318$; $\xi \approx 0.001616$; $\nu_1 \approx 0.99999$. The stable periodic solution (35) of differential equation (33)

$$x(t) \approx 0.001616 \cos 1570.791t$$

is presented in Fig. 4.

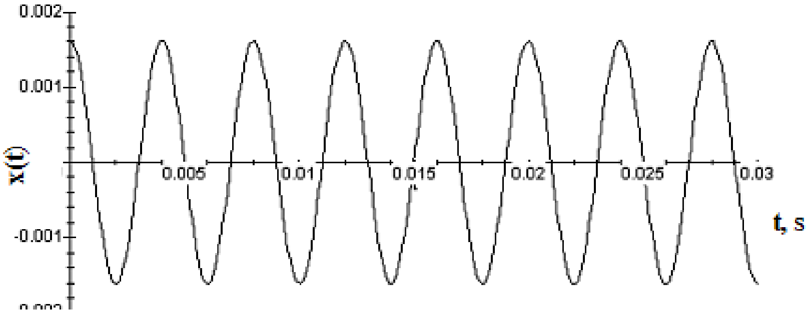

Fig. 4. The stable periodic solution of equation (33), when $h_N = 0.001\,\mathrm{s}$, $\varepsilon = 0.3$.

If $\varepsilon = \mu = 0$, then the characteristic equation (14) of the linear part of differential equations (33), (34) has two pairs of purely imaginary roots $\pm i\frac{\pi}{2h_N}$, $\pm i\sigma_*$, and the real parts of the other roots are negative. Then the system of differential equations (35), (36) under certain conditions has a stable periodic solution of two frequencies. The asymptotic expression of this solution is too complicated. In this case the oscillation of two frequencies is caused both by the perturbations in the feedback line and by the influence of delayed neutrons.

3.3 The reason of oscillation given by the influence of delayed neutrons

Let's assume that $r_N h_N < \frac{\pi}{2}$, but the characteristic quasi-polynomial (15) of the differential equation (4) in the neighbourhood of the equilibrium state $C(t) = C_0$ when $r_C = r_C^* + \mu$, parameter μ is assumed to be small, and $r_C^* = \frac{\sigma_*}{\sum_{j=1}^{6} \alpha_j \sin \sigma_* h_i}$, where σ_* is the unique root of the equation $\sum_{j=1}^{6} \alpha_j \cos \sigma h_j = 0$ belonging to the interval $(0, \frac{\pi}{2h_N})$, has one pair of purely imaginary roots $\pm i\sigma_*$, and real parts of other roots are negative.

Let's us analyse the equation (34). It is clear that $x(t) \to 0$, when $t \to \infty$. Then the problem leads to the finding a periodic solution of the equation

$$\dot{y}(t) + (r_C^* + \mu) \cdot \big[1 + y(t)\big] \cdot \sum_{j=1}^{6} \alpha_j y(t - h_j) = 0. \tag{38}$$

As shown in [3] it is possible to get an asymptotic expression of this solution [3].

4 Numerical experiments

The numerical solutions of the system of differential equations (3), (4) were also investigated using the MODEL MAKER software modelling system. The values of parameters α_j and h_j for the numerical model were taken from the Table 1. Then the numerical solutions were calculated for different values of parameters r_N and r_C (providing N_0 and C_0 are constant).

By changing the values of parameters r_N and r_C, the periodic solutions of the differential equations (3), (4) can be found (see Fig. 7 and Fig. 8).

The produced numerical results are also compared against asymptotic solutions as presented in Fig. 9, Fig. 10 for simple periodic solution and in Fig. 11, Fig. 12 for two frequencies case.

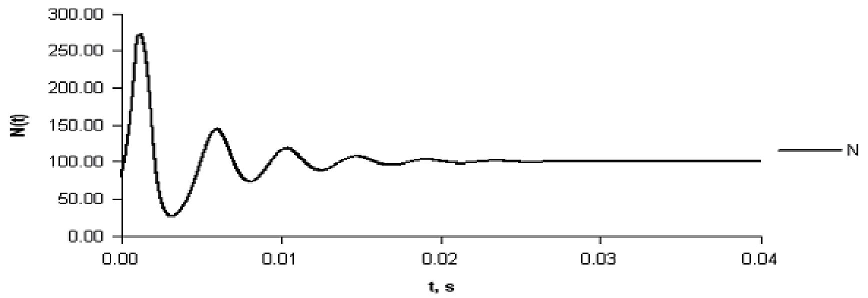

Fig. 5. The asymptotic stable solution of the differential equation (3), when $h_N = 0.001\,\mathrm{s}$, $N_0 = 100\,\mathrm{kW}$, $r_N = 1.200$.

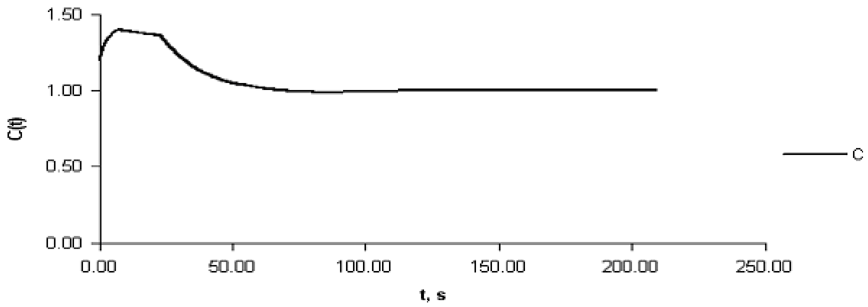

Fig. 6. The asymptotic stable solution of the differential equation (4), when $N_0 = 100\,\mathrm{kW}$, $C_0 = 1\,\mathrm{kW}$, $r_C = 0.05$.

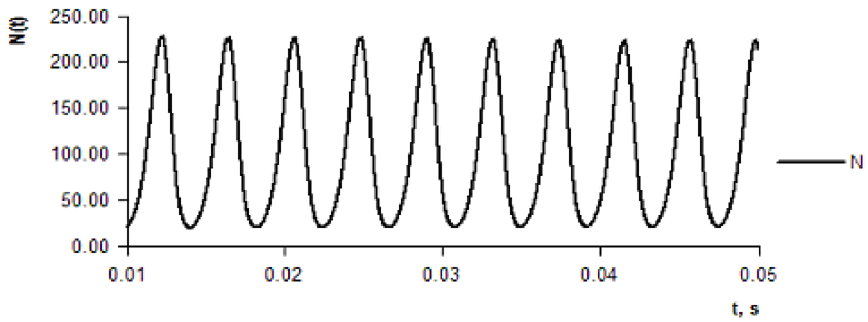

Fig. 7. The stable periodic solution of the differential equation (3), when $h_N = 0.001\,\mathrm{s}$, $N_0 = 100\,\mathrm{kW}$, $r_N = 1800$.

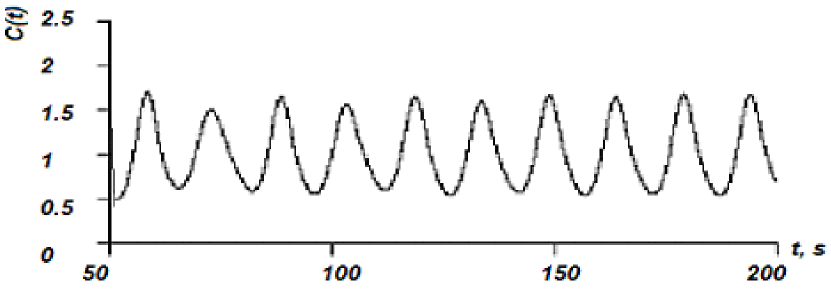

Fig. 8. The stable periodic solution of the differential equation (4), when $N_0 = 100\,\text{kW}$, $C_0 = 1\,\text{kW}$, $r_C = 0.1$.

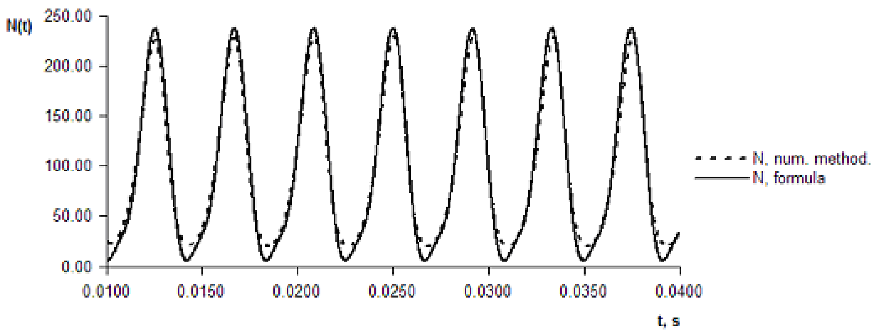

Fig. 9. The asymptotic stable solution of the differential equation (3), when $h_N = 0.001\,\text{s}$, $N_0 = 100\,\text{kW}$, $r_N = 1800$.

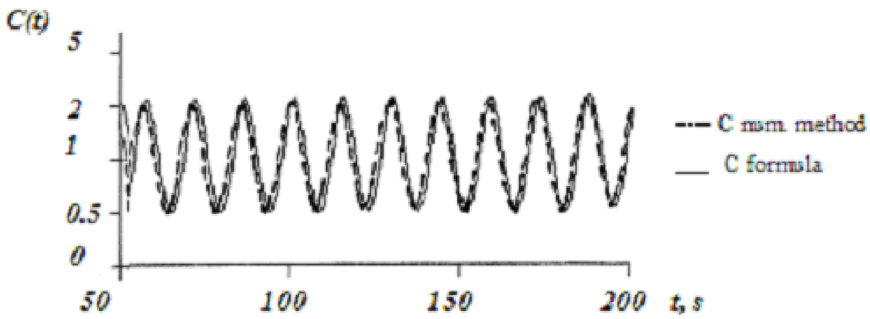

Fig. 10. The asymptotic stable solution of the differential equation (4), when $N_0 = 100\,\text{kW}$, $C_0 = 1\,\text{kW}$, $r_C = 0.1$.

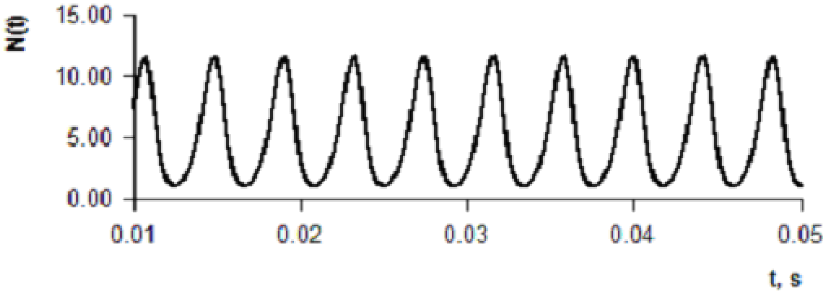

Fig. 11. The asymptotic stable solution of the differential equation (3), when $h_N = 0.001$ s, $N_0 = 5$ kW, $r_N = 1800$.

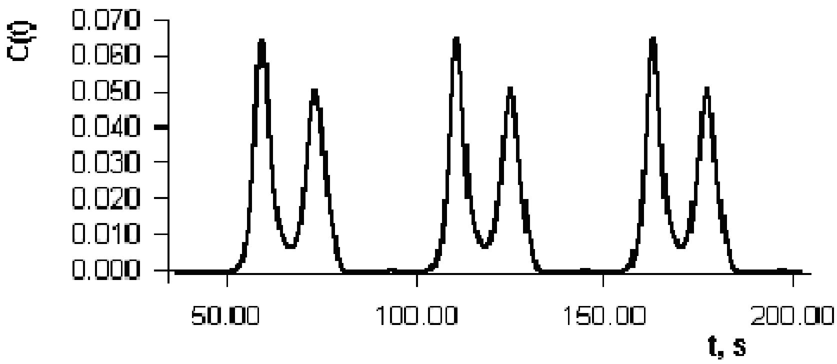

Fig. 12. The asymptotic stable solution of the differential equation (4), when $N_0 = 5$ kW, $C_0 = 0.03$ kW, $r_C = 0.8$.

5 Conclusion

The point model of a nuclear reactor with delayed feedback referring to the influence of delayed neutrons has been described by the system of two nonlinear differential equations with seven delays. The linear analysis of the system has been done by D-decomposition method. The nonlinear analysis is performed using the methods of the theory of bifurcations. The asymptotic solutions of the model coincide well enough with the results of the numerical experiments.

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