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Fixed point theorems for weakly β-admissible pair of F-contractions with application

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Abstract. In this paper, we introduce a new set $\mathcal{F}_s^b$ of nonlinear functions. We obtain unique common fixed point theorems for (β, F)-weak contractions under the effect of functions from $\mathcal{F}_s^b$. Moreover, we deduce new common fixed point results in ordered and graphic b-metric spaces. Our work generalizes several recent results existing in the literature. We set up an example to elucidate main result. We apply the main theorem to show the existence of common solution of the system of elliptic boundary value problems.

Keywords: common fixed point, (β, F)-weak contraction, β-complete b-metric space.

1 Introduction

The well-known Banach's contraction principle has many fruitful generalizations in various directions. One of these generalizations is for F-contraction presented by Wardowski [29]: every F-contraction defined on a complete metric space has a unique fixed point. So the concept of an F-contraction proved to be a milestone in fixed point theory. Numerous research papers on F-contractions have been published (see, for instant, [2, 10, 24]).

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In 2012, Samet et al. [26] introduced the idea of (α, ψ)-contractive and α-admissible mappings and evinced some significant fixed point results for such kind of mappings defined on complete metric spaces. Subsequently, Salimi et al. [25], Ćirić et al. [6] and Hussain et al. [13, 14] improved the concept of α-admissible mapping and proved some important (common) fixed point theorems.

In recent times, b-metric spaces were studied by many authors, especially fixed point theory on b-metric spaces [5, 11, 16, 23], [19, Chap. 12], [20]. Some authors have also studied topological properties of b-metric spaces. In [28], An et al. showed that every b-metric space with the topology induced by its convergence is a semi-metrizable space and thus many properties of b-metric spaces used in the literature are obvious. Then the authors proved the Stone-type theorem on b-metric spaces and get a sufficient condition for a b-metric space to be metrizable. Notice that a b-metric space is always understood to be a topological space with respect to the topology induced by its convergence, and a b-metric need not be continuous [28, Exs. 3.9 and 3.10].

Our objective, in this article, is to study fixed point theorems for (β, F)-weak contractions and their consequences. In order to achieve this, we consider Wardowski's paper [29]. In the said article, Wardowski imposed three conditions on function F along with contractive condition involving a self-mapping. In our work, we reduce the conditions imposed on F by omitting (WF_2) and establish common fixed point theorems in β-complete b-metric spaces. We give examples, which establish the significance of our work. We apply our main result to show the existence of solution of the system of elliptic boundary value problems.

2 The b-metric space and auxiliary lemmas

Throughout this paper, we denote the intervals $(0, \infty)$, $[0, \infty)$, $(-\infty, +\infty)$, by $\mathbb{R}^+$, $\mathbb{R}_0^+$ and $\mathbb{R}$, respectively.

Following concepts and results will be required for the proof of main result.

Czerwik [9] generalized metric function as follows.

Definition 1. (See [9].) Let $\Im$ be a nonempty set and $s \geqslant 1$ be a real number. The mapping $d^* : \Im \times \Im \to \mathbb{R}_0^+$ is said to be a b-metric if for all $\varsigma, \upsilon, \xi \in \Im$, we have:

(d_1^*) $\varsigma = \upsilon$ if and only if $d^*(\varsigma, \upsilon) = 0$.
(d_2^*) $d^*(\varsigma, \upsilon) = d^*(\upsilon, \varsigma)$.
(d_3^*) $d^*(\varsigma, \xi) \leqslant s[d^*(\varsigma, \upsilon) + d^*(\upsilon, \xi)]$.

The triplet $(\Im, d^*, s)$ is called a b-metric space (with coefficient $s \geqslant 1$).

Definition 1 allows us to remark that a b-metric space is more general than a metric space. One can see that for $s = 1$, d^* defines a metric and b-metric is discontinuous function, and it has different topological structure as compared to metric (see, for example, [21]).

Following lemma (Lemma 1), proved by Aghajani et al. [1], will be helpful in our work to establish the fixed point theorems for (β, F)-weak contractions.

Lemma 1. *Let $(\Im, d^*, s)$ be a b-metric space. If $r^*, t^* \in \Im$ and $\{r_n\}_{n\in\mathbb{N}}$ is a convergent sequence in $\Im$ with $\lim_{n\to\infty} r_n = r^*$, then*

$$\frac{1}{s}d^*(r^*, t^*) \leqslant \lim_{n\to\infty} \inf d^*(r_n, t^*) \leqslant \lim_{n\to\infty} \sup d^*(r_n, t^*) \leqslant sd^*(r^*, t^*).$$

For the notions like convergence, completeness, Cauchy sequence in the setting of b-metric spaces, the reader is referred to [1, 4, 9, 12, 15, 18].

Our investigations are based on following class of functions denoted by $\mathcal{F}_s$ and defined as follows.

Definition 2. A function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}_s$ if it satisfies following axioms:

(WF_1) F is strictly increasing;
(WF_3) There exists $\kappa \in (0, 1)$ such that $\lim_{r\to 0^+}(r)^\kappa F(r) = 0$.

Note that we have dropped the Wardowski's (WF_2) condition in Definition 2.

(WF_2) For each sequence $\{r_n\}$ of positive real numbers,

$$\lim_{n\to\infty} r_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(r_n) = -\infty.$$

Example 1. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by

(a) $F(r) = \ln r$; (d) $F(r) = -r^{-1/2}$;
(b) $F(r) = r + \ln r$; (e) $F(r) = r^a$, $a > 0$;
(c) $F(r) = \ln(r^2 + r)$; (f) $F(r) = \ln(r + 1)$.

It is easy to check that the functions given in (a), (b), (c) and (d) satisfy (WF_1), (WF_2) and (WF_3). The functions given in (e) and (f) belong to the family $\mathcal{F}_s$, which do not satisfy WF_2.

The reason to omit (WF_2), is following lemma.

Lemma 2. *If the function F satisfies (WF_1) and $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is a decreasing sequence such that $\lim_{n\to\infty} F(r_n) = -\infty$, then $\lim_{n\to\infty} r_n = 0$.*

Proof. We note that $\{r_n\}_{n\in\mathbb{N}}$ is bounded below and decreasing sequence, so it is convergent. Let $\lim_{n\to\infty} r_n = \zeta \geqslant 0$. Suppose on contrary $\zeta > 0$. Since, $r_n \geqslant \zeta$, therefore, $F(r_n) \geqslant F(\zeta)$. Thus, $F(\zeta) \leqslant \lim_{n\to\infty} F(r_n) = -\infty$, a contradiction. Hence, $\lim_{n\to\infty} r_n = 0$. □

In [7], authors introduce following compatibility condition to work in b-metric spaces.

(CF_4) For each $n \in \mathbb{N}$, there exists $\tau > 0$ such that

$$\tau + F(sr_n) \leqslant F(r_{n-1}) \quad \Longrightarrow \quad \tau + F(s^n r_n) \leqslant F(s^{n-1} r_{n-1}).$$

We denote class of functions satisfying (WF_1), (WF_3) and (CF_4) by $\mathcal{F}_s^b$.

Remark 1. The class $\mathcal{F}_s^b$ is nonempty.

Indeed, if we define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r+1)$ for all $r \in \mathbb{R}^+$, then (WF_1) and (WF_3) are obvious. We establish (CF_4): let $\tau + F(sr_n) \leqslant F(r_{n-1})$, then, for $\tau = \ln(s^{n-1})$, we have

$$\begin{aligned}
&\ln(s^{n-1}) + \ln(sr_n + 1) \leqslant \ln(r_{n-1}+1) \\
&\quad \implies \quad \ln\bigl(s^n r_n + s^{n-1}\bigr) \leqslant \ln(r_{n-1}+1) \quad \implies \quad s^n r_n \leqslant r_{n-1} + 1 - s^{n-1}.
\end{aligned}$$

Now consider

$$\begin{aligned}
&\tau + F\bigl(s^n r_n\bigr) \\
&\quad = \ln\bigl(s^{n-1}\bigr) + \ln\bigl(s^n r_n + 1\bigr) \leqslant \ln\bigl(s^{n-1}\bigr) + \ln\bigl(r_{n-1} - s^{n-1} + 1\bigr) \\
&\quad = \ln\bigl(s^{n-1} r_{n-1} - s^{2n-2} + s^{n-1}\bigr) \leqslant \ln\bigl(s^{n-1} r_{n-1} + s^{n-1}\bigl(1 - s^{n-1}\bigr)\bigr) \\
&\quad \leqslant \ln\bigl(s^{n-1} r_{n-1}\bigr) = F\bigl(s^{n-1} r_{n-1}\bigr).
\end{aligned}$$

Hence, $F \in \mathcal{F}_s^b$.

Definition 3. Let $(\Im, d^*, s)$ be a b-metric space. We say the mapping $T : \Im \to \Im$ is an F_{d^*}-contraction if there exist $F \in \mathcal{F}_s^b$ and $\tau > 0$ such that

$$\begin{aligned}
&d^*\bigl(T(\gamma), T(\eta)\bigr) > 0 \\
&\quad \implies \quad \tau + F\bigl(sd^*\bigl(T(\gamma), T(\eta)\bigr)\bigr) \leqslant F\bigl(d^*(\gamma,\eta)\bigr) \quad \forall \gamma, \eta \in \Im.
\end{aligned}$$

Remark 2. It can be seen from following example that there exists at least one self-mapping which satisfies F_{d^*}-contraction whereas it does not satisfy Banach contraction in b-metric spaces.

Example 2. Let $\Im = \{\gamma_n = 2^{n/2} n,\ n \in \mathbb{N}\}$. Define $d^* : \Im \times \Im \to \mathbb{R}_0^+$ by $d^*(\gamma,\eta) = |\gamma - \eta|^2$, then $(\Im, d^*, s = 2)$ is a b-metric space. Define the mapping $f : \Im \to \Im$ by

$$f(\gamma) = \begin{cases} 2^{(n-1)/2}(n-1) & \text{if } \gamma = \gamma_n; \\ \gamma_0 & \text{if } \gamma = \gamma_0. \end{cases}$$

It is easy to show that there exists $\tau > 0$ such that f is an F_{d^*}-contraction for $F : \mathbb{R}^+ \to \mathbb{R}$, $F(r) = r$, while f is not a Banach contraction in the b-metric sense. Following arguments justify our remark:

$$\lim_{n\to\infty} \frac{2d^*(f(\gamma_n), f(\gamma_0))}{d^*(\gamma_n, \gamma_0)} = \lim_{n\to\infty} \frac{2(\gamma_{n-1} - \gamma_0)^2}{(\gamma_n - \gamma_0)^2} = \lim_{n\to\infty} \frac{(2^{n/2}(n-1))^2}{(2^{n/2} n)^2} = 1.$$

This shows that Banach contraction principle cannot be applied for self-mapping f. Next arguments will show that f is an F_{d^*}-contraction. Indeed, by definition of function F, for every $\gamma, \eta \in \Im$ such that $f(\gamma) \neq f(\eta)$, we have

$$2d^*\bigl(f(\gamma), f(\eta)\bigr) - d^*(\gamma,\eta) \leqslant -\tau.$$

For $\gamma = \gamma_{n+k}$ and $\eta = \gamma_n$, consider

$$\begin{aligned}
&2d^*\big(f(\gamma_{n+k}), f(\gamma_n)\big) - d^*(\gamma_{n+k}, \gamma_n)\\
&\quad = \big(2^{(n+k)/2}(n+k-1) - 2^{n/2}(n-1)\big)^2 - \big(2^{(n+k)/2}(n+k) - 2^{n/2}(n)\big)^2\\
&\quad = 2^n\big(1 - 2^{k/2}\big)\big(2^{k/2}(2n+2k-1) - (2n-1)\big) \leqslant -1.
\end{aligned}$$

Also we see that $F \in \mathcal{F}_s^b$. Indeed, for $F(r) = r$, (WF_1) and (WF_3) can be easily worked out. For property (CF_4), we have following arguments: let $\tau + F(sr_n) \leqslant F(r_{n-1})$, that is, $\tau + sr_n \leqslant r_{n-1}$. Consider

$$\begin{aligned}
\tau + F\big(s^n r_n\big) &= \tau + s^n r_n = \tau + s^{n-1}(sr_n) \leqslant \tau + s^{n-1}(r_{n-1} - \tau)\\
&= \tau + s^{n-1} r_{n-1} - \tau s^{n-1} = \tau\big(1 - s^{n-1}\big) + s^{n-1} r_{n-1}\\
&\leqslant s^{n-1} r_{n-1} = F\big(s^{n-1} r_{n-1}\big).
\end{aligned}$$

This shows that for $\tau = 1$, f is an F_{d^*}-contraction.

Definition 4. Let $\{r_n\}$ be a sequence in $\mathbb{R}_0^+$, and let $\{a_n\}$ be a sequence in $\mathbb{R}^+$. We say that $\{r_n\} \in O(a_n)$ if there exists $C > 0$ such that $r_n \leqslant Ca_n$ for all $n \in \mathbb{N}$.

We use following Lemma (appeared in [27]) as formula to prove a sequence to be a Cauchy sequence.

Lemma 3. *Let $(\Im, d^*, s)$ be a b-metric space. Let $\{r_n\}$ be a sequence in $\Im$. Assume that*

$$\big\{d^*(r_n, r_{n+1})\big\} \in \cup\big\{O\big(n^{-t}\big)\colon t > 1 + \log_2 s\big\}.$$

Then $\{r_n\}$ is Cauchy.

For proof, see [27]. We apply Lemma 3, in our case, as follows:

Lemma 4. *Let $\{b_n\}$ be a decreasing sequence in $\mathbb{R}^+$. Assume that there exist a mapping $F : \mathbb{R}^+ \to \mathbb{R}$, $\tau \in \mathbb{R}^+$ and $\kappa \in (0, 1)$ satisfying (WF_3) and the following*:

$$n\tau + F\big(s^n b_n\big) \leqslant F(b_0). \tag{1}$$

Then $\{b_n\} \in O(n^{1/k})$.

Proof. We note that the condition (1) implies $\lim_{n\to\infty} F(s^n b_n) = -\infty$ and hence, Lemma 2 allows us to have $\lim_{n\to\infty} s^n b_n = 0$. By (WF_3), we infer that

$$\lim_{n\to\infty} \big(s^n b_n\big)^\kappa F\big(s^n b_n\big) = 0.$$

Again, by condition (1) we have

$$\big(s^n b_n\big)^\kappa F\big(s^n b_n\big) - \big(s^n b_n\big)^\kappa F(b_0) \leqslant -\big(s^n b_n\big)^\kappa n\tau \leqslant 0. \tag{2}$$

On taking limit $n \to \infty$ in (2), we have

$$\lim_{n\to\infty} n\big(s^n b_n\big)^{\kappa} = 0.$$

Then there exists $n_1 \in \mathbb{N}$ such that $n(s^n b_n)^{\kappa} \leqslant 1$ for $n \geqslant n_1$. It then follows that for $n \geqslant n_1$,

$$s^n b_n \leqslant \frac{1}{n^{1/\kappa}} \quad \Longrightarrow \quad b_n \leqslant \frac{1}{s^n} n^{-1/\kappa} \leqslant \frac{1}{s} n^{-1/\kappa} \quad \Longrightarrow \quad b_n \leqslant C n^{-1/\kappa},$$

where $C = s^{-1}$. Hence $\{b_n\} \in O(n^{-1/\kappa})$. □

3 Fixed point theorems

Recently, Mínak et al. [22] and Cosentino et al. [8] have employed Ćirić-type and Hardy–Rogers-type contractive conditions, respectively, on T in their definition of an F-contraction and found a unique fixed point of T in the context of a metric space. We introduce the notion of (β, F)-weak contraction by imposing a Ćirić-type contractive condition in terms of two mappings defined on a b-metric space and find their unique common fixed point.

In [3], authors introduced following terms.

Definition 5. (See [3].) Let $(\Im, d^*, s)$ be a b-metric space and $f : \Im \to \Im$ and $\alpha_s : \Im \times \Im \to \mathbb{R}_0^+$ be two functions. The mapping f is said to be an α_s-admissible mapping if

$$\alpha_s(\gamma,\eta) \geqslant s^2 \quad \Longrightarrow \quad \alpha_s\big(f(\gamma), f(\eta)\big) \geqslant s^2 \quad \forall \gamma,\eta \in \Im.$$

Definition 6. (See [3].) Let $(\Im, d^*, s)$ be a b-metric space and $f : \Im \to \Im$ and $\alpha_s : \Im \times \Im \to \mathbb{R}_0^+$ be two functions. The mapping f is said to be a triangular α_s-admissible mapping if

(i) $\alpha_s(\gamma,\eta) \geqslant s^2$ implies $\alpha_s(f(\gamma), f(\eta)) \geqslant s^2$, $\gamma,\eta \in \Im$;
(ii) $\alpha_s(\gamma,\chi) \geqslant s^2$, $\alpha_s(\chi,\eta) \geqslant s^2$ imply $\alpha_s(\gamma,\eta) \geqslant s^2$ for all $\gamma,\eta,\chi \in \Im$.

Remark 3. We observe that for the given function α_s, there exists a function $\beta : \Im \times \Im \to \mathbb{R}_0^+$ defined by $\beta(\gamma,\eta) = \alpha_s(\gamma,\eta)/s^2$ having following properties:

(i) For function f defined above, f is α_s-admissible if and only if f is β-admissible, that is,
$$\beta(\gamma,\eta) \geqslant 1 \quad \Longrightarrow \quad \beta\big(f(\gamma), f(\eta)\big) \geqslant 1 \quad \forall \gamma,\eta \in \Im;$$
(ii) The b-metric space $(\Im, d^*, s)$ is α_s-complete if and only if it is β-complete.

Considering Remark 3, we proceed as follows:

Definition 7. Let $(\Im, d^*, s)$ be a b-metric space and $f, g : \Im \to \Im$ and $\beta : \Im \times \Im \to \mathbb{R}_0^+$ be three mappings. The pair (f, g) is said to be weakly β-admissible pair of mappings if

$$\beta(\gamma,\eta) \geqslant 1 \quad \Longrightarrow \quad \beta\big(f(\gamma), gf(\gamma)\big) \geqslant 1, \quad \beta\big(g(\eta), fg(\eta)\big) \geqslant 1 \quad \forall \gamma,\eta \in \Im.$$

Definition 8. Let $(\Im, d^*, s)$ be a b-metric space and $f, g : \Im \to \Im$ and $\beta : \Im \times \Im \to \mathbb{R}_0^+$ be three mappings. The pair of mappings (f, g) is said to be a triangular weakly β-admissible pair of mappings if

(i) $\beta(\gamma, \eta) \geqslant 1$ implies $\beta(f(\gamma), gf(\gamma)) \geqslant 1$ and $\beta(g(\eta), fg(\eta)) \geqslant 1$ for all $\gamma, \eta \in \Im$;
(ii) $\beta(\gamma, \chi) \geqslant 1$, $\beta(\chi, \eta) \geqslant 1$ imply $\beta(\gamma, \eta) \geqslant 1$ for all $\gamma, \eta, \chi \in \Im$.

Example 3. Let $\Im = [0, \infty)$ and

$$f(r) = \begin{cases} r & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1, \infty), \end{cases} \qquad g(r) = \begin{cases} r^{1/3} & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1, \infty). \end{cases}$$

Define $\beta : \Im \times \Im \to \mathbb{R}_0^+$ by

$$\beta(\gamma, \eta) = \begin{cases} 1 + \eta - \gamma & \text{if } \gamma, \eta \in [0,1); \\ 0 & \text{if } \gamma, \eta \in [1, \infty). \end{cases}$$

Then the pair (f, g) is triangular weakly β-admissible pair of mappings. Indeed, if $\beta(\gamma, \eta) \geqslant 1$ and $\beta(\eta, \chi) \geqslant 1$, then $\gamma - \eta \leqslant 0$ and $\eta - \chi \leqslant 0$, which implies that $\gamma - \chi \leqslant 0$. Hence, for all $\gamma, \eta \in [0, 1)$, $\beta(\gamma, \chi) = 1 + \chi - \gamma \geqslant 1$,

$$\beta\big(f(\gamma), gf(\gamma)\big) = \beta\big(\gamma, \gamma^{1/3}\big) \geqslant 1, \qquad \beta\big(g(\eta), fg(\eta)\big) = \beta\big(\eta^{1/3}, \eta^{1/3}\big) \geqslant 1.$$

Definition 9. Let $(\Im, d^*, s)$ be a b-metric space, and let $\beta : \Im \times \Im \to \mathbb{R}_0^+$ and $f : \Im \to \Im$ be two mappings. We say the mapping f is an β-continuous mapping if for given $r \in \Im$ and sequence $\{r_n\}$,

$$\lim_{n \to \infty} d^*(r_n, r) = 0 \quad \text{and} \quad \beta(r_n, r_{n+1}) \geqslant 1$$
$$\implies \quad \lim_{n \to \infty} d^*\big(f(r_n), f(r)\big) = 0 \quad \forall n \in \mathbb{N}.$$

Example 4. Let $\Im = [0, \infty)$ and $d^* : \Im \times \Im \to [0, \infty)$ be defined by $d^*(\gamma, \eta) = |\gamma - \eta|^2$ for all $\gamma, \eta \in \Im$. Define

$$f(r) = \begin{cases} \sin(\pi r) & \text{if } r \in [0,1]; \\ \cos(\pi r) + 2 & \text{if } r \in (1, \infty), \end{cases} \qquad \beta(\gamma, \eta) = \begin{cases} \gamma^3 + \eta^3 + 1 & \text{if } \gamma, \eta \in [0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, f is not continuous on $\Im$, however, f is a β-continuous.

Definition 10. Let $(\Im, d^*, s)$ be a b-metric space and β as defined in Definition 5. The b-metric space $\Im$ is said to be β-complete if and only if every Cauchy sequence $\{r_n\}$ in $\Im$ such that $\beta(r_n, r_{n+1}) \geqslant 1$ converges in $\Im$ for all $n \in \mathbb{N}$.

If $(\Im, d^*)$ is a complete b-metric space, then $(\Im, d)$ is also a β-complete b-metric space, but the converse is not true. Following example explains this fact.

Example 5. Let $\Im = (0, \infty)$ and the b-metric $d^* : \Im \times \Im \to [0, \infty)$ defined by $d^*(\gamma, \eta) = |\gamma - \eta|^2$ for all $\gamma, \eta \in \Im$. Define $\beta : \Im \times \Im \to [0, \infty)$:

$$\beta(\gamma, \eta) = \begin{cases} \mathrm{e}^{d^*(\gamma,\eta)} & \text{if } \gamma, \eta \in [1, 3]; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $(\Im, d^*, s)$ is not a complete b-metric space, but $(\Im, d^*, s)$ is a β-complete b-metric space. Indeed, if $\{r_n\}$ is a Cauchy sequence in $\Im$ such that $\beta(r_n, r_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, then $r_n \in [1, 3]$. Since $[1, 3]$ is a closed subset of $\mathbb{R}$, we see that $([1, 3], d^*, 2)$ is a complete b-metric space, and then there exists $r \in [1, 3]$ such that $r_n \to r$ as $n \to \infty$.

Definition 11. Let $(\Im, d^*, s)$ be a b-metric space and let $\beta : \Im \times \Im \to \mathbb{R}_0^+$ be a mapping. The space $(\Im, d^*, s)$ is said to be β -regular if for any sequence $\{r_n\} \subset \Im$ such that $\beta(r_n, r_{n+1}) \geqslant 1$ and $r_n \to r$ as $n \to \infty$, we have $\beta(r_n, r) \geqslant 1$ for all $n \in \mathbb{N}$.

Definition 12. Let $(\Im, d^*, s)$ be a b-metric space. The self-mappings $f, g : \Im \to \Im$ are called (β, F)-weak contractions if there exist $F \in \mathcal{F}_s^b$ and $\tau > 0$ such that

$$\tau + F\big(s\beta(\gamma, \eta)\, d^*\big(f(\gamma), g(\eta)\big)\big) \leqslant F\big(\mathcal{M}_1(\gamma, \eta)\big) \quad \forall \gamma, \eta \in \Im \tag{3}$$

with $\beta(\gamma, \eta) \geqslant 1$ whenever

$$\min\big\{\beta(\gamma, \eta)\, d^*\big(f(\gamma), g(\eta)\big), \mathcal{M}_1(\gamma, \eta)\big\} > 0,$$

where

$$\mathcal{M}_1(\gamma, \eta) = \max\left\{ d^*(\gamma, \eta), d^*\big(\gamma, f(\gamma)\big), d^*\big(\eta, g(\eta)\big), \frac{d^*(\gamma, g(\eta)) + d^*(\eta, f(\gamma))}{2s} \right\}.$$

Following existence theorem is our main result.

Theorem 1. *Let $f, g : \Im \to \Im$ be (β, F)-weak contractions defined on an β-complete b-metric space $(\Im, d^*, s)$. Assume that $\kappa \in (0, 1/(1 + \log_2 s))$ and*

(i) *(f, g) is a weakly β-admissible pair of mappings*;
(ii) *There exists $r_0 \in \Im$ such that $\beta(r_0, f(r_0)) \geqslant 1$;*
(iii) (a) *Either one of f and g is a β-continuous mapping or*
  (b) *$\Im$ is β-regular space and the mapping F is continuous.*

Then we can construct a sequence $\{r_n\}$ in $\Im$ such that $r_n \to \upsilon \in \Im$. If $\beta(\upsilon, \upsilon) \geqslant 1$, then υ is a common fixed point of the pair (f, g). In addition, if ω is also a common fixed point of the pair (f, g) such that $\beta(\upsilon, \omega) \geqslant 1$, then $\upsilon = \omega$.

Proof. Firstly, we prove that the self-mappings f, g have at most one common fixed point. Suppose that υ and ω are two different common fixed points of f and g. Then $f(\upsilon) =$

$\upsilon \neq \omega = g(\omega)$. It follows that $d^*(f(\upsilon), g(\omega)) = d^*(\upsilon, \omega) > 0$. Since $\beta(\upsilon, \omega) \geqslant 1$, so the contractive condition (3) implies

$$\begin{aligned}
&\tau + F\big(s\beta(\upsilon,\omega)\, d^*\big(f(\upsilon), g(\omega)\big)\big) \\
&\quad \leqslant F\left(\max\left\{d^*(\upsilon,\omega), d^*\big(\upsilon, f(\upsilon)\big), d^*\big(\omega, g(\omega)\big), \frac{d^*(\upsilon, g(\omega)) + d^*(\omega, f(\upsilon))}{2s}\right\}\right) \\
&\quad \leqslant F\left(\max\left\{d^*(\upsilon,\omega), d^*(\upsilon,\upsilon), d^*(\omega,\omega), \frac{d^*(\upsilon,\omega) + d^*(\omega,\upsilon)}{2s}\right\}\right) \\
&\quad = F\big(d^*(\upsilon,\omega)\big) \leqslant F\big(s\beta(\upsilon,\omega) d^*(\upsilon,\omega)\big).
\end{aligned}$$

It shows that $\tau \leqslant 0$, a contradiction. Hence, the pair (f, g) has at most one common fixed point.

(i) We note that for $r_1 \neq r_2$, $\mathcal{M}_1(r_1, r_2) > 0$. Let $r_0 \in \Im$ be as in (ii). We construct an iterative sequence $\{r_n\}$ of points in $\Im$ such that $r_1 = f(r_0)$, $r_2 = g(r_1)$ and, generally, $r_{2n+1} = f(r_{2n})$, $r_{2n} = g(r_{2n-1})$ for all $n \in \mathbb{N}$. By assumption (i) we have

$$\begin{aligned}
&\beta\big(f(r_0), gf(r_0)\big) = \beta(r_1, r_2) \geqslant 1 \quad \text{and} \quad \beta\big(g(r_1), fg(r_1)\big) = \beta(r_2, r_3) \geqslant 1, \\
&\beta\big(f(r_2), gf(r_2)\big) = \beta(r_3, r_4) \geqslant 1 \quad \text{and} \quad \beta\big(g(r_3), fg(r_3)\big) = \beta(r_4, r_5) \geqslant 1.
\end{aligned}$$

Continuing on a same pattern, we have

$$\beta\big(f(r_{2n}), gf(r_{2n})\big) = \beta(r_{2n+1}, r_{2n+2}) \geqslant 1$$

and

$$\beta\big(g(r_{2n-1}), fg(r_{2n-1})\big) = \beta(r_{2n}, r_{2n+1}) \geqslant 1.$$

Hence, $\beta(r_n, r_{n+1}) \geqslant 1$ for all $n \in \mathbb{W}$. If $d^*(f(r_{2n}), g(r_{2n+1})) = 0$, then r_{2n} is a common fixed point of mappings f, g. Let $d^*(f(r_{2n}), g(r_{2n+1})) > 0$, then by contractive condition (3) we get

$$\begin{aligned}
F\big(sd^*(r_{2n+1}, r_{2n+2})\big) &\leqslant F\big(s\beta(r_{2n}, r_{2n+1})\, d^*\big(f(r_{2n}), g(r_{2n+1})\big)\big) \\
&\leqslant F\big(\mathcal{M}_1(r_{2n}, r_{2n+1})\big) - \tau \quad \forall n \in \mathbb{W},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_1(r_{2n}, r_{2n+1}) &= \max\bigg\{d^*(r_{2n}, r_{2n+1}), d^*\big(r_{2n}, f(r_{2n})\big), d^*\big(r_{2n+1}, g(r_{2n+1})\big), \\
&\qquad \frac{d^*(r_{2n}, g(r_{2n+1})) + d^*(r_{2n+1}, f(r_{2n}))}{2s}\bigg\} \\
&= \max\bigg\{d^*(r_{2n}, r_{2n+1}), d^*(r_{2n}, r_{2n+1}), d^*(r_{2n+1}, r_{2n+2}), \\
&\qquad \frac{d^*(r_{2n}, r_{2n+2}) + d^*(r_{2n+1}, r_{2n+1})}{2s}\bigg\} \\
&\leqslant \max\big\{d^*(r_{2n}, r_{2n+1}), d^*(r_{2n+1}, r_{2n+2})\big\}.
\end{aligned}$$

If $\mathcal{M}_1(r_{2n}, r_{2n+1}) = d^*(r_{2n+1}, r_{2n+2})$, then

$$F\big(sd^*(r_{2n+1}, r_{2n+2})\big) \leqslant F\big(d^*(r_{2n+1}, r_{2n+2})\big) - \tau,$$

which is a contradiction to (WF_1). Therefore,

$$F\big(sd^*(r_{2n+1}, r_{2n+2})\big) \leqslant F\big(d^*(r_{2n}, r_{2n+1})\big) - \tau, \quad n \in \mathbb{W}. \tag{4}$$

Similarly, we can have

$$F\big(sd^*(r_{2n+2}, r_{2n+3})\big) \leqslant F\big(d^*(r_{2n+1}, r_{2n+2})\big) - \tau, \quad n \in \mathbb{W}. \tag{5}$$

Hence, from (4) and (5) we have

$$F\big(sd^*(r_n, r_{n+1})\big) \leqslant F\big(d^*(r_{n-1}, r_n)\big) - \tau, \quad n \in \mathbb{N}. \tag{6}$$

Let $b_n = d^*(r_n, r_{n+1})$ for each $n \in \mathbb{W}$, by (6) and (CF_4) we have

$$\tau + F\big(s^n b_n\big) \leqslant F\big(s^{n-1} b_{n-1}\big), \quad n \in \mathbb{N}.$$

Repeating above process, we obtain

$$F\big(s^n b_n\big) \leqslant F(b_0) - n\tau, \quad n \in \mathbb{N}. \tag{7}$$

By Lemma 4, $\{b_n\} \in O(n^{-1/\kappa})$. Since $1/\kappa \in (1 + \log_2 s, \infty)$ holds, by Lemma 3 we infer that $\{r_n\}$ is Cauchy sequence. Since $\Im$ is β-complete b-metric space, there exists (say) $\upsilon \in \Im$ such that $r_{2n+1} \to \upsilon$ and $r_{2n+2} \to \upsilon$ as $n \to \infty$ with respect to topology induced by its convergence. The β-continuity of g implies

$$\begin{aligned} \upsilon &= \lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{2n+1} = \lim_{n\to\infty} r_{2n+2} = \lim_{n\to\infty} g(r_{2n+1}) \\ &= g\Big(\lim_{n\to\infty} r_{2n+1}\Big) = g(\upsilon). \end{aligned}$$

If $d^*(\upsilon, f(\upsilon)) > 0$ as $\beta(\upsilon, \upsilon) \geqslant 1$, so by contractive condition (3) we have

$$\begin{aligned} \tau + F\big(sd^*\big(f(\upsilon), \upsilon\big)\big) &\leqslant \tau + F\big(s\beta(\upsilon, \upsilon)\, d^*\big(f(\upsilon), g(\upsilon)\big)\big) \\ &\leqslant F\big(\mathcal{M}_1(\upsilon, \upsilon)\big) = F\big(d^*\big(f(\upsilon), \upsilon\big)\big), \end{aligned}$$

a contradiction. Thus, $d^*(f(\upsilon), \upsilon) = 0$ and (d_1^*) implies $\upsilon = f(\upsilon)$. Thus, we have $f(\upsilon) = g(\upsilon) = \upsilon$. Hence (f, g) has a common fixed point υ.

(ii) We have two different cases. First, if there exists a subsequence $\{r_{n_i}\}_{i\in\mathbb{N}} \subset \{r_n\}_{n\in\mathbb{N}}$ such that

$$r_{n_i} = \begin{cases} f(\upsilon) & \text{for all even } i; \\ g(\upsilon) & \text{for all odd } i, \end{cases}$$

then

$$\upsilon = \lim_{i\to\infty} r_{n_i} = \lim_{i\to\infty} f(\upsilon) = f(\upsilon) \quad \text{and} \quad \upsilon = \lim_{i\to\infty} r_{n_i} = \lim_{i\to\infty} g(\upsilon) = g(\upsilon).$$

So, we have done. Second, if there is no such subsequence of $\{r_n\}_{n\in\mathbb{N}}$, then there exists a natural number η_0 such that for every $n \geqslant \eta_0$, we have $d^*(f(r_{2n}), g(\upsilon)) > 0$ and $d^*(g(r_{2n+1}), f(\upsilon)) > 0$. It is given that the space $\Im$ is β-regular, thus, $\beta(r_{2n+1}, \upsilon) \geqslant 1$, $\beta(r_{2n}, \upsilon) \geqslant 1$. By contractive condition (3) we have

$$\begin{aligned}
&\tau + F\big(s\beta(r_{2n},\upsilon)\, d^*\big(f(r_{2n}), g(\upsilon)\big)\big)\\
&\quad\leqslant F\bigg(\max\bigg\{d^*(r_{2n},\upsilon), d^*\big(r_{2n}, f(r_{2n})\big), d^*\big(\upsilon, g(\upsilon)\big),\\
&\qquad\qquad \frac{d^*(r_{2n}, g(\upsilon)) + d^*(\upsilon, f(r_{2n}))}{2s}\bigg\}\bigg). \qquad (8)
\end{aligned}$$

We show that $d^*(\upsilon, g(\upsilon)) = 0$. Suppose on contrary that $d^*(\upsilon, g(\upsilon)) = p > 0$. Put $\gamma_n = d^*(r_n, \upsilon)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} r_n = \upsilon$, there exists $\eta_1 \in \mathbb{N}$ such that for every $n \geqslant \eta_1$, both $\gamma_n < p/2$ and $b_n < p/2$ hold. Consequently, by (8) we have

$$\begin{aligned}
&\tau + F\big(s\beta(r_{2n},\upsilon)(r_{2n},\upsilon)\, d^*\big(f(r_{2n}), g(\upsilon)\big)\big)\\
&\quad\leqslant F\bigg(\max\bigg\{\gamma_{2n}, b_{2n}, p, \frac{d^*(r_{2n}, g(\upsilon)) + \gamma_{2n+1}}{2s}\bigg\}\bigg)\\
&\quad\leqslant F\bigg(\max\bigg\{\gamma_{2n}, b_{2n}, p, \frac{s\gamma_{2n} + sp + \gamma_{2n+1}}{2s}\bigg\}\bigg)\\
&\quad\leqslant F\bigg(\max\bigg\{\frac{p}{2}, \frac{p}{2}, p, \frac{s\frac{p}{2} + sp + \frac{p}{2}}{2s}\bigg\}\bigg) = F(p).
\end{aligned}$$

Thus, for every $n \geqslant \max\{\eta_0, \eta_1\}$, we obtain

$$\tau + F\big(s\beta(r_{2n},\upsilon)\, d^*\big(f(r_{2n}), g(\upsilon)\big)\big) \leqslant F\big(d^*\big(\upsilon, g(\upsilon)\big)\big). \qquad (9)$$

Since F is continuous and increasing, by Lemma 1 and inequality (9) we have

$$\begin{aligned}
\tau + F\big(d^*\big(\upsilon, g(\upsilon)\big)\big) &\leqslant \tau + F\Big(s\beta(r_{2n},\upsilon) \lim_{n\to\infty} \inf d^*\big(f(r_{2n}), g(\upsilon)\big)\Big)\\
&\leqslant \tau + \lim_{n\to\infty} \inf F\big(s\beta(r_{2n},\upsilon)\, d^*\big(f(r_{2n}), g(\upsilon)\big)\big)\\
&\leqslant F\big(d^*\big(\upsilon, g(\upsilon)\big)\big).
\end{aligned}$$

The above inequality shows that $\tau \leqslant 0$ which is a contradiction. Thus, $d^*(g(\upsilon), \upsilon) = 0$ and hence $\upsilon = g(\upsilon)$. Similarly, we can prove that $\upsilon = f(\upsilon)$, and consequently, υ is a common fixed point of f and g. □

Following example illustrates Theorem 1.

Example 6. Let $\Im = [0, \infty)$ and define $d^* : \Im \times \Im \to \mathbb{R}_0^+$ by $d^*(\gamma, \eta) = |\gamma - \eta|^2$. Define $\beta : \Im \times \Im \to [0, \infty)$ by

$$\beta(\gamma, \eta) = \begin{cases} \mathrm{e}^{d^*(\gamma,\eta)}, & \gamma, \eta \in \Im,\ \gamma \geqslant \eta;\\ 5, & \gamma, \eta \in \Im,\ \eta > \gamma. \end{cases}$$

So, $(\Im, d^*, s)$ is a β-complete b-metric space with $s = 2$. Define the mappings $f, g\colon \Im \to \Im$ for all $r \in \Im$ by

$$f(r) = \ln\left(1 + \frac{r}{6}\right), \qquad g(r) = \ln\left(1 + \frac{r}{7}\right).$$

Clearly, f, g are β-continuous self-mappings. To prove that (f, g) is weakly β-admissible pair of mappings, let $\gamma, \eta \in \Im$ be such that $\eta = f(\gamma)$, thus we have $\eta = \ln(1 + \gamma/6)$. As

$$\begin{aligned} f(\gamma) &= \ln\left(1 + \frac{\gamma}{6}\right) \geqslant \ln\left(1 + \frac{\ln(1 + \frac{\gamma}{6})}{7}\right) = \ln\left(1 + \frac{\eta}{7}\right) \\ &= g(\eta) = gf(\gamma). \end{aligned}$$

Thus, $\beta(f\gamma, gf(\gamma)) \geqslant 1$. Again, let $\eta, \chi \in \Im$ be such that $\chi = g(\eta)$, thus we have $\chi = \ln(1 + \eta/7)$. Since

$$\begin{aligned} g(\eta) &= \ln\left(1 + \frac{\eta}{7}\right) \geqslant \ln\left(1 + \frac{\ln(1 + \frac{\eta}{7})}{6}\right) = \ln\left(1 + \frac{\chi}{6}\right) \\ &= f(\chi) = fg(\eta), \end{aligned}$$

thus, $\beta(g\eta, fg(\eta)) \geqslant 1$. Hence, (f, g) is a weakly β-admissible pair of mappings. Now for each $\gamma, \eta \in \Im$ with $\gamma \geqslant \eta$ and choosing ξ such that $\xi/(2\mathrm{e}) > 1 + \log_2 s$, we have

$$\begin{aligned} &2\beta(\gamma, \eta)\, d^*\big(f(\gamma), g(\eta)\big) \\ &\quad = 2\mathrm{e}\big|f(\gamma) - g(\eta)\big|^2 = 2\mathrm{e}\left|\ln\left(1 + \frac{\gamma}{6}\right) - \ln\left(1 + \frac{\eta}{7}\right)\right|^2 \\ &\quad \leqslant 2\mathrm{e}\left(\frac{\gamma}{6} - \frac{\eta}{7}\right)^2 \leqslant \frac{2\mathrm{e}}{\xi} d^*(\gamma, \eta) \leqslant \frac{2\mathrm{e}}{\xi} \mathcal{M}_1(\gamma, \eta). \end{aligned}$$

The above inequality can be written as

$$\ln\frac{\xi}{2\mathrm{e}} + \ln\big(2\beta(\gamma, \eta)\, d^*\big(f(\gamma), g(\eta)\big) + 1\big) \leqslant \ln\big(\mathcal{M}_1(\gamma, \eta) + 1\big).$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r + 1)$ for all $r \in \mathbb{R}^+$, then $F \in \mathcal{F}_s^b$ (as shown above). Hence, for all $\gamma, \eta \in \Im$ such that $d^*(f(\gamma), g(\eta)) > 0$, $\tau = \ln(\xi/(2\mathrm{e}))$, we obtain

$$\tau + F\big(s\beta(\gamma, \eta)\, d^*\big(f(\gamma), g(\eta)\big)\big) \leqslant F\big(\mathcal{M}(\gamma, \eta)\big).$$

Thus, the contractive condition (3) is satisfied for all $\gamma, \eta \in \Im$. Hence, all the hypotheses of the Theorem 1 are satisfied, note that f, g have a unique common fixed point $r = 0$.

Corollary 1. *Let $(\Im, d^*, s)$ be a β-complete b-metric space and $f, g : \Im \to \Im$ be self-mappings such that*

$$\begin{aligned} s^3 d^*\big(f(\gamma), g(\eta)\big) \leqslant k\Bigg(\max\Bigg\{ & d^*(\gamma, \eta), d^*\big(\gamma, f(\gamma)\big), d^*\big(\eta, g(\eta)\big), \\ & \frac{d^*(\gamma, g(\eta)) + d^*(\eta, f(\gamma))}{2s}\Bigg\}\Bigg) \end{aligned}$$

for all $\gamma, \eta \in \Im$, $k \in (0, 1/(1 + \log_2 s))$. *If* f *or* g *is continuous, then* f, g *have a unique common fixed point in* $\Im$.

Proof. Set $\beta(\gamma, \eta) = s^2$ for all $\gamma, \eta \in \Im$, and let $\tau > 0$ be such that $k = \mathrm{e}^{-\tau}$. Then, for $F(r) = \ln(r)$, the given inequality reduces to (3). Thus, conclusion follows from Theorem 1. $\square$

Definition 13. Let $(\Im, d^*, s)$ be a b-metric space and $f : \Im \to \Im$ and $\beta : \Im \times \Im \to \mathbb{R}_0^+$ be two mappings. We say the mapping f is a weakly β-admissible if

$$\beta(\gamma, \eta) \geqslant 1 \quad \Longrightarrow \quad \beta\bigl(f(\gamma), f^2(\gamma)\bigr) \geqslant 1, \quad \beta\bigl(f(\eta), f^2(\eta)\bigr) \geqslant 1 \quad \forall \gamma, \eta \in \Im.$$

Definition 14. Let $(\Im, d^*, s)$ be a b-metric space. The mappings $f : \Im \to \Im$ is called an (β, F)-weak contraction if there exist $F \in \mathcal{F}_s^b$, $\tau > 0$ such that

$$\tau + F\bigl(s\beta(\gamma, \eta)\, d^*\bigl(f(\gamma), f(\eta)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_2(\gamma, \eta)\bigr) \quad \forall \gamma, \eta \in \Im$$

whenever $\min\{\beta(\gamma, \eta) d^*(f(\gamma), f(\eta)), \mathcal{M}_2(\gamma, \eta)\} > 0$, where

$$\mathcal{M}_2(\gamma, \eta) = \max\biggl\{d^*(\gamma, \eta), d^*\bigl(\gamma, f(\gamma)\bigr), d^*\bigl(\eta, f(\eta)\bigr), \frac{d^*(\gamma, f(\eta)) + d^*(\eta, f(\gamma))}{2s}\biggr\}.$$

The Corollary 2, generalizes the result given by Mínak et al. [22].

Corollary 2. *Let* $f : \Im \to \Im$ *be a* (β, F)*-weak contraction mapping defined on* β*-complete* b*-metric space* $(\Im, d^*, s)$. *Assume that* $\kappa \in (0, 1/(1 + \log_2 s))$ *and*

(i) f *is a weakly* β*-admissible mapping*;
(ii) *There exists* r_0 *in* $\Im$ *such that* $\beta(r_0, f(r_0)) \geqslant 1$;
(iii) (a) *Either* f *is* β *continuous mapping or*
(b) $\Im$ *is* β*-regular space and the mapping* F *is continuous.*

Then we can construct a sequence $\{r_n\}$ *in* $\Im$ *such that* $r_n \to \upsilon \in \Im$. *If* $\beta(\upsilon, \upsilon) \geqslant 1$, *then* υ *is a fixed point of* f. *In addition, if* ω *is also a fixed point of* f *such that* $\beta(\upsilon, \omega) \geqslant 1$, *then* $\upsilon = \omega$.

Proof. Setting $g = f$ in the proof of Theorem 1, we obtain required result. $\square$

Example 7. Let f be defined as in Example 2. Then it follows that the result given by Mínak et al. [22] is not applicable, nevertheless, f is an F_{d^*}-weak contraction, and hence Corollary 2 follows if we take $\beta(\gamma, \eta) = 1$ for all $\gamma, \eta \in \Im$.

Definition 15. Let $(\Im, d^*, s)$ be a b-metric space. The mappings $f, g : \Im \to \Im$ are called Hardy–Rogers-type (β, F)-contraction if there exist $F \in \mathcal{F}_s^b$, $\tau > 0$ such that

$$\tau + F\bigl(s\beta(\gamma, \eta)\, d^*\bigl(f(\gamma), g(\eta)\bigr)\bigr) \leqslant F\bigl(\mathcal{R}(\gamma, \eta)\bigr) \quad \forall \gamma, \eta \in \Im \tag{10}$$

with $\beta(\gamma,\eta) \geqslant 1$ whenever

$$\min\bigl\{\beta(\gamma,\eta)\, d^*\bigl(f(\gamma), g(\eta)\bigr), \mathcal{R}(\gamma,\eta)\bigr\} > 0,$$

where

$$\begin{aligned}\mathcal{R}(\gamma,\eta) = {} & a_1 d^*(\gamma,\eta) + a_2 d^*\bigl(\gamma, f(\gamma)\bigr) + a_3 d^*\bigl(\eta, g(\eta)\bigr) \\ & + a_4\bigl[d^*\bigl(\gamma, g(\eta)\bigr) + d^*\bigl(f(\gamma),\eta\bigr)\bigr],\end{aligned}$$

where $a_i \geqslant 0$, $i = 1,2,3,4$, such that $a_1 + a_2 + a_3 + 2sa_4 = 1$.

Theorem 2. *Let $f, g : \Im \to \Im$ be a pair of Hardy–Rogers-type (β, F)-contractions defined on β-complete b-metric space $(\Im, d^*, s)$. Assume that $\kappa \in (0, 1/(1+\log_2 s))$ and*

(i) *f, g are weakly β-admissible self-mappings*;
(ii) *There exists r_0 in $\Im$ such that $\beta(r_0, f(r_0)) \geqslant 1$*;
(iii) (a) *Either one of f and g is β-continuous mapping or*
(b) *$\Im$ is β-regular space and the mapping F is continuous.*

Then we can construct a sequence $\{r_n\}$ in $\Im$ such that $r_n \to \upsilon \in \Im$ (say). If $\beta(\upsilon,\upsilon) \geqslant 1$, then υ is a fixed point of f. In addition, if ω is also a fixed point of f such that $\beta(\upsilon,\omega) \geqslant 1$, then $\upsilon = \omega$.

Proof. Since

$$\begin{aligned}\mathcal{R}(\gamma,\eta) &= a_1 d^*(\gamma,\eta) + a_2 d^*\bigl(\gamma, f(\gamma)\bigr) + a_3 d^*\bigl(\eta, g(\eta)\bigr) \\ &\quad + a_4\bigl[d^*\bigl(\gamma, g(\eta)\bigr) + d^*\bigl(\eta, f(\gamma)\bigr)\bigr] \\ &= a_1 d^*(\gamma,\eta) + a_2 d^*\bigl(\gamma, f(\gamma)\bigr) + a_3 d^*\bigl(\eta, g(\eta)\bigr) \\ &\quad + 2sa_4\left[\frac{d^*(\gamma, g(\eta)) + d^*(\eta, f(\gamma))}{2s}\right] \\ &\leqslant a_1\mathcal{M}_1(\gamma,\eta) + a_2\mathcal{M}_1(\gamma,\eta) + a_3\mathcal{M}_1(\gamma,\eta) \\ &\quad + 2sa_4\mathcal{M}_1(\gamma,\eta) \\ &= (a_1 + a_2 + a_3 + 2sa_4)\mathcal{M}_1(\gamma,\eta) = \mathcal{M}_1(\gamma,\eta).\end{aligned}$$

Inequality (10) implies inequality (3), so the proof of Theorem 2 follows from Theorem 1. ☐

4 Consequences

Let $(\Im, d^*, s)$ be a b-metric space, and let $\preccurlyeq$ be a binary relation over $\Im$. Let $\mathcal{M}_1(\gamma,\eta)$ and $\mathcal{R}(\gamma,\eta)$ are defined as in Definition 12 and Definition 15, respectively.

Definition 16. Let f and g be two self-mappings defined on $\Im$ and $\preccurlyeq$ be a binary relation over $\Im$. The mappings f, g are said to be weakly increasing with respect to $\preccurlyeq$ if for all $\gamma,\eta \in \Im$, we have

$$\gamma \preccurlyeq \eta \quad \Longrightarrow \quad f(\gamma) \preccurlyeq gf(\gamma) \quad \text{and} \quad g(\eta) \preccurlyeq fg(\eta).$$

Assume that the mapping $\beta : \Im \times \Im \to \mathbb{R}_0^+$ is defined by

$$\beta(\gamma, \eta) = \begin{cases} s^2 & \text{if } \gamma \preccurlyeq \eta; \\ 0 & \text{otherwise.} \end{cases}$$

We see that Definition 16 is a special case of Definition 7.

Definition 17. The b-metric space $(\Im, d^*, s)$ is said to be regular with respect to $\preccurlyeq$ if for any sequence $\{r_n\} \subset \Im$ such that $r_n \preccurlyeq r_{n+1}$ and $r_n \to r$ as $n \to \infty$, we have $r_n \preccurlyeq r$ for all $n \in \mathbb{N}$.

Now we are able to rewrite Theorems 1 and 2 in the framework of ordered b-metric spaces.

Theorem 3. *Let $f, g : \Im \to \Im$ be two weakly increasing mappings defined on complete ordered b-metric space $(\Im, d^*, s, \preccurlyeq)$. If there exist $F \in \mathcal{F}_s^b$, $\tau > 0$ and $\kappa \in (0, 1/(1 + \log_2 s))$ such that*

$$\tau + F\bigl(s^3 d^*\bigl(f(\gamma), g(\eta)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(\gamma, \eta)\bigr) \quad \forall \gamma, \eta \in \Im,$$

with $\gamma \preccurlyeq \eta$ whenever $\min\{s^2 d^*(f(\gamma), g(\eta)), \mathcal{M}_1(\gamma, \eta)\} > 0$ and following conditions hold:

(i) *There exists $r_0 \in \Im$ such that $r_0 \preccurlyeq f(r_0)$;*
(ii) (a) *Either one of f and g is continuous or*
  (b) *$\Im$ is $\preccurlyeq$-regular space and the mapping F is continuous.*

Then we can construct a sequence $\{r_n\}$ in $\Im$ such that $r_n \to \upsilon \in \Im$ (say), which is a common fixed point of f and g. In addition, if ω is also a common fixed point of the pair (f, g) such that $\upsilon \preccurlyeq \omega$, then $\upsilon = \omega$.

Proof. Define

$$\beta(\gamma, \eta) = \begin{cases} s^2 & \text{if } \gamma \preccurlyeq \eta; \\ 0 & \text{otherwise.} \end{cases}$$

The arguments follow the same lines as in proof of Theorem 1. □

Theorem 4. *Let $f, g : \Im \to \Im$ be two weakly increasing mappings defined on a complete ordered b-metric space $(\Im, d^*, s, \preccurlyeq)$. If there exist $F \in \mathcal{F}_s^b$, $\tau > 0$ and $\kappa \in (0, 1/(1 + \log_2 s))$ such that*

$$\tau + F\bigl(s^3 d^*\bigl(f(\gamma), g(\eta)\bigr)\bigr) \leqslant F\bigl(\mathcal{R}(\gamma, \eta)\bigr) \quad \forall \gamma, \eta \in \Im$$

with $\gamma \preccurlyeq \eta$ whenever $\min\{s^2 d^*(f(\gamma), g(\eta)), \mathcal{R}(\gamma, \eta)\} > 0$. Assume that following conditions hold:

(i) *There exists $r_0 \in \Im$ such that $r_0 \preccurlyeq f(r_0)$;*
(ii) (a) *Either one of f and g is b-continuous or*
  (b) *$\Im$ is $\preccurlyeq$-regular space and the mapping F is continuous.*

Then we can construct a sequence $\{r_n\}$ in $\Im$ such that $r_n \to \upsilon \in \Im$ (say), which is a common fixed point of f and g. In addition, if ω is also a common fixed point of the pair (f, g) such that $\upsilon \preccurlyeq \omega$, then $\upsilon = \omega$.

Proof. Define

$$\beta(\gamma, \eta) = \begin{cases} s^2 & \text{if } \gamma \preccurlyeq \eta; \\ 0 & \text{otherwise.} \end{cases}$$

The arguments follow the same lines as in proof of Theorem 2. □

Recently, some results have appeared in the setting of metric spaces endowed with a graph. The first result in this direction was given by Jachymski [17].

Definition 18. Let f and g be two self-mappings on a graphic b-metric space $(V(G), d^*, s)$. A pair (f, g) is said to be, weakly G-increasing if $(\gamma, \eta) \in E(G)$ implies $(f(\gamma), gf(\gamma)) \in E(G)$ and $(g(\eta), fg(\eta)) \in E(G)$ for all $\gamma, \eta \in V(G)$,

Let $(V(G), d^*, s)$ be a graphic b-metric space, and let

$$\beta(\gamma, \eta) = \begin{cases} s^2 & \text{if } (\gamma, \eta) \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

By this assumption we see that the above definition is a special case of the definition of weak β-admissible mappings.

Definition 19. Let $(V(G), d^*, s)$ be a graphic b-metric space. It is said to be G-complete if and only if every Cauchy sequence $\{r_n\}$ in V(G) such that $(r_n, r_{n+1}) \in E(G)$, converges in V(G).

Definition 20. Let $(V(G), d^*, s)$ be a graphic b-metric space and $T : V(G) \to V(G)$ be a mapping. We say that T is a G-continuous mapping if for given $r \in V(G)$ and sequence $\{r_n\}$,

$$\lim_{n\to\infty} d^*(r_n, r) = 0 \quad \text{and} \quad (r_n, r_{n+1}) \in E(G) \quad \forall n \in \mathbb{N}$$
$$\implies \quad \lim_{n\to\infty} d^*\bigl(T(r_n), T(r)\bigr) = 0.$$

Definition 21. The graphic b- metric space $(V(G), d^*, s)$ is said to be regular if for any sequence $\{r_n\} \subset V(G)$ such that $(r_n, r_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $r_n \to r$ as $n \to \infty$, we have $(r_n, r) \in E(G)$ for all $n \in \mathbb{N}$.

Now we are able to rewrite Theorems 1 and 2 in the framework of the graphic metric space.

Theorem 5. *Let $f, g : V(G) \to V(G)$ be self-mappings defined on G-complete graphic b-metric space $(V(G), d^*, s)$. If there exist $F \in \mathcal{F}_s^b$, $\tau > 0$ and $\kappa \in (0, 1/(1 + \log_2 s))$ such that*

$$\tau + F\bigl(s^3 d^*\bigl(f(\gamma), g(\eta)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(\gamma, \eta)\bigr) \quad \forall \gamma, \eta \in V(G)$$

with $(\gamma, \eta) \in E(G)$ whenever $\min\{s^2 d^*(f(\gamma), g(\eta)), \mathcal{M}_1(\gamma, \eta)\} > 0$ and following conditions hold:

(i) *f, g are weakly G-increasing self-mappings*;
(ii) *There exists $r_0 \in V(G)$ such that $(r_0, f(r_0)) \in E(G)$*;
(iii) (a) *either one of f and g is a G-continuous self-mapping or*
(b) *$V(G)$ is regular space and the mapping F is continuous.*

Then we can construct a sequence $\{r_n\}$ in $V(G)$ such that $r_n \to \upsilon \in V(G)$ (say), which is a common fixed point of f and g. In addition, if ω is also a common fixed point of the pair (f, g) such that $\upsilon \preccurlyeq \omega$, then $\upsilon = \omega$.

Proof. Define

$$\beta(\gamma, \eta) = \begin{cases} s^2 & \text{if } (\gamma, \eta) \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

The arguments follow the same lines as in proof of Theorem 1. $\square$

Theorem 6. *Let $f, g : V(G) \to V(G)$ be self-mappings defined on G-complete graphic b-metric space $(V(G), d^*, s)$. If there exist $F \in \mathcal{F}_s^b$, $\tau > 0$ and $\kappa \in (0, 1/(1 + \log_2 s))$ such that*

$$\tau + F\bigl(s^3 d^*\bigl(f(\gamma), g(\eta)\bigr)\bigr) \leqslant F\bigl(\mathcal{R}(\gamma, \eta)\bigr) \quad \forall \gamma, \eta \in V(G)$$

with $(\gamma, \eta) \in E(G)$ whenever $\min\{s^2 d^*(f(\gamma), g(\eta)), \mathcal{R}(\gamma, \eta)\} > 0$ and following conditions hold:

(i) *f, g are weakly G-increasing self-mappings*;
(ii) *There exists $r_0 \in V(G)$ such that $(r_0, f(r_0)) \in E(G)$*;
(iii) (a) *Either one of f and g is a G-continuous self-mapping or*
(b) *$V(G)$ is regular space and the mapping F is continuous.*

Then we can construct a sequence $\{r_n\}$ in $V(G)$ such that $r_n \to \upsilon \in V(G)$ (say), which is a common fixed point of f and g. In addition, if ω is also a common fixed point of the pair (f, g) such that $\upsilon \preccurlyeq \omega$, then $\upsilon = \omega$.

Proof. Define

$$\beta(\gamma, \eta) = \begin{cases} s^2 & \text{if } (\gamma, \eta) \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

The arguments follow the same lines as in proof of Theorem 2. $\square$

5 Application of Theorem 1

This section contains an existence result, which shows the application of Theorem 1 in establishing the existence of solution to the system of elliptic boundary value problems given below. Let $C(I)$ be the space of all continuous real valued mappings defined on $I = [0, 1]$.

$$\begin{aligned} -\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= H\bigl(t, x(t)\bigr), \quad t \in [0, 1], \qquad x(0) = x(1) = 0, \\ -\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= K\bigl(t, y(t)\bigr), \quad t \in [0, 1], \qquad y(0) = y(1) = 0, \end{aligned} \tag{11}$$

where $H, K : [0,1] \times X \to \mathbb{R}$ are continuous mappings. The Green function associated with (11) is defined by

$$G(t,u) = \begin{cases} t(1-u), & 0 \leqslant t \leqslant u \leqslant 1 \\ u(1-t), & 0 \leqslant u \leqslant t \leqslant 1. \end{cases}$$

Let $X = (C(I), \mathbb{R})$, and define $d^* : X \times X \to [0, \infty)$ by

$$d^*(x,y) = \sup_{t \in I} \left| x(t) - y(t) \right|^2.$$

It is known that (X, d^*, s) is a complete b-metric space with constant $s = 2$. Now, consider the operators $f, g : X \to X$ given by

$$\begin{aligned} f\bigl(x(t)\bigr) &= \int_0^1 G(t,u) H\bigl(u, x(u)\bigr)\,\mathrm{d}u, \\ g\bigl(y(t)\bigr) &= \int_0^1 G(t,u) K\bigl(u, y(u)\bigr)\,\mathrm{d}u \end{aligned} \tag{12}$$

for all $t \in I$. It is remarked that (11) has a common solution if and only if the operator f and g have a common fixed point.

Theorem 7. *Assume that*

(i) *There exist $\tau > 0$ and continuous mappings $H, K : [0,1] \times X \to \mathbb{R}$, $\beta : X \times X \to [1, \infty)$ such that*

$$\left| H\bigl(t, x(t)\bigr) - K\bigl(t, y(t)\bigr) \right|^2 \leqslant \frac{64\mathrm{e}^{-\tau}}{s\beta(x,y)} \mathcal{M}_1\bigl(x(t), y(t)\bigr) \quad \forall t \in I;$$

(ii) *There exists $x_0 \in C(I)$ such that $\beta(x_0, f(x_0)) \geqslant 1$;*

(iii) *$\beta(x,y) \geqslant 1$ implies $\beta(f(x), gf(x)) \geqslant 1$ and $\beta(g(y), fg(y)) \geqslant 1$ for all $x, y \in X$.*

Then (11) *has at least one solution $x^*(\cdot) \in C^2(I)$.*

Proof. We note that $x^*(\cdot) \in C^2(I)$ is a solution of (11) if and only if $x^*(\cdot) \in C(I)$ is a common solution of the integral equations (12).

Let $x(\cdot), y(\cdot) \in C(I)$. By assumption (i) we get

$$\begin{aligned} \left| f\bigl(x(t)\bigr) - g\bigl(y(t)\bigr) \right|^2 &= \left[\left| \int_0^1 G(t,u) \bigl[H\bigl(u, x(u)\bigr) - K\bigl(u, y(u)\bigr) \bigr]\,\mathrm{d}u \right| \right]^2 \\ &\leqslant \left[\int_0^1 G(t,u) \left| H\bigl(u, x(u)\bigr) - K\bigl(u, y(u)\bigr) \right| \mathrm{d}u \right]^2 \end{aligned}$$

$$\leqslant \left[\int_0^1 G(t,u) \sqrt{64 \frac{\mathrm{e}^{-\tau}}{s\beta(x,y)} \mathcal{M}_1\big(x(u), y(u)\big)}\, \mathrm{d}u \right]^2$$
$$\leqslant \left[8 \int_0^1 G(t,u) \sqrt{\frac{\mathrm{e}^{-\tau}}{s\beta(x,y)} \mathcal{M}_1\big(x(u), y(u)\big)}\, \mathrm{d}u \right]^2$$
$$\leqslant \frac{8^2 \mathrm{e}^{-\tau}}{s\beta(x,y)} \sup_{u\in I} \mathcal{M}_1\big(x(u), y(u)\big) \left[\int_0^1 G(t,u)\, \mathrm{d}u \right]^2. \tag{13}$$

Let $\mathcal{M}_1(x,y)$ be defined as in Definition 12. Then it can easily be proved that $\mathcal{M}_1(x,y) = \sup_{u\in I} \mathcal{M}_1(x(u), y(u))$. Since $\int_0^1 G(t,u)\, \mathrm{d}u = -t^2/2 + t/2$ for all $t \in I$, we have

$$\sup_{t\in I} \left[\int_0^1 G(t,u)\, \mathrm{d}u \right]^2 = \frac{1}{8^2}.$$

Taking the supremum over t in inequality (13), we get

$$d^*\big(f(x), g(y)\big) \leqslant \frac{\mathrm{e}^{-\tau}}{s\beta(x,y)} \mathcal{M}_1(x,y) \quad \forall x, y \in C(I).$$

Consequently, we have

$$\beta(x,y)\, d^*\big(f(x), g(y)\big) \leqslant \mathrm{e}^{-\tau} \mathcal{M}_1(x,y).$$

Thus, for $F(r) = \ln(r)$, all assumptions of Theorem 1 are satisfied. Hence f and g have a common fixed point $x^*(\cdot) \in C(I)$, that is, $fx^*(t) = gx^*(t) = x^*(t)$, which shows that system (11) have a solution. □

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