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Cyclic (noncyclic) φ-condensing operator and its application to a system of differential equations

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Abstract. We establish a best proximity pair theorem for noncyclic φ-condensing operators in strictly convex Banach spaces by using a measure of noncompactness. We also obtain a counterpart result for cyclic φ-condensing operators in Banach spaces to guarantee the existence of best proximity points, and so, an extension of Darbo's fixed point theorem will be concluded. As an application of our results, we study the existence of a global optimal solution for a system of ordinary differential equations.

Keywords: best proximity pair, noncyclic φ-condensing operator, ordinary differential equations, strictly convex Banach space.

1 Introduction

Let X be a Banach space and $C \subseteq X$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. It is well known that if C is a nonempty, compact and convex subset of a Banach space X, then any nonexpansive mapping of C into C has a fixed point.

Let A and B be two nonempty subsets of a normed linear space X. A mapping $T : A \cup B \to A \cup B$ is called *relatively nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for any $(x, y) \in A \times B$. Also, T is called *noncyclic* whenever $T(A) \subseteq A$, $T(B) \subseteq B$. Obviously, the class of noncyclic relatively nonexpansive mappings contains the class of nonexpansive mappings as a subclass. In fact, noncyclic relatively nonexpansive mappings may not be continuous, necessarily. A point $(p, q) \in A \times B$ is said to be a *best proximity pair* if this point is a solution of the following minimization problem:

$$\min_{x \in A} \|x - Tx\|, \quad \min_{y \in B} \|y - Ty\|, \quad \text{and} \quad \min_{(x,y) \in A \times B} \|x - y\|. \tag{1}$$

Clearly, $(p, q) \in A \times B$ is a solution of problem (1) if and only if

$$p = Tp, \quad q = Tq, \quad \text{and} \quad \|p - q\| = \operatorname{dist}(A, B).$$

In 2005, the following existence theorem of best proximity pairs for noncyclic relatively nonexpansive mappings was established.

Theorem 1. (*See* [8].) *Let A, B be nonempty, compact, and convex subsets of a strictly convex Banach space X. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping, then T has a best proximity pair.*

A counterpart result of Theorem 1 was obtained for *cyclic mappings*. We mention that a mapping $T : A \cup B \to A \cup B$ is said to be a cyclic mapping if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $p \in A \cup B$ is called a *best proximity point* for the mapping T, provided that $\|p - Tp\| = \operatorname{dist}(A, B)$, that is, the point $p \in A \cup B$ is a best proximity point for the mapping T if p is a solution of the following nonlinear minimization problem:

$$\min_{x \in A \cup B} \|x - Tx\|.$$

Next theorem is a cyclic version of Theorem 1 to find best proximity points.

Theorem 2. (*See* [8].) *Let A, B be nonempty, compact, and convex subsets of a Banach space X. If $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping, then T has a best proximity point.*

It is remarkable to note that the strictly convexity condition of the Banach space X is not necessary.

In this article, we introduce new classes of noncyclic (cyclic) mappings, called noncyclic (cyclic) φ-condensing operators, and investigate the existence of best proximity pairs (points) by using a notion of measure of noncompactness. In this way, we show that the results alike to the celebrated *Darbo's fixed point theorem* for condensing mappings can be obtained for cyclic φ-condensing operators. As an application of our main conclusions, we prove the existence of a *global optimal solution* for a system of differential equations.

2 Preliminaries

Let X and Y be Banach spaces and K a subset of X. A mapping $T : K \to Y$ is said to be a *compact operator* if T is continuous and maps bounded sets into relatively compact sets. It is worth noticing that in finite dimensional Banach spaces, continuous mappings defined on closed sets are compact.

The well-know *Schauder's fixed point theorem* states that every compact self-mapping defined on a nonempty, bounded, closed, and convex subset of a Banach space has at least one fixed point. The Schauder's fixed point theorem is a very useful tool for proving the existence of solutions to many nonlinear problems, especially, problems concerning ordinary and partial differential equations.

Here, we recall the notion of measure of noncompactness as below.

Definition 1. Let (X, d) be a complete metric space and Σ the family of bounded subsets of X. A function $\mu : \Sigma \to [0, \infty)$ is called a measure of noncompactness (MNC) if it satisfies the following conditions:

(i) $\mu(A) = 0$ iff A is relatively compact;
(ii) $\mu(A) = \mu(\overline{A})$ for all $A \in \Sigma$;
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$.

If μ is an MNC on Σ, then the following properties will be concluded immediately (see [3] for more information):

(a) If $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$;
(b) $\mu(A \cap B) \leqslant \min\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$;
(c) If A is a finite set, then $\mu(A) = 0$;
(d) If $\{A_n\}$ is a decreasing sequence of nonempty, bounded and closed subsets of X such that $\lim_{n\to\infty} \mu(A_n) = 0$, then $A_\infty := \cap_{n\geqslant 1} A_n$ is nonempty and compact.

Also, if X is a Banach space, then

(e) $\mu(\overline{\mathrm{con}}(A)) = \mu(A)$ for all $A \in \Sigma$;
(f) $\mu(tA) = |t|\mu(A)$, for any number t and $A \in \Sigma$;
(g) $\mu(A + B) \leqslant \mu(A) + \mu(B)$, for all $A, B \in \Sigma$.

From now on, $\mathcal{B}(x; r)$ will denote the closed ball in the Banach space X centered at $x \in X$ with radius $r > 0$. The following proposition provides two well-known examples of MNCs.

Proposition 1. (*See* [3].) *Let (X, d) be a complete metric space. For any $A \in \Sigma$, define*

$$\alpha(A) := \inf\bigl\{\varepsilon > 0\colon\ A \subseteq \cup_{j=1}^{n} E_j,\ E_j \in \Sigma,\ \mathrm{diam}(E_j) \leqslant \varepsilon \ \textit{for all}\ 1 \leqslant j \leqslant n < \infty\bigr\},$$

$$\chi(A) := \inf\bigl\{\varepsilon > 0\colon\ A \subseteq \cup_{j=1}^{n} \mathcal{B}(x_j; r_j),\ x_j \in X,\ r_j \leqslant \varepsilon \ \textit{for all}\ 1 \leqslant j \leqslant n < \infty\bigr\}.$$

Then α and χ are MNC*, which were introduced by Kuratowski and Hausdorff, respectively. Moreover, these two* MNC*s are related by the inequalities*

$$\chi(A) \leqslant \alpha(A) \leqslant 2\chi(A) \quad \textit{for all}\ A \in \Sigma.$$

Definition 2. Let X be a Banach space, A a nonempty subset of X, and μ a measure of noncompactness on X. An operator $T : A \to X$ is said to be an r-condensing operator for some $r \in [0, 1)$ if $\mu(T(K)) \leqslant r\mu(K)$ for all nonempty and bounded $K \subseteq A$.

For example, if A is a subset of a Banach space X and $T, S : A \to X$ are two mappings so that T is compact, and S is a contraction with the contraction constant $r \in [0, 1)$, then the operator $T + S$ is an r-condensing operator. We refer to [2] for some interesting examples about condensing operators.

In 1955, Darbo proved the following fixed point theorem for condensing self-mappings by using the Schauder's fixed point theorem (see also [1, 11] for some generalizations of Darbo's fixed point problem).

Theorem 3. (*See* [4].) *Let A be a nonempty, bounded, closed, and convex subset of a Banach space X and $T : A \to A$ be a continuous mapping, which is an r-condensing operator for some $r \in [0, 1)$. Then T has a fixed point.*

Let A and B be two nonempty subsets of a normed linear space X. We shall say that a pair (A, B) of subsets of a Banach space X satisfies a property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D)$ if and only if $A \subseteq C$, $B \subseteq D$. The *closed and convex hull* of a set A will be denoted by $\overline{\mathrm{con}}(A)$. Also, $\mathrm{diam}(A)$ stands for the diameter of the set A.

The next lemma, due to Mazur, will be used in our main conclusions.

Lemma 1. (*See* [6].) *Let A be a nonempty and compact subset of a Banach space X. Then $\overline{\mathrm{con}}(A)$ is compact.*

For a nonempty pair (A, B) in a normed linear space X, we define

$$A_0 = \{x \in A\colon \text{ there is } y' \in B\colon \|x - y'\| = \mathrm{dist}(A, B)\},$$
$$B_0 = \{y \in B\colon \text{ there is } x' \in A\colon \|x' - y\| = \mathrm{dist}(A, B)\}.$$

We mention that if (A, B) is a nonempty, weakly compact, and convex pair in a Banach space X, then the pair (A_0, B_0) is also nonempty, weakly compact, and convex.

Definition 3. A nonempty pair (A, B) in a normed linear space X is said to be proximinal if $A = A_0$ and $B = B_0$.

Here, we recall the following geometric property of Banach spaces, which will be useful in our main results.

Definition 4. A Banach space X is said to be strictly convex if the following implication holds for $x, y, p \in X$ and $R > 0$:

$$\begin{cases} \|x - p\| \leqslant R, \\ \|y - p\| \leqslant R, \\ x \neq y \end{cases} \implies \left\| \frac{x + y}{2} - p \right\| < R.$$

For example, Hilbert spaces and L^p spaces ($1 < p < \infty$) are strictly convex Banach spaces. Finally, we mention the recent works [5, 7].

3 Main results

3.1 Noncyclic φ-condensing operators

We begin our discussion by recalling the following concept.

Definition 5. (See [9, 14].) Let (A, B) be a nonempty pair in a normed linear space X. A mapping $T : A \cup B \to A \cup B$ is said to be a noncyclic relatively u-continuous mapping if T is noncyclic on $A \cup B$ and, for all $(x, y) \in A \times B$, satisfies the following condition:

$$\text{for all } \varepsilon > 0, \text{ there is } \delta > 0\text{: if } \|x - y\|^* < \delta, \text{ then } \|Tx - Ty\|^* < \varepsilon,$$

where $\|x - y\|^* = \|x - y\| - \operatorname{dist}(A, B)$.

It is clear that the class of noncyclic relatively u-continuous mappings contains the class of noncyclic relatively nonexpansive mappings as a subclass.

The following existence theorem, which is an extension of Theorem 1, was established in [9] and [14] with different approaches.

Theorem 4. *Let (A, B) be nonempty, compact, and convex pair in a strictly convex Banach space X. If $T : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping, then T has a best proximity pair.*

Definition 6. (See [10].) Let (A, B) be a nonempty and bounded pair in a normed linear space X and $T : A \cup B \to A \cup B$ a noncyclic (cyclic) mapping. We say that T is compact whenever both $T|_A$ and $T|_B$ are compact, that is, the pair $(\overline{T(A)}, \overline{T(B)})$ is compact.

Here, we present a generalization of Theorem 4.

Theorem 5. *Let (A, B) be a nonempty, bounded, closed, and convex pair in a strictly convex Banach space X such that A_0 is nonempty. Assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping. If T is compact, then T has a best proximity pair.*

Proof. Put $H_1 = \overline{\operatorname{con}}(\overline{T(A)})$, $H_2 = \overline{\operatorname{con}}(\overline{T(B)})$. Let $x \in A_0$. Then there is a point $y \in B$ so that $\|x - y\| = \operatorname{dist}(A, B)$. By the fact that T is noncyclic relatively u-continuous, $\|Tx - Ty\| = \operatorname{dist}(A, B)$. This implies that $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(A, B)$. Since $\overline{\operatorname{con}}(\overline{T(A)}) \subseteq A$,

$$T(H_1) = T\big(\overline{\operatorname{con}}\big(\overline{T(A)}\big)\big) \subseteq T(A) \subseteq \overline{T(A)} \subseteq \overline{\operatorname{con}}\big(\overline{T(A)}\big) = H_1.$$

By a similar argument, $T(H_2) \subseteq H_2$, that is, T is noncyclic on $H_1 \cup H_2$. Besides, from Lemma 1, (H_1, H_2) is compact and convex in a strictly convex Banach space X. Thus, from Theorem 4, T has a best proximity pair. $\square$

The following example guarantees that Theorem 5 cannot be concluded from Theorem 4.

Example 1. Let us consider $X = (\mathbb{R}^2, \|\cdot\|_2)$ and suppose $A = [0,\infty) \times \{0\}, B = [0,\infty) \times \{1\}$. Then (A,B) is a closed, convex, and unbounded pair. Define the mapping $T : A \cup B \to A \cup B$ with

$$T(x,0) = \left(\frac{x}{x+1}, 0\right), \qquad T(y,1) = \left(\frac{y}{y+1}, 1\right).$$

Then $(\overline{T(A)}, \overline{T(B)}) = ([0,1/2] \times \{0\}, [0,1/2] \times \{1\})$, which implies that T is compact. On the other hand, it is easy to see that T is relatively u-continuous. Thus T has a best proximity pair, which is the point $((0,0),(0,1))$.

Next example shows that the strictly convexity condition of Theorem 5 cannot be dropped.

Example 2. Let X be the real space l_2 renormed as $\|x\| = \max\{\|x\|_2, \sqrt{2}\|x\|_\infty\}$, where $\|x\|_\infty$ denotes the l_∞-norm, and $\|x\|_2$ the l_2 norm. Suppose that $\{e_n\}$ is the canonical basis of l_2. Note that X is not a strictly convex Banach space. Suppose

$$A := \{x = (x_n)\colon\ x_3 = 1,\ \|x\| \leqslant \sqrt{2}\} \quad \text{and} \quad B := \{y := e_1 + e_2\}.$$

Then (A,B) is a bounded, closed, and convex pair in X. Put $u := e_1+e_3$ and $v := e_2+e_3$. Then $\|u - v\| = \|v - y\| = \sqrt{2}$. Also, for each $x = (x_1, x_2, 1, x_4, \dots) \in A$, we have $\|x\|_2 \leqslant \sqrt{2}$, which implies that $\sum_{i\neq 3} |x_i|^2 \leqslant 1$, and by the fact that $\|x\|_\infty \leqslant 1$ we obtain $|x_i| \leqslant 1$ for each $i \in \mathbb{N}$. So, for all $x \in A$, we have $\|x - y\| \geqslant \sqrt{2}$, which implies that $\operatorname{dist}(A,B) = \sqrt{2}$. Now, define the mapping $T : A \cup B \to A \cup B$ with

$$Ty = y \quad \text{and} \quad Tx = \begin{cases} v & \text{if } x = u, \\ u & \text{if } x \neq u \end{cases} \quad \text{for each } x \in A.$$

Then T is noncyclic, and for each $\alpha \in [0,1)$ and $x \in A$, we have

$$\|Tx - Ty\| = \sqrt{2} \leqslant \|x - y\|,$$

that is, T is relatively u-continuous. Moreover, $(\overline{T(A)}, \overline{T(B)}) = (\{u,v\}, \{y\})$, which implies that T is compact. Note that $\operatorname{Prox}_{A\times B}(T) = \emptyset$ since the fixed point set of T in A is empty.

In what follows, let Φ denote the set of all functions $\varphi : [0,\infty) \to [0,1)$ such that

$$\varphi(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0.$$

A self-mapping $T : A \to A$ is said to be a φ-contraction [12], provided that

$$\|Tx - Ty\| \leqslant \varphi\big(\|x - y\|\big)\|x - y\| \quad \text{for all } x, y \in A,$$

where A is nonempty subset of a normed linear space X, and $\varphi \in \Phi$.

Definition 7. Let (A,B) be a nonempty and convex pair in a Banach space X and μ be an MNC on X. A mapping $T: A\cup B \to A\cup B$ is said to be a noncyclic (cyclic) φ-condensing operator for some $\varphi \in \Phi$, provided that for any nonempty, bounded, closed, convex, proximinal, and T-invariant pair $(K_1,K_2)\subseteq (A,B)$ such that $\operatorname{dist}(K_1,K_2)=\operatorname{dist}(A,B)$, we have

$$\mu\bigl(T(K_1)\cup T(K_2)\bigr) \leqslant \varphi\bigl(\mu(K_1\cup K_2)\bigr)\mu(K_1\cup K_2).$$

Example 3. Let (A,B) be a nonempty and convex pair in a Banach space X such that B is compact and α is the Kuratowski measure of noncompactness on X. If $T: A\cup B\to A\cup B$ is a noncyclic mapping for which $T|_A$ is φ-contraction, then T is a noncyclic φ-condensing operator.

Proof. Let $(K_1,K_2)\subseteq (A,B)$ be a nonempty, bounded, closed, convex, and proximinal pair, which is T-invariant, and $\operatorname{dist}(K_1,K_2)=\operatorname{dist}(A,B)$. By the fact that B is compact, $\alpha(K_2)=\alpha(T(K_2))=0$ and so

$$\begin{aligned}\alpha\bigl(T(K_1)\cup T(K_2)\bigr) &= \max\bigl\{\alpha\bigl(T(K_1)\bigr),\alpha\bigl(T(K_2)\bigr)\bigr\}\\ &= \alpha\bigl(T(K_1)\bigr) \leqslant \varphi\bigl(\alpha(K_1)\bigr)\alpha(K_1)\\ &= \varphi\bigl(\max\bigl\{\alpha(K_1),\alpha(K_2)\bigr\}\bigr)\max\bigl\{\alpha(K_1),\alpha(K_2)\bigr\}\\ &= \varphi\bigl(\alpha(K_1\cup K_2)\bigr)\alpha(K_1\cup K_2),\end{aligned}$$

and hence, the result follows. □

We now present the main result of this paper.

Theorem 6. *Let (A,B) be a nonempty, bounded, closed, and convex pair in a strictly convex Banach space X such that A_0 is nonempty and μ is an* MNC *on X. Let $T: A\cup B\to A\cup B$ be a noncyclic relatively u-continuous mapping, which is φ-condensing in the sense of Definition* 7. *Then T has a best proximity pair.*

Proof. Note that (A_0,B_0) is nonempty, closed, convex, and proximinal. Let $x\in A_0$. Then there exists $y\in B_0$ such that $\|x-y\|=\operatorname{dist}(A,B)$. Since T is relatively u-continuous, $\|Tx-Ty\|=\operatorname{dist}(A,B)$ and so $Tx\in A_0$. Thus $T(A_0)\subseteq A_0$. Similarly, $T(B_0)\subseteq B_0$, which implies that (A_0,B_0) is T-invariant. Set $\mathcal{C}^0=A_0$ and $\mathcal{D}^0=B_0$ and, for all $n\in\mathbb{N}$, define

$$\mathcal{C}^n=\overline{\operatorname{con}}\bigl(T\bigl(\mathcal{C}^{n-1}\bigr)\bigr),\qquad \mathcal{D}^n=\overline{\operatorname{con}}\bigl(T\bigl(\mathcal{D}^{n-1}\bigr)\bigr).$$

Then we have

$$\mathcal{C}^1=\overline{\operatorname{con}}\bigl(T(\mathcal{C}^0)\bigr)=\overline{\operatorname{con}}\bigl(T(A_0)\bigr)\subseteq A_0=\mathcal{C}^0.$$

Continuing this process and by induction we obtain $\mathcal{C}^{n-1}\supseteq\mathcal{C}^n$ for all $n\in\mathbb{N}$. Equivalently, $\mathcal{D}^{n-1}\supseteq\mathcal{D}^n$ for all $n\in\mathbb{N}$. Suppose that there exists $k\in\mathbb{N}$ for which $\max\{\mu(\mathcal{C}^k),\mu(\mathcal{D}^k)\}=0$. Then $(\mathcal{C}^k,\mathcal{D}^k)$ is a compact pair. Also, we have

$$T\bigl(\mathcal{C}^k\bigr)\subseteq\overline{\operatorname{con}}\bigl(T\bigl(\mathcal{C}^k\bigr)\bigr)=\mathcal{C}^{k+1}\subseteq\mathcal{C}^k.$$

A similar argument implies that $T(\mathcal{D}^k) \subseteq \mathcal{D}^k$, and so T is noncyclic relatively u-continuous on $\mathcal{C}^k \cup \mathcal{D}^k$, where $(\mathcal{C}^k, \mathcal{D}^k)$ is a compact and convex pair in a strictly convex Banach space X. Thus, from Theorem 5, T has a best proximity pair, and we are finished.

So, we assume that $\max\{\mu(\mathcal{C}^n), \mu(\mathcal{D}^n)\} > 0$ for any $n \in \mathbb{N}$. If there exist $l_1, l_2 \in \mathbb{N}$ with $l_1 < l_2$ such that $\mu(\mathcal{C}^{l_1}) = \mu(\mathcal{D}^{l_2}) = 0$, then, by the fact that the sequence $\{\mathcal{C}^n\}_{n\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence, we have $\mathcal{C}^{l_2} \subseteq \mathcal{C}^{l_1}$ and so $\mu(\mathcal{C}^{l_2}) \leqslant \mu(\mathcal{C}^{l_1})$, which leads to $\mu(\mathcal{C}^{l_2}) = 0$. Hence, $\max\{\mu(\mathcal{C}^{l_2}), \mu(\mathcal{D}^{l_2})\} = 0$, which is a contradiction, and so,

$$\min\{\mu(\mathcal{C}^n), \mu(\mathcal{D}^n)\} > 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Also, for the pair $(x, y) \in \mathcal{C}^0 \times \mathcal{D}^0$ with $\|x-y\| = \operatorname{dist}(A, B)$, we have $\|T^n x - T^n y\| = \operatorname{dist}(A, B)$ for all $n \in \mathbb{N}$ because of the fact that T is noncyclic relatively u-continuous. From the definition of the pair $(\mathcal{C}^n, \mathcal{D}^n)$ we obtain $(T^n x, T^n y) \in \mathcal{C}^n \times \mathcal{D}^n$, which implies that

$$\operatorname{dist}(\mathcal{C}^n, \mathcal{D}^n) = \operatorname{dist}(A, B) \quad \text{for all } n \in \mathbb{N}.$$

Now suppose that $u \in \mathcal{C}^1 = \overline{\operatorname{con}}(T(\mathcal{C}^0))$. Then $u = \sum_{j=1}^m c_j T(u_j)$, where $u_j \in \mathcal{C}^0$ for all $1 \leqslant j \leqslant m$ such that $c_j \geqslant 0$ and $\sum_{j=1}^m c_j = 1$. Since $(\mathcal{C}^0, \mathcal{D}^0)$ is proximinal, for all $1 \leqslant j \leqslant m$, there exists $v_j \in \mathcal{D}^0$ such that $\|u_j - v_j\| = \operatorname{dist}(\mathcal{C}^0, \mathcal{D}^0)$ $(= \operatorname{dist}(A, B))$ and so $\|Tu_j - Tv_j\| = \operatorname{dist}(A, B)$. Put $v := \sum_{j=1}^m c_j T(v_j)$. Then $v \in \mathcal{D}^1$ and

$$\begin{aligned}\|u - v\| &= \left\| \sum_{j=1}^m c_j T(u_j) - \sum_{j=1}^m c_j T(v_j) \right\| \leqslant \sum_{j=1}^m \|T(u_j) - T(v_j)\| \\ &= \operatorname{dist}(A, B).\end{aligned}$$

Therefore, the pair $(\mathcal{C}^1, \mathcal{D}^1)$ is proximinal. Using a similar discussion, we can see that the pair $(\mathcal{C}^n, \mathcal{D}^n)$ is proximinal for all $n \in \mathbb{N} \cup \{0\}$. Thus $(\mathcal{C}^n, \mathcal{D}^n)$ is a nonempty, bounded, closed, convex, and proximinal pair, which is T-invariant. Since T is noncyclic φ-condensing, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\mu(\mathcal{C}^{n+1} \cup \mathcal{D}^{n+1}) &= \max\{\mu(\mathcal{C}^{n+1}), \mu(\mathcal{D}^{n+1})\} \\ &= \max\{\mu(\overline{\operatorname{con}}(T(\mathcal{C}^n))), \mu(\overline{\operatorname{con}}(T(\mathcal{D}^n)))\} \\ &= \max\{\mu((T(\mathcal{C}^n))), \mu((T(\mathcal{D}^n)))\} \\ &= \mu(T(\mathcal{C}^n) \cup T(\mathcal{D}^n)) \\ &\leqslant \varphi(\mu(\mathcal{C}^n \cup \mathcal{D}^n))\mu(\mathcal{C}^n \cup \mathcal{D}^n) \\ &\leqslant \mu(\mathcal{C}^n \cup \mathcal{D}^n).\end{aligned}$$

Then $\{\mu(\mathcal{C}^n \cup \mathcal{D}^n)\}$ is a decreasing sequence and bounded below, so, there exists a real number $r \geqslant 0$ such that $\lim_{n\to\infty} \mu(\mathcal{C}^n \cup \mathcal{D}^n) = r$. We claim that $r = 0$. Suppose the contrary. Thus, for all $n \in \mathbb{N}$, we have

$$\frac{\mu(\mathcal{C}^{n+1} \cup \mathcal{D}^{n+1})}{\mu(\mathcal{C}^n \cup \mathcal{D}^n)} \leqslant \varphi(\mu(\mathcal{C}^n \cup \mathcal{D}^n)).$$

The above inequality yields $\lim_{n\to\infty}\varphi(\mu(\mathcal{C}^n\cup\mathcal{D}^n)) = 1$. In view of the fact that $\varphi\in\Phi$, we conclude that $r=0$, which is impossible. Hence,

$$\lim_{n\to\infty}\mu\big(\mathcal{C}^n\cup\mathcal{D}^n\big)=\max\Big\{\lim_{n\to\infty}\mu\big(\mathcal{C}^n\big),\lim_{n\to\infty}\mu\big(\mathcal{D}^n\big)\Big\}=0.$$

It now follows from condition (d) of Definition 1 that the pair $(\mathcal{C}_\infty,\mathcal{D}_\infty)$ is nonempty, closed, and convex, which is T-invariant, where $\mathcal{C}_\infty=\bigcap_{n=0}^\infty\mathcal{C}^n$ and $\mathcal{D}_\infty=\bigcap_{n=0}^\infty\mathcal{D}^n$. Furthermore, $\operatorname{dist}(\mathcal{C}_\infty,\mathcal{D}_\infty)=\operatorname{dist}(A,B)$, and it is easy to check that $(\mathcal{C}_\infty,\mathcal{D}_\infty)$ is proximinal. On the other hand, $\max\{\mu(\mathcal{C}_\infty),\mu((\mathcal{D}_\infty))\}=0$, which ensures that the pair $(\mathcal{C}_\infty,\mathcal{D}_\infty)$ is compact. Finally, the result follows from Theorem 5. □

Remark 1. It is worth noticing that if the Banach space X in Theorem 6 is reflexive, then A_0 is nonempty. Indeed, if $\{(x_n,y_n)\}$ is a sequence in $A\times B$ for which $\|x_n-y_n\|\to\operatorname{dist}(A,B)$, then, by the fact that (A,B) is bounded and X is reflexive, there exists a subsequence $\{(x_{n_j},y_{n_j})\}$ such that $x_{n_j}\rightharpoonup p\in A$ and $y_{n_j}\rightharpoonup q\in B$, where "$\rightharpoonup$" denotes the weakly convergence. Now, from the lower semicontinuity of the norm we have

$$\|p-q\|\leqslant\liminf_{j\to\infty}\|x_{n_j}-y_{n_j}\|=\operatorname{dist}(A,B),$$

which ensures that the pair (A_0,B_0) is nonempty.

Next corollaries are straightforward consequences of Theorem 6.

Corollary 1. *Let (A,B) be a nonempty, bounded, closed, and convex pair in a strictly convex Banach space X such that A_0 is nonempty and μ is an* MNC *on X. Let $T: A\cup B\to A\cup B$ be a noncyclic relatively u-continuous mapping such that for any nonempty, closed, convex, proximinal, and T-invariant pair with* $\operatorname{dist}(K_1,K_2)=\operatorname{dist}(A,B)$

$$\mu\big(T(K_1)\cup T(K_2)\big)\leqslant\psi\big(\mu(K_1\cup K_2)\big)$$

for some $\psi\in\Psi$, where Ψ denotes the set of all functions $\psi:[0,\infty)\to[0,\infty)$ such that ψ is a nondecreasing and upper semi-continuous function for which $\psi(t)<t$ for all $t>0$. Then T has a best proximity pair.

Proof. Let $a\in(0,1)$. It is sufficient to define the function $\varphi:[0,\infty)\to[0,1)$ with

$$\varphi(t)=\begin{cases} a & \text{if } t=0,\\ \frac{\psi(t)}{t} & \text{if } 0<t\leqslant\mu(A\cup B),\\ \frac{\psi(\mu(A\cup B))}{t} & \text{if } t>\mu(A\cup B).\end{cases}$$

Now suppose that $\varphi(t_n)\to1$. If the sequence $\{t_n\}$ is unbounded, then $\varphi(t_n)\to0$, which is a contradiction. So, $\{t_n\}$ is bounded. Let $\{t_{n_k}\}$ be an arbitrary subsequence of $\{t_n\}$ such that $t_{n_k}\to l$. Because of the fact that ψ is upper semi-continuous, we obtain

$$l=\lim_{k\to\infty}t_{n_k}=\limsup_{k\to\infty}\psi\big(t_{n_k}\big)\leqslant\psi(l),$$

and so, we must have $l = 0$. Note that if $(K_1, K_2) \subseteq (A, B)$ is a nonempty, closed, convex, proximinal, and T-invariant pair such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$, then we have

$$\mu\big(T(K_1) \cup T(K_2)\big) \leqslant \psi\big(\mu(K_1 \cup K_2)\big) = \varphi\big(\mu(K_1 \cup K_2)\big)\mu(K_1 \cup K_2),$$

where $\varphi \in \Phi$. Now the result follows from Theorem 6. □

The next corollary is a main result of [10].

Corollary 2. *Let (A, B) be a nonempty, bounded, closed, and convex pair in a strictly convex Banach space X such that A_0 is nonempty and μ is an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping such that for any nonempty, closed, convex, proximinal, and T-invariant pair with* $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$,

$$\mu\big(T(K_1) \cup T(K_2)\big) \leqslant r\mu(K_1 \cup K_2)$$

for some $r \in (0, 1)$. Then T has a best proximity pair.

Proof. The result follows by taking $\psi(t) = rt$ in Corollary 1. □

3.2 Cyclic φ-condensing operators

The main purpose of this section is to obtain the cyclic version of Theorem 6 for best proximity points in Banach spaces, which do not have the geometric property of strictly convexity. To this end, we need the following result of [10].

Theorem 7. *Let (A, B) be a nonempty, bounded, closed, and convex pair in a Banach space X such that A_0 is nonempty. Assume that $T : A \cup B \to A \cup B$ is a cyclic, relatively nonexpansive mapping. If T is compact, then T has a best proximity point.*

Obviously, Theorem 7 is a real generalization of Theorem 2. We now state the main conclusion of this section.

Theorem 8. *Let (A, B) be a nonempty, bounded, closed, and convex pair in a Banach space X such that A_0 is nonempty and μ is an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping, which is φ-condensing in the sense of Definition 7 for some $\varphi \in \Phi$. Then T has a best proximity point.*

Proof. Using a similar argument of Theorem 6, we have that (A_0, B_0) is closed, convex, proximinal, and T-invariant, that is, $T(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$. For all $n \in \mathbb{N}$, define

$$\mathcal{C}^n = \overline{\operatorname{con}}\big(T\big(\mathcal{C}^{n-1}\big)\big), \qquad \mathcal{D}^n = \overline{\operatorname{con}}\big(T\big(\mathcal{D}^{n-1}\big)\big),$$

where, $\mathcal{C}^0 := A_0$ and $\mathcal{D}^0 := B_0$, then we have

$$\mathcal{C}^1 = \overline{\operatorname{con}}\big(T\big(\mathcal{C}^0\big)\big) = \overline{\operatorname{con}}\big(T(A_0)\big) \subseteq B_0 = \mathcal{D}^0,$$

and so, $T(\mathcal{C}^1) \subseteq T(\mathcal{D}^0)$, which implies that

$$\mathcal{C}^2 = \overline{\operatorname{con}}\big(T\big(\mathcal{C}^1\big)\big) \subseteq \overline{\operatorname{con}}\big(T\big(\mathcal{D}^0\big)\big) = \mathcal{D}^1.$$

Continuing this process, we obtain $\mathcal{C}^{n+1} \subseteq \mathcal{D}^n$. Also, we have

$$\mathcal{D}^1 = \overline{\operatorname{con}}\big(T\big(\mathcal{D}^0\big)\big) = \overline{\operatorname{con}}\big(T(B_0)\big) \subseteq A_0 = \mathcal{C}^0,$$

and hence, $T(\mathcal{D}^1) \subseteq T(\mathcal{C}^0)$. Thus

$$\mathcal{D}^2 = \overline{\operatorname{con}}\big(T\big(\mathcal{D}^1\big)\big) \subseteq \overline{\operatorname{con}}\big(T\big(\mathcal{C}^0\big)\big) = \mathcal{C}^1.$$

Then by induction we conclude that $\mathcal{D}^n \subseteq \mathcal{C}^{n-1}$ for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{C}^{n+2} \subseteq \mathcal{D}^{n+1} \subseteq \mathcal{C}^n \subseteq \mathcal{D}^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Thereby, $\{(\mathcal{C}^{2n}, \mathcal{D}^{2n})\}_{n \geqslant 0}$ is a decreasing sequence consisting of closed and convex pairs in $A_0 \times B_0$. Furthermore, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$T\big(\mathcal{D}^{2n}\big) \subseteq T\big(\mathcal{C}^{2n-1}\big) \subseteq \overline{\operatorname{con}}\big(T\big(\mathcal{C}^{2n-1}\big)\big) = \mathcal{C}^{2n},$$
$$T\big(\mathcal{C}^{2n}\big) \subseteq T\big(\mathcal{D}^{2n-1}\big) \subseteq \overline{\operatorname{con}}\big(T\big(\mathcal{D}^{2n-1}\big)\big) = \mathcal{D}^{2n}.$$

So, we deduce that $(\mathcal{C}^{2n}, \mathcal{D}^{2n})$ is T-invariant. Let $(x, y) \in \mathcal{C}^0 \times \mathcal{D}^0$ be such that $\|x - y\| = \operatorname{dist}(A, B)$. Then $(T^{2n}x, T^{2n}y) \in \mathcal{C}^{2n} \times \mathcal{D}^{2n}$, and by the fact that T is relatively nonexpansive, we have

$$\operatorname{dist}\big(\mathcal{C}^{2n}, \mathcal{D}^{2n}\big) \leqslant \big\|T^{2n}x - T^{2n}y\big\| \leqslant \|x - y\| = \operatorname{dist}(A, B).$$

Similar to Theorem 6, we can see that $(\mathcal{C}^{2n}, \mathcal{D}^{2n})$ is also proximinal for all $n \in \mathbb{N}$. Notice that if $\max\{\{\mu(\mathcal{C}^{2k}), \mu(\mathcal{D}^{2k})\} = 0$ for some $k \in \mathbb{N}$, then the result follows from Theorem 7. So, we assume that $\max\{\{\mu(\mathcal{C}^{2n}), \mu(\mathcal{D}^{2n})\} > 0$ for all $n \in \mathbb{N}$. Again using similar discussion of Theorem 6, we conclude that $\min\{\{\mu(\mathcal{C}^{2n}), \mu(\mathcal{D}^{2n})\} > 0$ for all $n \in \mathbb{N}$. Since T is cyclic φ-condensing, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\mu\big(\mathcal{C}^{2n+2} \cup \mathcal{D}^{2n+2}\big) &= \max\big\{\mu\big(\mathcal{C}^{2n+2}\big), \mu\big(\mathcal{D}^{2n+2}\big)\big\} \leqslant \max\big\{\mu\big(\mathcal{D}^{2n+1}\big), \mu\big(\mathcal{C}^{2n+1}\big)\big\} \\
&= \max\big\{\mu\big(\overline{\operatorname{con}}\big(T\big(\mathcal{D}^{2n}\big)\big)\big),\ \mu\big(\overline{\operatorname{con}}\big(T\big(\mathcal{C}^{2n}\big)\big)\big)\big\} \\
&= \max\big\{\mu\big(\big(T\big(\mathcal{C}^{2n}\big)\big)\big),\ \mu\big(\big(T\big(\mathcal{D}^{2n}\big)\big)\big)\big\} \\
&= \mu\big(T\big(\mathcal{C}^{2n}\big) \cup T\big(\mathcal{D}^{2n}\big)\big) \leqslant \varphi\big(\mu\big(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}\big)\big)\mu\big(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}\big) \\
&\leqslant \mu\big(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}\big).
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \mu\big(\mathcal{C}^{2n} \cup \mathcal{D}^{2n}\big) = \max\Big\{\lim_{n\to\infty} \mu\big(\mathcal{C}^{2n}\big), \lim_{n\to\infty} \mu\big(\mathcal{D}^{2n}\big)\Big\} = 0.$$

If we set $\mathcal{C}_\infty = \bigcap_{n=0}^{\infty} \mathcal{C}^{2n}$ and $\mathcal{D}_\infty = \bigcap_{n=0}^{\infty} \mathcal{D}^{2n}$, then $(\mathcal{C}_\infty, \mathcal{D}_\infty)$ is nonempty, closed, convex, and T-invariant with $\operatorname{dist}(A, B) = \operatorname{dist}(\mathcal{C}_\infty, \mathcal{D}_\infty)$ for which we have $\max\{\mu(\mathcal{C}_\infty), \mu((\mathcal{D}_\infty))\} = 0$. Hence, T has a best proximity point. □

The next result is immediate.

Corollary 3. *Let (A, B) be a nonempty, bounded, closed, and convex pair in a Banach space X such that A_0 is nonempty and μ is an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping, which is ψ-condensing for some $\psi \in \Psi$. Then T has a best proximity point.*

Remark 2. If in Corollary 3, we consider $\psi(t) = rt$ for some $r \in (0, 1)$ and $A = B$, then we retrieve Darbo's fixed point theorem [4].

4 Existence of optimal solution for a system of differential equations

To provide an application of our results, we recall an extension of *mean-value theorem*.

Lemma 2. (*See* [13].) *Let J be a real interval, X a Banach space, and $f : J \to X$ a differentiable mapping. Let $a, b \in J$ with $a < b$. Then*

$$f(b) - f(a) \in (b - a)\overline{\text{con}}\big(\big\{f'(t)\colon t \in [a, b]\big\}\big).$$

Definition 8. Let a, b be real positive numbers, I be the real interval $[t_0 - a,\, t_0 + a]$, and $V_1 = \mathcal{B}(x_0; b)$, $V_2 = \mathcal{B}(x_1; b)$ be closed balls in a Banach space X, where t_0 is a real number, and $x_0, x_1 \in X$. Assume that $f : I \times V_1 \to X$ and $g : I \times V_2 \to X$ are continuous mappings. Consider the following system of differential equations:

$$\begin{aligned} x'(t) &= f\big(t, x(t)\big), \quad x(t_0) = x_0, \\ y'(t) &= g\big(t, y(t)\big), \quad y(t_0) = x_1, \end{aligned} \tag{2}$$

defined on a closed real interval $J = [t_0 - h,\, t_0 + h]$ for some real positive number h. Let $\mathcal{C}(J, X)$ consist of all continuous mappings from J into X with the supremum norm, and let

$$\begin{aligned} \mathcal{C}(J, V_1) &= \big\{x \in \mathcal{C}(J, X)\colon x(t_0) = x_0\big\}, \\ \mathcal{C}(J, V_2) &= \big\{y \in \mathcal{C}(J, X)\colon y(t_0) = x_1\big\}. \end{aligned}$$

Thus, for all $(x, y) \in \mathcal{C}(J, V_1) \times \mathcal{C}(J, V_2)$, we have

$$\|x - y\|_\infty = \sup_{t \in J} \big\|x(t) - y(t)\big\| \geqslant \|x_0 - x_1\|,$$

and so, $\operatorname{dist}(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2)) = \|x_0 - x_1\|$. Let

$$T : \mathcal{C}(J, V_1) \cup \mathcal{C}(J, V_1) \to \mathcal{C}(J, X)$$

be an operator defined as

$$\begin{aligned} Tx(t) &= x_1 + \int_{t_0}^{t} g\big(s, x(s)\big)\, \mathrm{d}s, \quad x \in \mathcal{C}(J, V_1), \\ Ty(t) &= x_0 + \int_{t_0}^{t} f\big(s, y(s)\big)\, \mathrm{d}s, \quad y \in \mathcal{C}(J, V_2). \end{aligned}$$

We say that $z \in \mathcal{C}(J, V_1) \cup \mathcal{C}(J, V_2)$ is an optimal solution for the system of differential equations given in (2), provided that

$$\|z - Tz\|_\infty = \operatorname{dist}(A, B).$$

Now we are ready to state and prove the following theorem.

Theorem 9. *Under the assumptions of Definition* 8*, we suppose that*

$$\alpha\big(f(I \times W_2) \cup g(I \times W_1)\big) \leqslant \psi\big(\alpha(W_1 \cup W_2)\big),$$
$$\big\|f(t,x) - g(t,y)\big\| \leqslant \frac{1}{h}\big(\big\|x(t) - y(t)\big\| - \|x_1 - x_0\|\big)$$

for some $\psi \in \Psi$ and for any $(W_1, W_2) \subseteq (V_1, V_2)$ and $h \leqslant \min\{a, b/M_1, b/M_2, 1/2b\}$, where

$$M_1 = \sup\big\{\big\|f(t,x)\big\|\colon (t,x) \in I \times V_1\big\},$$
$$M_2 = \sup\big\{\big\|g(t,y)\big\|\colon (t,y) \in I \times V_2\big\}.$$

Then system (2) *has an optimum solution.*

Proof. Notice that $(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ is a bounded, closed, and convex pair in $\mathcal{C}(J, X)$ and that T is cyclic on $\mathcal{C}(J, V_1) \cup \mathcal{C}(J, V_2)$. We assert that $T(\mathcal{C}(J, V_1))$ is a bounded and equicontinuous subset of $\mathcal{C}(J, V_2)$. Let $t, t' \in J$ and $x \in \mathcal{C}(J, V_1)$. Then

$$\begin{aligned}\big\|Tx(t)\big\| &= \left\| x_1 + \int_{t_0}^{t} g\big(s, x(s)\big)\,\mathrm{d}s \right\| \leqslant \|x_1\| + \int_{t_0}^{t} \big\|g\big(s, x(s)\big)\big\|\,\mathrm{d}s \\ &\leqslant \|x_1\| + M_2 h \leqslant \|x_1\| + b,\end{aligned}$$

and this leads to boundedness of $T(\mathcal{C}(J, V_1))$. On the other hand,

$$\begin{aligned}\big\|Tx(t) - Tx(t')\big\| &= \left\| \int_{t_0}^{t} g\big(s, x(s)\big)\,\mathrm{d}s - \int_{t_0}^{t'} g\big(s, x(s)\big)\,\mathrm{d}s \right\| \\ &\leqslant \int_{t}^{t'} \big\|g\big(s, x(s)\big)\big\|\,\mathrm{d}s \leqslant M_2|t - t'|,\end{aligned}$$

that is, $T(\mathcal{C}(J, V_1))$ is equicontinuous. Similarly, we can see that $T(\mathcal{C}(J, V_2))$ is also bounded and equicontinuous. Therefore, by Arzela–Ascoli's theorem we conclude that the pair $(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ is relatively compact. We now prove that T is a φ-condensing operator for some $\varphi \in \Phi$. To this end, suppose that $(K_1, K_2) \subseteq (\mathcal{C}(J, V_1), \mathcal{C}(J, V_2))$ is a nonempty, closed, convex, and proximinal pair, which is T-invariant and that

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}\big(\mathcal{C}(J, V_1), \mathcal{C}(J, V_2)\big) \ \big(= \|x_0 - x_1\|\big).$$

Using Theorem 2.11 of [3], we deduce that

$$
\begin{aligned}
&\alpha\bigl(T(K_1)\cup T(K_2)\bigr)\\
&\quad=\max\bigl\{\alpha\bigl(T(K_1)\bigr),\alpha\bigl(T(K_2)\bigr)\bigr\}\\
&\quad=\max\Bigl\{\sup_{t\in J}\bigl\{\alpha\bigl(\{Tx(t)\colon x\in K_1\}\bigr)\bigr\},\ \sup_{t\in J}\bigl\{\alpha\bigl(\{Ty(t)\colon y\in K_2\}\bigr)\bigr\}\Bigr\}\\
&\quad=\max\Biggl\{\sup_{t\in J}\Biggl\{\alpha\Biggl(\Biggl\{x_1+\int\limits_{t_0}^{t}g\bigl(s,x(s)\bigr)\,\mathrm{d}s\colon x\in K_1\Biggr\}\Biggr)\Biggr\},\\
&\qquad\qquad\sup_{t\in J}\Biggl\{\alpha\Biggl(\Biggl\{x_0+\int\limits_{t_0}^{t}f\bigl(s,y(s)\bigr)\,\mathrm{d}s\colon y\in K_2\Biggr\}\Biggr)\Biggr\}\Biggr\}.
\end{aligned}
$$

Now from Lemma 2 we obtain

$$
\begin{aligned}
x_1+\int\limits_{t_0}^{t}g\bigl(s,x(s)\bigr)\,\mathrm{d}s&\in x_1+(t-t_0)\overline{\mathrm{con}}\bigl(\bigl\{g\bigl(s,x(s)\bigr)\colon s\in[t_0,t]\bigr\}\bigr),\\
x_0+\int\limits_{t_0}^{t}f\bigl(s,y(s)\bigr)\,\mathrm{d}s&\in x_0+(t-t_0)\overline{\mathrm{con}}\bigl(\bigl\{f\bigl(s,y(s)\bigr)\colon s\in[t_0,t]\bigr\}\bigr),
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&\alpha\bigl(T(K_1)\cup T(K_2)\bigr)\\
&\quad\leqslant\max\Bigl\{\sup_{t\in J}\bigl\{\alpha\bigl(\bigl\{x_1+(t-t_0)\overline{\mathrm{con}}\bigl(\bigl\{g\bigl(s,x(s)\bigr)\colon s\in[t_0,t]\bigr\}\bigr)\bigr\}\bigr)\bigr\},\\
&\qquad\qquad\sup_{t\in J}\bigl\{\alpha\bigl(\bigl\{x_0+(t-t_0)\overline{\mathrm{con}}\bigl(\bigl\{f\bigl(s,y(s)\bigr)\colon s\in[t_0,t]\bigr\}\bigr)\bigr\}\bigr)\bigr\}\Bigr\}\\
&\quad\leqslant\max\Bigl\{\sup_{0\leqslant\lambda\leqslant h}\bigl\{\alpha\bigl(\bigl\{x_1+\lambda\overline{\mathrm{con}}\bigl(\bigl\{g(J\times K_1)\bigr\}\bigr)\bigr\}\bigr)\bigr\},\\
&\qquad\qquad\sup_{0\leqslant\lambda\leqslant h}\bigl\{\alpha\bigl(\bigl\{x_0+\lambda\overline{\mathrm{con}}\bigl(\bigl\{f(J\times K_2)\bigr\}\bigr)\bigr\}\bigr)\bigr\}\Bigr\}\\
&\quad=\max\bigl\{h\alpha\bigl(g(J\times K_1)\bigr),h\alpha\bigl(f(J\times K_2)\bigr)\bigr\}\\
&\quad=h\alpha\bigl(\bigl\{g(J\times K_1)\cup f(J\times K_2)\bigr\}\bigr)\\
&\quad\leqslant\frac{1}{2b}\psi\bigl(\alpha(K_1\cup K_2)\bigr)\\
&\quad\leqslant\frac{1}{\alpha(K_1\cup K_2)}\psi\bigl(\alpha(K_1\cup K_2)\bigr),
\end{aligned}
$$

which imples that T is a φ-condensing operator, where $\varphi(t):=\psi(t)/t\in\Phi$. Finally, we show that T is cyclic relatively nonexpansive. Indeed, for any $(x,y)\in\mathcal{C}(J,V_1)\times\mathcal{C}(J,V_1)$, we have

$$\begin{aligned}\big\|Tx(t)-Ty(t)\big\| &= \left\|\left(x_1+\int\limits_{t_0}^{t} g\big(s,x(s)\big)\,\mathrm{d}s\right)-\left(x_0+\int\limits_{t_0}^{t} f\big(s,y(s)\big)\,\mathrm{d}s\right)\right\| \\ &\leqslant \|x_1-x_0\|+\int\limits_{t_0}^{t}\big\|g\big(s,x(s)\big)-f\big(s,y(s)\big)\big\|\,\mathrm{d}s \\ &\leqslant \|x_1-x_0\|+\frac{1}{h}\int\limits_{t_0}^{t}\big(\big\|x(s)-y(s)\big\|-\|x_1-x_0\|\big)\,\mathrm{d}s \\ &\leqslant \|x_1-x_0\|+\big(\|x-y\|_\infty-\|x_1-x_0\|\big)=\|x-y\|_\infty,\end{aligned}$$

and thereby, $\|Tx-Ty\|_\infty \leqslant \|x-y\|_\infty$. Now the result follows from Theorem 8. □

As an application of Theorem 9, we obtain the next existence result of a solution for a system of differential equations satisfying the same initial conditions.

Corollary 4. *Under the notations of Definition* 8 *and the assumptions of Theorem* 9*, if* $x_0 = x_1$*, then the system*

$$\begin{aligned} x'(t) &= f\big(t,x(t)\big), \quad x(t_0)=x_0, \\ y'(t) &= g\big(t,y(t)\big), \quad y(t_0)=x_0, \end{aligned}$$

has a solution.

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