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Existence theory for nonlocal boundary value problems involving mixed fractional derivatives*

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Abstract. In this paper, we develop the existence theory for a new kind of nonlocal three-point boundary value problems for differential equations and inclusions involving both left Caputo and right Riemann–Liouville fractional derivatives. The Banach and Krasnoselskii fixed point theorems and the Leray–Schauder nonlinear alternative are used to obtain the desired results for the single-valued problem. The existence of solutions for the multivalued problem concerning the upper semicontinuous and Lipschitz cases is proved by applying nonlinear alternative for Kakutani maps and Covitz and Nadler fixed point theorem. Examples illustrating the main results are also presented.

Keywords: fractional differential equations, fractional differential inclusion, fractional derivative, boundary value problem, existence, fixed point theorems.

1 Introduction

Fractional differential equations and inclusions involving different kinds of fractional derivatives (Caputo, Riemann–Liouville, Hadamard to name a few) supplemented with a variety of boundary conditions have been investigated by many researchers, and one can find many interesting results on the topic in the related literature. For examples and details, we refer the reader to a series of articles [1–5, 9, 13, 21, 22] and the references cited therein. However, there are fewer results on boundary value problems of fractional-order differential equations involving both right and left fractional derivatives. It is imperative to mention that fractional differential equations containing left and right Riemann–Liouville fractional derivatives appear as the Euler–Lagrange equations in the study of variational principles, for details, see [6] and the references cited therein.

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In [23], the existence of an extremal solution to a nonlinear system with the right-handed Riemann–Liouville fractional derivative was discussed. In [15], the authors studied the existence of solutions for a nonlinear higher-order fractional boundary value problem involving both the left Riemann–Liouville and the right Caputo fractional derivatives:

$$
\begin{aligned}
&(-1)^m\, {}^C D_{1-}^{\alpha} D_{0+}^{\beta} + f\bigl(t, u(t)\bigr) = 0, \quad 0 \leqslant t \leqslant 1,\\
&u(0) = u^{(i)}(0) = 0, \quad i = 1, \dots, m+n-2, \qquad D_{0+}^{\beta+m-1} u(1) = 0,
\end{aligned}
$$

where ${}^C D_{1-}^{\alpha}$ and D_{0+}^{β} respectively denote the left Caputo fractional derivative of order $\alpha \in (m-1, m)$ and the right Riemann–Liouville fractional derivative of order $\beta \in (n-1, n)$, $m, n \geqslant 2$, are integers. In [19], the authors proved the existence of solutions for the following boundary value problem involving both left Caputo and right Riemann–Liouville fractional derivatives:

$$
\begin{aligned}
&-{}^C D_{1-}^{\alpha} D_{0+}^{\beta} y(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 \leqslant t \leqslant 1,\\
&u(0) = u'(0) = u(1) = 0,
\end{aligned}
$$

where ${}^C D_{1-}^{\alpha}$ and D_{0+}^{β} denote the left Caputo fractional derivative of order $\alpha \in (0, 1]$ and the right Riemann–Liouville fractional derivative of order $\beta \in (1, 2]$. The existence of solutions for a nonlinear fractional oscillator equation with both left Riemann–Liouville and right Caputo fractional derivatives was studied in [12]. To the best of our knowledge, the study of nonlocal boundary value problems involving mixed fractional-order derivatives is yet to be initiated.

In this paper, we introduce a new class of nonlocal boundary value problems (BVP for short) of mixed fractional differential equations and inclusions involving both left Caputo and right Riemann–Liouville fractional derivatives and obtain some existence and uniqueness results for the problems at hand. In precise terms, we investigate the problems

$$
\begin{aligned}
&{}^C D_{1-}^{\alpha} D_{0+}^{\beta} y(t) = f\bigl(t, y(t)\bigr), \quad t \in J := [0, 1],\\
&y(0) = y'(0) = 0, \qquad y(1) = \delta y(\eta), \quad 0 < \eta < 1,
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
&D_{1-}^{\alpha} D_{0+}^{\beta} y(t) \in F\bigl(t, y(t)\bigr), \quad t \in J := [0, 1],\\
&y(0) = y'(0) = 0, \qquad y(1) = \delta y(\eta), \quad 0 < \eta < 1,
\end{aligned}
\tag{2}
$$

where ${}^C D_{1-}^{\alpha}$ and D_{0+}^{β} denote the left Caputo fractional derivative of order $\alpha \in (1, 2]$ and the right Riemann–Liouville fractional derivative of order $\beta \in (0, 1]$, respectively, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\delta \in \mathbb{R}$ is an appropriate constant.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and prove a basic result that plays a key role in the forthcoming analysis. Section 3 contains the existence and uniqueness results for problem (1), which rely on fixed point theorems due to Banach, Krasnoselskii and Leray–Schauder nonlinear alternative. Section 4 deals with the existence results for the multivalued problem, concerning the upper semicontinuous and Lipschitz cases, which are based on nonlinear alternative for Kakutani maps and Covitz and Nadler fixed point theorem for multivalued maps. Illustrative examples for the obtained results are also presented. Though the tools of the fixed point theory employed in the present analysis are the standard ones, their exposition is proved to be of substantial value in achieving the desired results.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts [16] that we need in the sequel.

Definition 1. We define the left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ as

$$I_{0+}^{\alpha} g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,\mathrm{d}s,$$
$$I_{1-}^{\alpha} g(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,\mathrm{d}s, \tag{3}$$

provided the right-hand sides are point-wise defined on $(0, \infty)$, where Γ is the gamma function.

Definition 2. The left Riemann–Liouville fractional derivative and the right Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : (0, \infty) \to \mathbb{R}$ such that $g \in C^n((0, \infty), \mathbb{R})$ are respectively given by

$$D_{0+}^{\alpha} g(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(I_{0+}^{n-\alpha} g\big)(t),$$
$${}^{C}D_{1-}^{\alpha} g(t) = (-1)^n I_{1-}^{n-\alpha} g^{(n)}(t),$$

where $n - 1 < \alpha < n$.

The following lemma, dealing with a linear variant of problem (1), plays an important role in the forthcoming analysis.

Lemma 1. *Let $h \in C(J, \mathbb{R})$ and $\delta \neq \eta^{-(\beta+1)}$. The function y is a solution of the problem*

$${}^{C}D_{1-}^{\alpha} D_{0+}^{\beta} y(t) = h(t), \quad t \in J := [0, 1],$$
$$y(0) = y'(0) = 0, \qquad y(1) = \delta y(\eta), \quad 0 < \eta < 1,$$

if and only if

$$y(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} I_{1-}^{\alpha} h(s)\,\mathrm{d}s + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})\Gamma(\beta)}\Bigg[\delta\int_0^{\eta}(\eta-s)^{\beta-1} I_{1-}^{\alpha} h(s)\,\mathrm{d}s - \int_0^1 (1-s)^{\beta-1} I_{1-}^{\alpha} h(s)\,\mathrm{d}s\Bigg], \tag{4}$$

where $I_{1-}^{\alpha} y(s)$ is defined by (3).

Proof. We first apply the right fractional integral I_{1-}^{α} to the equation ${}^{C}D_{1-}^{\alpha} D_{0+}^{\beta} y(t) = h(t)$ and then the left fractional integral I_{0+}^{β} to the resulting equation, and using the properties of Caputo and Riemann–Liouville fractional derivatives, we get

$$\begin{aligned} y(t) &= I_{0+}^{\beta}\big(I_{1-}^{\alpha} h(t) + c_0 + c_1 t\big) + c_2 t^{\beta-1} \\ &= I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + c_0\frac{t^{\beta}}{\Gamma(\beta+1)} + c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)} + c_2 t^{\beta-1}. \end{aligned} \tag{5}$$

Inserting the conditions $y(0)=0$ and $y'(0)=0$ in (5) yields $c_2=0$ and $c_0=0$, respectively, and consequently, (5) reduces to

$$y(t) = I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)}. \tag{6}$$

Making use of the condition $y(1)=\delta y(\eta)$ in equation (6) yields

$$c_1 = \frac{\Gamma(\beta+2)}{1-\delta\eta^{\beta+1}}\big[\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=\eta} - I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=1}\big],$$

which, on substituting in (6), completes the solution (4). The converse follows by direct computation. The proof is completed. □

3 Existence and uniqueness results for problem (1)

Let $\mathcal{X} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ equipped with the norm $\|y\| = \sup\{|y(t)| : t\in[0,1]\}$.

By Lemma 1, problem (1) can be transformed into a fixed point problem as

$$y = \mathcal{G}y,$$

where the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned}\mathcal{G}y(t) = {}& \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} I_{1-}^{\alpha} f\bigl(s, y(s)\bigr)\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})\Gamma(\beta)} \Biggl[\delta \int_0^{\eta} (\eta-s)^{\beta-1} I_{1-}^{\alpha} f\bigl(s, y(s)\bigr)\,\mathrm{d}s \\ & - \int_0^1 (1-s)^{\beta-1} I_{1-}^{\alpha} f\bigl(s, y(s)\bigr)\,\mathrm{d}s\Biggr]. \end{aligned} \tag{7}$$

Remark 1. The operator (7) can be written as

$$\begin{aligned}\mathcal{G}y(t) = {}& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u, y(u)\bigr)\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u, y(u)\bigr)\,\mathrm{d}u\,\mathrm{d}s \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u, y(u)\bigr)\,\mathrm{d}u\,\mathrm{d}s\Biggr]. \end{aligned}$$

Note that

$$\begin{aligned} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\ & \quad = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(u-s)^{\alpha}}{\Gamma(\alpha+1)}\Big|_s^1\,\mathrm{d}s = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\,\mathrm{d}s \\ & \quad \leqslant \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}\,\mathrm{d}s \quad \bigl((1-s)^{\alpha} \leqslant 1,\ 1 < \alpha \leqslant 2\bigr) \\ & \quad = \frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}. \end{aligned}$$

Thus we have the following estimate.

Lemma 2. *Let $\|f\| = \sup_{t\in[0,1]} |f(t, y(t))|$. Then we have $\|y\| \leqslant \|f\|\varOmega_1$, where*

$$\varOmega_1 = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \biggl[1 + \frac{|\delta|\eta^{\beta}+1}{|1-\delta\eta^{\beta+1}|}\biggr]. \tag{8}$$

3.1 Uniqueness result

Our first result deals with the existence and uniqueness of solutions for problem (1).

Theorem 1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition*

(H1) $|f(t,x) - f(t,y)| \leqslant L|x-y|$ *for all* $t \in [0,1]$, $x, y \in \mathbb{R}$, $L > 0$.

Then problem (1) *has a unique solution on* $[0,1]$ *if*

$$L\Omega_1 < 1,$$

where Ω_1 *is defined by* (8).

Proof. Let us define $\sup_{t\in[0,1]} |f(t,0)| = M$ and select $r \geqslant M\Omega_1/(1-L\Omega_1)$ to establish that $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{y \in \mathcal{X}\colon \|y\| \leqslant r\}$ and $\mathcal{G}$ is defined by (7). Using condition (H1), we have

$$\begin{aligned}\big|f(t,y)\big| &= \big|f(t,y) - f(t,0) + f(t,0)\big| \leqslant \big|f(t,y) - f(t,0)\big| + \big|f(t,0)\big| \\ &\leqslant L\|y\| + M \leqslant Lr + M.\end{aligned}$$

Then, for $y \in \mathcal{B}_r$, by using Lemma 2, we obtain

$$\begin{aligned}\|\mathcal{G}y\| \leqslant{}& \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,y(u)\big)\big|\,\mathrm{d}u\,\mathrm{d}s \\ &+ \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|}\Bigg[|\delta|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,y(u)\big)\big|\,\mathrm{d}u\,\mathrm{d}s \\ &+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,y(u)\big)\big|\,\mathrm{d}u\,\mathrm{d}s\Bigg]\Bigg\} \\ \leqslant{}& (Lr+M)\Omega_1 < r.\end{aligned}$$

This shows that $\mathcal{G}y \in \mathcal{B}_r$, $y \in \mathcal{B}_r$. Thus $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$. Next, we show that $\mathcal{G}$ is a contraction. For that, let $y, z \in \mathcal{X}$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned}&\big\|(\mathcal{G}y) - (\mathcal{G}z)\big\| \\ &\quad\leqslant \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,y(u)\big)-f\big(u,z(u)\big)\big|\,\mathrm{d}u\,\mathrm{d}s \\ &\qquad+ \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|}\Bigg[|\delta|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,y(u)\big)-f\big(u,z(u)\big)\big|\,\mathrm{d}u\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f\big(u, y(u)\big) - f\big(u, z(u)\big) \right| \mathrm{d}u \, \mathrm{d}s \Bigg] \Bigg\} \\
&\leqslant L\Omega_1 \|y - z\|,
\end{aligned}
$$

which, in view of the given condition $L\Omega_1 < 1$, implies that $\mathcal{G}$ is a contraction. In consequence, it follow by the contraction mapping principle that there exists a unique solution for problem (1) on $[0, 1]$. This completes the proof. □

3.2 Existence results

Our next existence result for problem (1) is based on Krasnoselskii fixed point theorem [18].

Theorem 2. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying condition* (H1). *In addition, we assume that*:

(H2) $|f(t, y)| \leqslant m(t)$ *for all* $(t, y) \in [0, 1] \times \mathbb{R}$ *and* $m \in C([0, 1], \mathbb{R}^+)$.

Then there exists at least one solution for problem (1) *on* $[0, 1]$ *if*

$$
\frac{L}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1. \tag{9}
$$

Proof. Setting $\sup_{t \in [0,1]} |m(t)| = \|m\|$, we fix

$$
\varrho \geqslant \|m\| \Omega_1, \tag{10}
$$

where Ω_1 is defined by (8), and consider $B_\varrho = \{y \in \mathcal{X}\colon \|y\| \leqslant \varrho\}$. Introduce the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on B_ϱ as follows:

$$
\mathcal{G}_1 y(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, y(u)\big) \, \mathrm{d}u \, \mathrm{d}s
$$

and

$$
\begin{aligned}
\mathcal{G}_2 y(t) = \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Bigg[&\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, y(u)\big) \, \mathrm{d}u \, \mathrm{d}s \\
&- \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, y(u)\big) \, \mathrm{d}u \, \mathrm{d}s \Bigg].
\end{aligned}
$$

Observe that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. Now we verify the hypotheses of Krasnoselskii fixed point theorem in the following steps.

(i) For $y, z \in B_\varrho$, we have

$$
\begin{aligned}
&\|\mathcal{G}_1 y + \mathcal{G}_2 z\| \\
&\quad = \sup_{t\in[0,1]} \bigl|(\mathcal{G}_1 y)(t) + (\mathcal{G}_2 z)(t)\bigr| \\
&\quad \leqslant \sup_{t\in[0,1]} \Biggl\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u, y(u)\bigr)\bigr| \,\mathrm{d}u \,\mathrm{d}s \\
&\qquad + \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|} \Biggl[|\delta| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u, z(u)\bigr)\bigr| \,\mathrm{d}u \,\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u, z(u)\bigr)\bigr| \,\mathrm{d}u \,\mathrm{d}s \Biggr] \Biggr\} \\
&\quad \leqslant \|m\| \varOmega_1 \leqslant \varrho,
\end{aligned}
$$

where we have used (10). Thus $\mathcal{G}_1 y + \mathcal{G}_2 z \in B_\varrho$.

(ii) It is easy to show that $\mathcal{G}_1$ is a contraction by using assumption (H1) together with (9).

(iii) Using the continuity of f, it is easy to show that the operator $\mathcal{G}_2$ is continuous. Further, $\mathcal{G}_2$ is uniformly bounded on B_ϱ as

$$
\|\mathcal{G}_2 x\| = \sup_{t\in[0,1]} \bigl|(\mathcal{G}_2 y)(t)\bigr| \leqslant \frac{\|m\|(|\delta|\eta^\beta + 1)}{|1-\delta\eta^{\beta+1}|\Gamma(\alpha+1)\Gamma(\beta+1)}.
$$

In order to establish that $\mathcal{G}_2$ is compact, we define $\sup_{(t,y)\in[0,1]\times B_\varrho} |f(t,y)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$
\begin{aligned}
&\bigl|(\mathcal{G}_2 y)(t_2) - (\mathcal{G}_1 2y)(t_1)\bigr| \\
&\quad \leqslant \frac{\overline{f}|t_2^{\beta+1} - t_1^{\beta+1}|}{|1-\delta\eta^{\beta+1}|} \Biggl[|\delta| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \,\mathrm{d}u \,\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \,\mathrm{d}u \,\mathrm{d}s \Biggr] \\
&\quad \leqslant \frac{\overline{f}|t_2^{\beta+1} - t_1^{\beta+1}|}{|1-\delta\eta^{\beta+1}|} \biggl[\frac{|\delta|\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \biggr] \to 0
\end{aligned}
$$

as $t_1 \to t_2$ independently of y. This shows that $\mathcal{G}_2$ is relatively compact on B_ϱ. As all the conditions of the Arzelà–Ascoli theorem are satisfied, so $\mathcal{G}_2$ is compact on B_ϱ. In view of steps (i)–(iii), the conclusion of Krasnoselskii fixed point theorem is applied, and hence, there exists at least one solution for problem (1) on $[0,1]$. The proof is finished. □

Remark 2. Interchanging the role of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the foregoing result, we can obtain a second result by requiring the condition $L(|\delta|\eta^\beta+1)/(\Gamma(\alpha+1)\Gamma(\beta+1)) < 1$, instead of (9).

The following existence result is based on Leray–Schauder nonlinear alternative.

Theorem 3. *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that*:

(H3) *There exist a function* $g \in C([0,1], \mathbb{R}^+)$*, and a nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,y)| \leqslant g(t)\psi(|y|)$ *for all* $(t,y) \in [0,1] \times \mathbb{R}$;

(H4) *There exists a constant* $K > 0$ *such that*

$$\frac{K}{\|g\|\psi(K)\Omega_1} > 1.$$

Then problem (1) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ defined by (7). We show that $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{X} = C([0,1], \mathbb{R})$. For a positive number r, let $\mathcal{B}_r = \{y \in C([0,1], \mathbb{R})\colon \|y\| \leqslant r\}$ be a bounded set in $\mathcal{X}$. Then, by using the fact that $(p-s)^{\alpha-1} \leqslant 1$ $(1 < \alpha \leqslant 2)$, we have

$$\begin{aligned}
\bigl|\mathcal{G}y(t)\bigr| &\leqslant \|g\|\psi(r)\Biggl\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\
&\quad + \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|}\Biggl[|\delta|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s\Biggr]\Biggr\} \\
&\leqslant \|g\|\psi(r)\Omega_1,
\end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields $\|\mathcal{G}y\| \leqslant \|g\|\psi(r)\Omega_1$.

Next, we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $\mathcal{X}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of $\mathcal{X}$. Then, using the fact that $(p-s)^{\alpha-1} \leqslant 1$ $(1 < \alpha \leqslant 2)$, we obtain

$$\begin{aligned}
&\bigl|\mathcal{G}y(t_2) - \mathcal{G}y(t_1)\bigr| \\
&\quad \leqslant \|g\|\psi(r)\Biggl\{\Biggl|\int_0^{t_1} \frac{[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]}{\Gamma(\beta)\Gamma(\alpha+1)}\,\mathrm{d}s + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}\,\mathrm{d}s\Biggr| \\
&\qquad + \frac{|t_2^{\beta+1} - t_1^{\beta+1}|}{|1-\delta\eta^{\beta+1}|}\Biggl[|\delta|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}$$

$$+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s\biggr]\biggr\}$$
$$\leqslant \|g\|\psi(r)\biggl\{\frac{2(t_2-t_1)^\beta + t_2^\beta - t_1^\beta}{\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{|t_2^{\beta+1}-t_1^{\beta+1}|}{|1-\delta\eta^{\beta+1}|}\biggl[\frac{|\delta|\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\biggr]\biggr\},$$

which tends to zero independently of $y \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As $\mathcal{G}$ satisfies the above assumptions, therefore it follows by the Arzelà–Ascoli theorem that $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is completely continuous.

The result will follow from the Leray–Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $y = \lambda\mathcal{G}y$ is bounded for $\lambda \in [0,1]$. For that, let y be a solution of $y = \lambda\mathcal{G}y$ for $\lambda \in [0,1]$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
\bigl|y(t)\bigr| &= \bigl|\lambda\mathcal{G}y(t)\bigr| \\
&\leqslant g(t)\psi\bigl(\|y\|\bigr)\biggl\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\
&\quad + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})}\biggl[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s\biggr]\biggr\} \\
&\leqslant \|g\|\psi\bigl(\|y\|\bigr)\varOmega_1,
\end{aligned}$$

which implies that

$$\frac{\|y\|}{\|g\|\psi(\|y\|)\varOmega_1} \leqslant 1.$$

In view of (H4), there is no solution y such that $\|y\| \neq K$. Let us set

$$U = \bigl\{y \in \mathcal{X}\colon\ \|y\| < K\bigr\}.$$

The operator $\mathcal{G} : \overline{U} \to \mathcal{X}$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda\mathcal{G}(y)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [11] we deduce that $\mathcal{G}$ has a fixed point $u \in \overline{U}$, which is a solution of problem (1). This completes the proof. □

3.3 Examples

In this subsection, we construct examples for the illustration of the results obtained in the last section. For that, we consider the following problem:

$$\begin{aligned}
&{}^{C}D_{1-}^{3/2}D_{0+}^{1/2}y(t) = f\bigl(t,y(t)\bigr), \quad t \in J := [0,1], \\
&y(0) = y'(0) = 0, \qquad y(1) = 2y\biggl(\frac{3}{4}\biggr).
\end{aligned} \tag{11}$$

Here $\alpha = 3/2$, $\beta = 1/2$, $\eta = 3/4$, $\delta = 2$ ($\delta \neq 1/\eta^{\beta+1}$), and

$$f(t, y) = \frac{\mathrm{e}^t}{51} \tan^{-1} y + \frac{|y|}{(t^2 + 51)(1 + |y|)} + \frac{1}{\sqrt{t^2 + 1}}.$$

With the given value of the parameters, it is found that

$$\Omega_1 = \frac{8}{3\pi} \left[1 + \frac{\sqrt{3} + 1}{|1 - \frac{3\sqrt{3}}{4}|} \right] \approx 8.84,$$

$|f(t, y_1) - f(t, y_2)| \leqslant L|y_1 - y_2|$, $L = (\mathrm{e} + 1)/51$, and $L\Omega_1 \approx 0.643 < 1$. Clearly, all the assumptions of Theorem 1 hold true, and consequently, its conclusion can be applied to problem (11).

In order to illustrate Theorem 2, we notice that (9) is satisfied as

$$\frac{L}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} = \frac{8(\mathrm{e} + 1)}{153\pi} \approx 0.061 < 1$$

and

$$\left| f(t, y) \right| \leqslant m(t) = \frac{\pi \mathrm{e}^t}{102} + \frac{1}{(t^2 + 9)} + \frac{1}{\sqrt{t^2 + 1}}.$$

As the hypothesis of Theorem 2 is satisfied, we deduce from the conclusion of Theorem 2 that problem (11) has at least one solution on $[0, 1]$.

Now we demonstrate the application of Theorem 3 by considering the nonlinear function

$$f(t, y) = \frac{1}{8\sqrt{t^2 + 4}} (\sin y + \cos y + 3). \tag{12}$$

Clearly, $|f(t, y)| \leqslant g(t)\psi(\|y\|)$, where $g(t) = 1/(8\sqrt{t^2 + 4})$, $\psi(\|y\|) = (4 + \|y\|)$. By condition (H4) we find that $K > 5.072625$. Thus all the conditions of Theorem 3 hold true, and consequently, problem (11) with $f(t, y)$ given by (12) has at least one solution on $[0, 1]$.

4 Existence results for problem (2)

Before presenting the existence results for problem (2), we outline the necessary concepts on multivalued maps [10, 14].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is closed}\}$, $\mathcal{P}_{\mathrm{b}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is bounded}\}$, $\mathcal{P}_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is compact}\}$, and $\mathcal{P}_{\mathrm{cp,c}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is compact and convex}\}$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$ (i.e., $\sup_{x \in \mathbb{B}}\{\sup\{|y|\colon y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply that $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\mathrm{Fix}G$. A multivalued map $G : [0,1] \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, G(t)) = \inf\{|y - z|\colon z \in G(t)\}$ is measurable.

For each $y \in \mathcal{X}$, define the set of selections of F by

$$S_{F,y} := \bigl\{v \in L^1\bigl([0,1], \mathbb{R}\bigr)\colon v(t) \in F\bigl(t, y(t)\bigr) \text{ for a.e. } t \in [0,1]\bigr\}.$$

Definition 3. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(i) $t \to F(t, y)$ is measurable for each $y \in \mathbb{R}$;
(ii) $y \to F(t, y)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1], \mathbb{R}^+)$ such that $\|F(t,y)\| = \sup\{|v|\colon v \in F(t,y)\} \leqslant \varphi_\rho(t)$ for all $y \in \mathbb{R}$ with $\|y\| \leqslant \rho$ and for a.e. $t \in [0,1]$.

We define the graph of G to be the set $\mathrm{Gr}(G) = \{(x,y) \in X \times Y\colon y \in G(x)\}$ and recall two results for closed graphs and upper semicontinuity.

Lemma 3. (*See* [10, *Prop.* 1.2].) *If $G : X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is u.s.c., then $\mathrm{Gr}(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 4. (*See* [20].) *Let X be a separable Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{\mathrm{cp,c}}(X)$ be an L^1-Carathéodory multivalued map, and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator $\Theta \circ S_{F,x} : C([0,1], X) \to \mathcal{P}_{\mathrm{cp,c}}(C([0,1], X))$,*

$$(\Theta \circ S_{F,y})(y) = \Theta(S_{F,y}),$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

For the forthcoming analysis, we need the following lemma.

Lemma 5 [Nonlinear alternative for Kakutani maps]. (*See* [11].) *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{\mathrm{cp,c}}(C)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ defined by

$$H_d(A, B) = \max\Big\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(A, b)\Big\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space, and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [17]).

Definition 4. A multivalued operator $N : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called:

(i) γ-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leqslant \gamma d(x, y)$ for each $x, y \in X$;
(ii) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 6. (*See* [8].) *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then* $\mathrm{Fix}N \neq \emptyset$.

Definition 5. A function $y \in C([0, 1], \mathbb{R})$ is said to be a solution of the boundary value problem (2) if $y(0) = y'(0) = 0$, $y(1) = \delta y(\eta)$, and there exists a function $v \in S_{F,y}$ such that $v(t) \in F(t, y(t))$ and

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s\Bigg]. \end{aligned}$$

4.1 The upper semicontinuous case

In the case when F has convex values, we prove an existence result based on nonlinear alternative of Leray–Schauder type.

Theorem 4. *Assume that*:

(H1) *$F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values*;
(H2) *there exist a function $\phi \in C([0, 1], \mathbb{R}^+)$ and a nondecreasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|F(t, y)\|_{\mathcal{P}} := \sup\{|w|{:}\ w \in F(t, y)\} \leqslant \phi(t)\Omega(|y|)$ for each $(t, y) \in [0, 1] \times \mathbb{R}$*;
(H3) *there exists a constant $M > 0$ such that $M/(\|\phi\|\Omega_1\Omega(M)) > 1$, where Ω_1 is defined by* (8).

Then the boundary value problem (2) *has at least one solution on* $[0, 1]$.

Proof. Define an operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by $\Omega_F(y) = \{h(t)\colon\ h \in \mathcal{X}\}$, where

$$h(t) = \Biggl\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \Biggr] \Biggr\}$$

for $v \in S_{F,y}$. We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each* $y \in C([0,1],\mathbb{R})$. This step is obvious since $S_{F,y}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets* (*balls*) *into bounded sets in* $\mathcal{X}$. For a positive number ρ, let $B_\rho = \{y \in \mathcal{X}\colon\ \|y\| \leqslant \rho\}$ be a bounded ball in $\mathcal{X}$. Then, for each $h \in \Omega_F(y)$, $y \in B_\rho$, there exists $v \in S_{F,y}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \Biggr].$$

Then, by using the fact that $(p-s)^{\alpha-1} \leqslant 1$ $(1 < \alpha \leqslant 2)$, we have

$$\begin{aligned} \bigl|h(t)\bigr| &\leqslant \|\phi\|\Omega\bigl(\|y\|\bigr) \Biggl\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\ &\quad + \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|} \Biggl[|\delta| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}u\,\mathrm{d}s \Biggr] \Biggr\} \\ &\leqslant \|\phi\|\Omega\bigl(\|y\|\bigr)\Omega_1, \end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|h\| \leqslant \|\phi\| \Omega(r) \Omega_1.$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $\mathcal{X}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in B_\rho$. For each $h \in \Omega_F(y)$, using the fact that $(p-s)^{\alpha-1} \leqslant 1$ $(1 < \alpha \leqslant 2)$, we obtain

$$\begin{aligned}
&\big|h(t_2) - h(t_1)\big| \\
&\quad \leqslant \|g\| \Omega(r) \Biggl\{ \Biggl| \int_0^{t_1} \frac{[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]}{\Gamma(\beta)\Gamma(\alpha+1)} \,\mathrm{d}s + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)} \,\mathrm{d}s \Biggr| \\
&\qquad + \frac{|t_2^{\beta+1} - t_1^{\beta+1}|}{|1 - \delta\eta^{\beta+1}|} \Biggl[|\delta| \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \,\mathrm{d}u \,\mathrm{d}s \\
&\qquad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \,\mathrm{d}u \,\mathrm{d}s \Biggr] \Biggr\} \\
&\quad \leqslant \|g\| \Omega(r) \Biggl\{ \frac{2(t_2-t_1)^{\beta} + t_2^{\beta} - t_1^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{|t_2^{\beta+1} - t_1^{\beta+1}|}{|1 - \delta\eta^{\beta+1}|} \frac{|\delta|\eta^{\beta} + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \Biggr\},
\end{aligned}$$

which tends to zero independently of $y \in \mathcal{B}_\rho$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above assumptions, therefore it follows by the Arzelà–Ascoli theorem that $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is completely continuous.

In our next step, we show that Ω_F *is upper semicontinuous.* To this end, it is sufficient to show that Ω_F *has a closed graph* by Lemma 3. Let $y_n \to y_*$, $h_n \in \Omega_F(y_n)$, and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(y_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(u) \,\mathrm{d}u \,\mathrm{d}s \\
& + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(u) \,\mathrm{d}u \,\mathrm{d}s \\
& - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(u) \,\mathrm{d}u \,\mathrm{d}s \Biggr].
\end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in [0,1]$,

$$h_*(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(u)\,\mathrm{d}u\,\mathrm{d}s + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})}\Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(u)\,\mathrm{d}u\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(u)\,\mathrm{d}u\,\mathrm{d}s\Biggr].$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to \mathcal{X}$ given by

$$\begin{aligned} v \to \Theta(v)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \\ &\quad + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})}\Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \\ &\quad - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s\Biggr]. \end{aligned}$$

Observe that, as $n \to \infty$,

$$\begin{aligned} &\bigl\|h_n(t) - h_*(t)\bigr\| \\ &= \Biggl\| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} (v_n - v_*)(u)\,\mathrm{d}u\,\mathrm{d}s \\ &\quad + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})}\Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} (v_n - v_*)(u)\,\mathrm{d}u\,\mathrm{d}s \\ &\quad - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} (v_n - v_*)(u)\,\mathrm{d}u\,\mathrm{d}s\Biggr]\Biggr\| \\ &\to 0. \end{aligned}$$

Thus it follows by Lemma 4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$.

Since $y_n \to y_*$, therefore we have

$$\begin{aligned} h_*(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(u)\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(u)\,\mathrm{d}u\,\mathrm{d}s \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(u)\,\mathrm{d}u\,\mathrm{d}s \Biggr] \end{aligned}$$

for some $v_* \in S_{F,y_*}$.

Finally, we show there exists an open set $U \subseteq C([0,1],\mathbb{R})$ with $y \notin \theta\Omega_F(y)$ for any $\theta \in (0,1)$ and all $y \in \partial U$. Let $\theta \in (0,1)$ and $y \in \theta\Omega_F(y)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in [0,1]$, we obtain

$$\begin{aligned} \bigl|y(t)\bigr| &= \bigl|\theta\Omega_F(y)(t)\bigr| \\ &\leqslant \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|\,\mathrm{d}u\,\mathrm{d}s \\ &\quad + \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|} \Biggl[|\delta| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|\,\mathrm{d}u\,\mathrm{d}s \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|v(u)\bigr|\,\mathrm{d}u\,\mathrm{d}s \Biggr] \\ &\leqslant \|\phi\|\Omega\bigl(\|y\|\bigr)\Omega_1, \end{aligned}$$

which implies that $\|y\|/(\|\phi\|\Omega(\|y\|)\Omega_1) \leqslant 1$. In view of (H3), there exists M such that $\|y\| \neq M$. Let us set $U = \{y \in \mathcal{X}\colon \|y\| < M\}$. Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(\mathcal{X})$ is upper semicontinuous and completely continuous. From the choice of U there is no $y \in \partial U$ such that $y \in \theta\Omega_F(y)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 5) we deduce that Ω_F has a fixed point $y \in \overline{U}$, which is a solution of problem (2). This completes the proof. □

4.2 The Lipschitz case

We prove in this subsection the existence of solutions for problem (2) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [8].

Theorem 5. *Assume that*:

(A1) $F : [0,1]\times\mathbb{R} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ *is such that* $F(\cdot, y(t)) : [0,1] \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ *is measurable for each* $y \in \mathbb{R}$;

(A2) $H_d(F(t,y), F(t,\bar{y}) \leqslant q(t)|y-\bar{y}|$ *for almost all* $t \in [0,1]$ *and* $y, \bar{y} \in \mathbb{R}$ *with* $q \in C([0,1],\mathbb{R}^+)$ *and* $d(0, F(t,0)) \leqslant q(t)$ *for almost all* $t \in [0,1]$.

Then problem (2) *has at least one solution on* $[0,1]$ *if*

$$\|q\|\Omega_1 < 1, \tag{13}$$

where Ω_1 *is defined by* (8).

Proof. Consider the operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined in the beginning of the proof of Theorem 4. Observe that the set $S_{F,y}$ is nonempty for each $y \in \mathcal{X}$ by assumption (A1). So F has a measurable selection (see [7, Thm. III.6]). Now we show that the operator Ω_F satisfies the assumptions of Lemma 6. To show that $\Omega_F(y) \in \mathcal{P}_{\mathrm{cl}}(\mathcal{X})$ for each $y \in \mathcal{X}$, let $\{u_n\}_{n\geqslant 0} \in \Omega_F(y)$ be such that $u_n \to u$ as $n \to \infty$ in $\mathcal{X}$. Then $u \in \mathcal{X}$, and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(u)\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(u)\,\mathrm{d}u\,\mathrm{d}s \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(u)\,\mathrm{d}u\,\mathrm{d}s \Biggr]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus $v \in S_{F,y}$, and for each $t \in [0,1]$, we have

$$\begin{aligned} & u_n(t) \to u(t) \\ & \quad = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \\ & \qquad + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})} \Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \\ & \qquad - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)\,\mathrm{d}u\,\mathrm{d}s \Biggr]. \end{aligned}$$

Hence, $u \in \Omega_F(y)$.

Next, we show that there exists $\hat{\theta} := \|q\|\Omega_1 < 1$ such that

$$H_d\bigl(\Omega_F(y), \Omega_F(\bar{y})\bigr) \leqslant \hat{\theta}\|y - \bar{y}\|, \quad y, \bar{y} \in \mathcal{X}.$$

Let $y, \bar{y} \in \mathcal{X}$ and $h_1 \in \Omega_F(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in [0, 1]$,

$$\begin{aligned} h_1(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(u)\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})}\Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(u)\,\mathrm{d}u\,\mathrm{d}s \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(u)\,\mathrm{d}u\,\mathrm{d}s\Biggr]. \end{aligned}$$

By (A2) we have

$$H_d\bigl(F(t, y), F(t, \bar{y})\bigr) \leqslant q(t)|y - \bar{y}|.$$

So, there exists $w \in F(t, \bar{y})$ such that

$$\bigl|v_1(t) - w\bigr| \leqslant (t)\bigl|y(t) - \bar{y}(t)\bigr|, \quad t \in [0, 1].$$

Define $U : [0, 1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \bigl\{w \in \mathbb{R}\colon \bigl|v_1(t) - w\bigr| \leqslant q(t)\bigl|y(t) - \bar{y}(t)\bigr|\bigr\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{y})$ is measurable [7, Prop. III.4], there exists a function $v_2(t)$, which is a measurable selection for $U(t) \cap F(t, \bar{y})$. So $v_2(t) \in F(t, \bar{y})$, and for each $t \in [0, 1]$, we have $|v_1(t) - v_2(t)| \leqslant q(t)|y(t) - \bar{y}(t)|$. For each $t \in [0, 1]$, let us define

$$\begin{aligned} h_2(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(u)\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{t^{\beta+1}}{(1-\delta\eta^{\beta+1})}\Biggl[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(u)\,\mathrm{d}u\,\mathrm{d}s \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(u)\,\mathrm{d}u\,\mathrm{d}s\Biggr]. \end{aligned}$$

Thus

$$\begin{aligned}
&\left|h_1(t) - h_2(t)\right| \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1 - v_2|(u)\,\mathrm{d}u\,\mathrm{d}s \\
&\qquad + \frac{t^{\beta+1}}{|1-\delta\eta^{\beta+1}|}\left[|\delta| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1 - v_2|(u)\,\mathrm{d}u\,\mathrm{d}s\right. \\
&\qquad \left. + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1 - v_2|(u)\,\mathrm{d}u\,\mathrm{d}s\right] \\
&\quad\leqslant \|q\| \Omega_1 \|y - \bar{y}\|,
\end{aligned}$$

which yields $\|h_1 - h_2\| \leqslant \|q\| \Omega_1 \|y - \bar{y}\|$.

Analogously, interchanging the roles of y and $\overline{y}$, we can obtain

$$H_d\big(\Omega_F(y), \Omega_F(\bar{y})\big) \leqslant \|q\| \Omega_1 \|y - \bar{y}\|.$$

By condition (13) it follows that Ω_F is a contraction, and hence, it has a fixed point y by Lemma 6, which is a solution of problem (2). This completes the proof. □

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