Weakly Nonlocal Supersymmetric KdV Hierarchy

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Abstract. The present work offers a weakly nonlocal generalization of superKdV, which possesses also an unlimited number of conservation laws and exact solutions.

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1 Introduction

The nonlinear Korteweg-de Vries (KdV) equation $\phi_t + \phi\phi_x - \alpha\phi_{xxx} = 0$ is a universal model to describe one-dimensional nonlinear waves in dispersion media without dissipation, in which the law of dispersion for nonlinear waves has the form $\omega = a_1 k + a_3 k^3$. The KdV equation is a basis for modelling magneto-acoustic and ion-sound waves in plasma, acoustic waves in crystals, surface and internal waves of a moderate amplitude in oceans [1]. The KdV equation is integrated by the inverse scattering method [2] and has an $N = 1$ supersymmetric extension [3].

In a few papers [4–9] the following nonlocal generalizations of the KdV as examples of the first group were proposed.

However, these generalizations result in destruction of integrability, and the conserved values disappear. It would be of interest to find integrable generalizations of the KdV equation that could preserve both the properties of nonlocality and the conservation laws.

The present work offers a weakly nonlocal generalization of the evolutionary KdV and super-KdV (sKdV), which possesses also an unlimited number of conservation laws and exact solutions.

2 The nonlocal supersymmetric KdV equation

Let the superfield [10] $\chi = \theta_a D_x^{1-p}\phi + \psi$ unite two fields with different properties: the "bosonic" field $\phi(x,t) \in C^1(\Omega) \subset \mathbf{R}^2$, $\Omega = (x,t)\colon x \in \mathbf{R}^1$, $t > 0$ and its spinorial superpartner $\psi(x,t) \in C^1(\Omega) \subset \mathbf{R}^2$; θ is the Grassmann variable (constant Majorana spinor), ${}_aD_x^p f(x)$ is the fractional derivative in the sense of Riemann-Liouville [11]:

$$ {}_aD_x^p\phi(x,t) = \frac{1}{\Gamma(1-p)}\,\frac{d}{dx}\int_a^x \frac{\phi(\xi,t)}{(x-\xi)^p}\,d\xi, \tag{1} $$

in which $0 < p < 1$, and a is the parameter of nonlocality.

The transformations of the fields ϕ, ψ, because of the fractional derivatives ${}_aD_x^p f(x)$, are nonlocal:

$$ \begin{cases} \delta_\eta \psi = \eta\, {}_aD_x^{1-p}\phi, \\ \delta_\eta\, {}_aD_x^{1-p}\phi = \eta\psi_x. \end{cases} \tag{2} $$

However, the commutator of the two transformations (2) is a spatial translation:

$$ [\delta_\eta, \delta_\xi] = 2\xi\eta\partial_x. \tag{3} $$

The supersymmetric equation

$$ \chi_t = \Big(\chi_{xx} + \frac{1}{2}\chi\mathcal{D}\chi\Big)_x, \tag{4} $$

(here $\mathcal{D} = \theta\partial_x + \partial_\theta$ is a supersymmetric derivative) is a system of two evolutionary equations,

$$ \begin{cases} \psi_t = \psi_{xxx} + \dfrac{1}{2}({}_aD_x^{1-p}\phi\cdot\psi)_x, \\ \phi_t = \phi_{xxx} + \dfrac{1}{2}{}_aD_x^p\big[({}_aD_x^{1-p}\phi)^2 - \psi\psi_x\big], \end{cases} \tag{5} $$

which is invariant in respect of the supertransformations (2). In the general case, the system (5) is a system of two nonlinear nonlocal evolution equations, which becomes local when

$$p = 0 \qquad \begin{cases} \psi_t = \psi_{xxx} + \dfrac{1}{2}(\phi_x\psi)_x, \\ \phi_t = \phi_{xxx} + \dfrac{1}{2}(\phi_x^2 - \psi\psi_x); \end{cases} \tag{6}$$

$$p = 1 \qquad \begin{cases} \psi_t = \psi_{xxx} + \dfrac{1}{2}(\phi\psi)_x, \\ \phi_t = \phi_{xxx} + \dfrac{1}{2}(\phi^2 - \psi\psi_x)_x. \end{cases} \tag{7}$$

The supersymmetric equation (4) and the corresponding system of equations (5) unites two fields of different nature, and only one of them is nonlocal.

At the parameter of fractional derivative $p = 1$ the supersymmetric equation (4) turns into the ordinary supersymmetric KdV equation (7), which allows us to designate equation (4) as a nonlocal supersymmetric KdV equation (nsKdV).

The analytical solutions of the classical KdV and the KdV with quadratic nonlinearity are well known. The nKdV relations with these equations allow us to express the solutions of nKdV through classical solutions of the corresponding equations. In particular,

$$\phi(x,t) = -\frac{u}{2}\,{}_aD_x^p\,\mathrm{ch}^{-2}\left[\frac{1}{2}\sqrt{u}(\xi - \xi_0)\right] \tag{8}$$

is the nonlocal generalization of the classical soliton solution.

3 Integrability and conservation laws

The supersymmetry of nsKdV does not mean its integrability. The most direct proof of the integrability of supersymmetric *local* KdV equation (4) is that it has a Lax representation. Let apply supersymmetric Lax representation in our *nonlocal* case:

$$\mathcal{L}_t = [-4\mathcal{L}_+^{3/2}, \mathcal{L}], \quad \mathcal{L} = \partial^2 - \phi\mathcal{D}, \tag{9}$$

and the conservation laws are obtained as follows:

$$\mathcal{H}_{2n+1} = \int \mathrm{sRes}\,\mathcal{L}^{(2n+1)/2}\,dxd\theta. \tag{10}$$

For the super pseudodifferential operator $\mathcal{P} = \sum_{i=-\infty}^{n} \alpha_i \mathcal{D}^i$

$$\mathcal{P}_+ = \sum_{i=0}^{n} \alpha_i \mathcal{D}^i, \quad \mathrm{sRes}\, \mathcal{P} = \alpha_{-1}, \tag{11}$$

where definite integration takes place over set $\mathbf{R}^1 \times \Xi$: $x \in \mathbf{R}^1,\ \theta \in \Xi$.

The first conservation law is the difference of asymptotic states:

$$\begin{aligned} \int \left[\mathcal{D}\chi(x,\theta)\right] dx d\theta &= \int (\theta\psi_x + {}_aD_x^{1-p}\phi)\, dx d\theta \\ &= \int \psi_x\, dx = \psi(+\infty) - \psi(-\infty) = 0. \end{aligned} \tag{12}$$

The second conservation law is

$$\mathcal{H}_3 = \int dx d\theta (\chi \mathcal{D} \chi) = \int dx \left[({}_aD_x^{1-p}\phi)^2 - \psi\psi_x\right]. \tag{13}$$

This list of conserved quantities could be continued according to the general expression (10).

The physical meaning of the corresponding conservation laws becomes clearer in the pure "bosonic" case. In the case of the nsKdV (4) without supersymmetry

$$\phi_t + \frac{1}{2} {}_aD_x^p ({}_aD_x^{1-p}\phi)^2 - \alpha\phi_{xxx} = 0, \tag{14}$$

for $x \in \mathbf{R},\ \forall\, t > 0,\ {}_aD_x^{3-p}\phi(\pm\infty, t) = {}_aD_x^{1-p}\phi(\pm\infty, t) = 0$, we deal with a conservation value:

$$I^{(p)} = \int_{-\infty}^{+\infty} {}_aD_x^{1-p}\phi\, dx = inv, \tag{15}$$

as

$$\begin{aligned} \frac{dI^{(p)}}{dt} &= \int_{-\infty}^{+\infty} \left[\alpha\, {}_aD_x^{3-p}\phi - \frac{1}{2}({}_aD_x^{1-p}\phi)^2\right]_x dx \\ &= \left[\alpha\, {}_aD_x^{3-p}\phi - \frac{1}{2}({}_aD_x^{1-p}\phi)^2\right]\Big|_{-\infty}^{+\infty} = 0. \end{aligned} \tag{16}$$

This conservation value shows that the difference in asymptotic values for any time moment remains unchanged. In the applications, at $p = 1$ this conservation law is called the "mass" conservation law, because $\phi(x,t)$ can be a one-dimensional density or gradient of any physical, chemical or biological magnitude.

Even this simple example highlights two important properties of the nonlocal conservation law (15): it interrelates the conservation values of two different dynamic systems, which can be of different mathematical nature (*e.g.*, in our case these values are integral and discrete).

As follows from the integrable hierarchy (9) or by a direct verification like above in the bosonic case, the momentum conservation law is

$$P = \frac{1}{2}\int\limits_{-\infty}^{+\infty} ({}_aD_x^{1-p}\phi)^2\,dx = inv. \tag{17}$$

The energy conservation law is

$$E = \int\limits_{-\infty}^{+\infty} \left[({}_aD_x^{1-p}\phi)^3 - \frac{1}{2}({}_aD_x^{2-p}\phi)^2\right] dx = inv. \tag{18}$$

These conservation laws at the $p = 1$ turn into the momentum and energy conservation laws of the classical KdV evolution equation.

4 Conclusions

Thus, we see that it is possible to construct a supersymmetric weakly nonlocal generalization of the evolutionary KdV equation. The nonlocal term is similar to the same one in the Burgers evolutionary equation [12] which could be applied for solving the problem of cold dust matter distribution [13].

We use deliberately the nonlocal term ${}_aD_x^p({}_aD_x^{1-p}\phi)^2/2$ and the supertransformation $[\delta_\eta, \delta_\xi] = 2\xi\eta\partial_x$ (3) for the construction of the nsKdV, nevertheless we suppose that it is possible to construct a nsKdV for the case of supersymmetric transformation of the form

$$[\delta_\eta, \delta_\xi] = 2\xi\eta\,{}_aD_x^p. \tag{19}$$

Considering the specific nature of the fractional differential operator, it would be extremely interesting to find such representation.

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