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Value Distribution of General Dirichlet Series. VI*

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Abstract. In the paper a limit theorem in the sense of weak convergence of probability measures on the complex plane for a new class of general Dirichlet series is obtained.

Keywords: distribution, general Dirichlet series, probability measure, weak convergence.

1 Introduction

Let $s = \sigma + it$ be a complex variable, and let $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, real and complex numbers, respectively. The series of the form

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \tag{1}$$

where $a_m \in \mathbb{C}$, and $\{\lambda_m\}$ is an increasing sequence of real numbers, $\lim\limits_{m\to\infty} \lambda_m = +\infty$, is called a general Dirichlet series. If $\lambda_m = \log m$, then we obtain an ordinary Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Suppose that series (1) converges absolutely for $\sigma > \sigma_a$ and has the sum $f(s)$. Then $f(s)$ is an analytic function in the region $\{s \in \mathbb{C}\colon \sigma > \sigma_a\}$.

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In [1] limit theorems on the complex plane for the function $f(s)$ were obtained. Suppose that $f(s)$ is meromorphically continuable to the half-plane $\{s \in \mathbb{C}\colon \sigma > \sigma_1\}$, $\sigma_1 < \sigma_a$, and that all poles of $f(s)$ in this region are included in a compact set. Moreover, we require that, for $\sigma > \sigma_1$, the estimates

$$f(s) = O\big(|t|^a\big), \quad |t| \geq t_0 > 0, \quad a > 0 \tag{2}$$

and

$$\int\limits_0^T \big|f(\sigma + it)\big|^2 dt = O(T), \quad T \to \infty, \tag{3}$$

should be satisfied. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\operatorname{meas}\big\{t \in [0,T]\colon \ \ldots\big\},$$

where $\operatorname{meas}\{A\}$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and in place of dots a condition satisfied by t is to be written. On $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ define the probability measure $P_T(A)$ by

$$P_{T,\sigma}(A) = \nu_T\big(f(\sigma + it) \in A\big).$$

The first result of [1] is the following theorem.

Theorem A. Suppose that for the function $f(s)$ conditions (2) and (3) are satisfied. Then on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ there exists a probability measure P_σ such that the measure $P_{T,\sigma}$ converges weakly to P_σ as $T \to \infty$.

For the identification of the limit measure P_σ in Theorem A some additional conditions are necessary. Also, for the definition of P_σ we need some topological structure. Let $\gamma = \big\{s \in \mathbb{C}\colon |s| = 1\big\}$ be the unit circle on the complex plane, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group.

Therefore, on $\big(\Omega, \mathcal{B}(\Omega)\big)$ the probability Haar measure m_H exists, and we obtain a probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Let $\omega(m)$ denote the projection of $\omega \in \Omega$ to the coordinate space γ_m. Suppose, that the exponents λ_m satisfy the inequality

$$\lambda_m \geq c(\log m)^\delta \tag{4}$$

with some positive constants c and δ. Then in [1] it was proved (Lemma 3) that, for $\sigma > \sigma_1$,

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m \sigma} \tag{5}$$

is a complex-valued random variable defined on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Then [1] contains the following statement.

Theorem B. Suppose that the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers, satisfies inequality (4), and for the function $f(s)$ conditions (2) and (3) are satisfied. Then the probability measure $P_{T,\sigma}$ converges weakly to the distribution of the random variable $f(\sigma, \omega)$ as $T \to \infty$.

Condition (4) restricts the choice of sequence of exponents $\{\lambda_m\}$ for which Theorem B is true. The aim of this note is to replace condition (4) by a certain average condition. Suppose that, for $\sigma > \sigma_1$, the series

$$\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} \log^2 m \tag{6}$$

converges. Later, it will be proved that the convergence of series (6) is a sufficient condition that $f(\sigma, \omega)$ defined by (5) should be a complex-valued random variable for $\sigma > \sigma_1$.

Theorem 1. *Suppose that the system of exponents* $\{\lambda_m\}$ *is linearly independent over the field of rational numbers, series* (6) *converges, and for the function* $f(s)$ *conditions* (2) *and* (3) *are satisfied. Then the probability measure* $P_{T,\sigma}$ *converges weakly to the distribution of the random variable* $f(\sigma, \omega)$ *as* $T \to \infty$.

2 The random variable $f(\sigma, \omega)$

In this section we will prove that, if series (6) converges, then $f(\sigma, \omega), \sigma > \sigma_1$, is a complex-valued random variable. For this, we will use Rademacher's theorem on series of pairwise orthogonal random variables, for the proof, see, for example, [2]. Denote by $\mathbb{E}\xi$ the expectation of the random element ξ.

Lemma 2 [2]. Suppose that $\{X_m\}$ is a sequence of pairwise orthogonal random variables and that $\sum_{m=1}^{\infty} \mathbb{E}|X_m|^2 \log^2 m < \infty$. Then the series $\sum_{m=1}^{\infty} X_m$ converges almost surely.

Theorem 2. *Let $\sigma > \sigma_1$. Then $f(\sigma, \omega)$ is a complex-valued random variable defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.*

Proof. Let, for $\sigma > \sigma_1$,

$$\xi_m(\omega) = a_m \omega(m) e^{-\lambda_m \sigma}.$$

Then $\{\xi_m : m \in \mathbb{N}\}$ is a sequence of complex-valued random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. We have

$$\begin{aligned} \mathbb{E}(\xi_m \overline{\xi}_k) &= \int_\Omega \xi_m(\omega) \overline{\xi_k(\omega)} dm_H = a_m \overline{a_k} e^{-(\lambda_m + \lambda_k)\sigma} \int_\Omega \omega(m) \overline{\omega(k)} dm_H \\ &= \begin{cases} 0, & \text{if} \quad m \neq k, \\ |a_m|^2 e^{-2\lambda_m \sigma}, & \text{if} \quad m = k. \end{cases} \end{aligned} \tag{7}$$

This means that $\{\xi_m : m \in \mathbb{N}\}$ is a sequence of pairwise orthogonal complex-valued random variables. Since by (7)

$$\mathbb{E}|\xi_m|^2 = |a_m|^2 e^{-2\lambda_m \sigma}$$

and series (6) converges, we have that, for $\sigma > \sigma_1$,

$$\sum_{m=1}^{\infty} \mathbb{E}|\xi_m|^2 \log^2 m < \infty.$$

Therefore, by Lemma 2 for $\sigma > \sigma_1$ the series

$$\sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m \sigma}$$

converges for almost all $\omega \in \Omega$ with respect to the Haar measure m_H. This shows that $f(\sigma, \omega)$, for $\sigma > \sigma_1$, is a random variable on $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. $\square$

Clearly, there exists general Dirichlet series with small exponents, for which series (6) converges. For example, if $\lambda_m = \log\log^2 m$ and $a_m = O(1/m)$, then series (1) converges absolutely for $\sigma \geq 1$. Suppose that it is analytically continuable to some region $\sigma > \sigma_1$ with $\sigma_1 < 1$. Then we have that

$$\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} \log^2 m \ll \sum_{m=1}^{\infty} \frac{\log^{2-2\sigma} m}{m^2} < \infty.$$

3 Limit theorems for Dirichlet polynomials

In the sequel we suppose that the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers. Let, for $n \in \mathbb{N}$ and fixed $\widehat{\omega} \in \Omega$,

$$g_{N,n}(s) = \sum_{m=1}^{N} a_m v(m,n) e^{-\lambda_m s}$$

and

$$g_{N,n}(s, \omega) = \sum_{m=1}^{N} a_m v(m,n) \omega(m) e^{-\lambda_m s},$$

where $v(m,n) = \exp\{-e^{(\lambda_m - \lambda_n)\sigma_2}\}, \sigma_2 > 0$, and on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ define two probability measures

$$P_{T,N,n,\sigma}(A) = \nu_T\big(g_{N,n}(\sigma + it) \in A\big)$$

and

$$\widehat{P}_{T,N,n,\sigma}(A) = \nu_T\big(g_{N,n}(\sigma + it, \widehat{\omega}) \in A\big).$$

Theorem 3. *On* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *there exists a probability measure* $P_{N,n,}$ *such that the probability measures* $P_{T,N,\sigma}$ *and* $\widehat{P}_{T,N,\sigma}$ *both converge weakly to* $P_{N,n,\sigma}$ *as* $T \to \infty$.

Proof. For the proof of Theorem 3, we introduce one more probability measure

$$Q_T(A) = \nu_T\big((e^{it\lambda_m})_{m \in \mathbb{N}} \in A\big), \quad A \in \mathcal{B}(\Omega).$$

The dual group of Ω is izomorphic to

$$\bigoplus_{m\in\mathbb{N}} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}$. $\underline{k} = \{k_m : m \in \mathbb{N}\} \in \bigoplus\limits_{m\in\mathbb{N}} \mathbb{Z}_m$, where only a finite number of integers k_m are non-zero, acts on Ω by

$$\underline{k} \to \omega^{\underline{k}} = \prod_{m=1}^{\infty} \omega^{k_m}(m), \quad \omega \in \Omega.$$

Therefore, the linear independence of $\{\lambda_m\}$ shows that the Fourier transform $g_T(\underline{k})$ of the measure Q_T is of the form

$$g_T(\underline{k}) = \int\limits_{\Omega} \Big(\prod_{m=1}^{\infty} \omega^{k_m}(m)\Big) dQ_T = \frac{1}{T}\int\limits_{0}^{T} \Big(\prod_{m=1}^{\infty} e^{it\lambda_m k_m}\Big) dt$$
$$= \begin{cases} 1, & \text{if} \quad k_m = 0 \quad \text{for all} \quad m \in \mathbb{N}, \\ \dfrac{\exp\{iT \sum\limits_{m=1}^{\infty} \lambda_m k_m\} - 1}{iT \sum\limits_{m=1}^{\infty} \lambda_m k_m}, & \text{otherwise.} \end{cases}$$

Hence we find that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1, & \text{if} \quad k_m = 0 \quad \text{for all} \quad m \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and a continuity theorem for probability measures on locally compact group, see, for example, [3], implies that the probability measure Q_T converges weakly to the measure m_H.

Let $h \colon \Omega \to \mathbb{C}$ be given by the formula

$$h\big(\{\omega(m) \colon m \in \mathbb{N}\}\big) = \sum_{m=1}^{\infty} \frac{a_m v(m,n) e^{-\lambda_m \sigma}}{\omega(m)}.$$

The function h is continuous and satisfies

$$h\big(\{e^{i\lambda_m t} \colon m \in \mathbb{N}\}\big) = g_{N,n}(\sigma + it).$$

Therefore, Theorem 5.1 of [4] and the weak convergence of the probability measure Q_T show that $P_{T,N,n,\sigma} = Q_T h^{-1}$ converges weakly to $m_H h^{-1}$ as $T \to \infty$.

Now define $h_1 : \Omega \to \Omega$ by

$$h_1\bigl(\{\omega(m) : m \in \mathbb{N}\}\bigr) = \bigl(\{\omega(m)\widehat{\omega}^{-1}(m) : m \in \mathbb{N}\}\bigr),$$

where $\widehat{\omega}$ is a fixed element of Ω. Then we have that

$$g_{N,n}(\sigma + it, \widehat{\omega}) = \sum_{m=1}^{N} \frac{a_m v(m,n) e^{-\lambda_m(\sigma+it)}}{\widehat{\omega}^{-1}(m)} = h\bigl(h_1(\{e^{i\lambda_m t} : m \in \mathbb{N}\})\bigr).$$

Hence, similarly as above, we obtain that the probability $P_{T,N,n,\sigma} = Q_T(hh_1)^{-1}$ converges weakly to the measure $m_H(hh_1)^{-1} = (m_H h_1^{-1})h^{-1} = m_H h^{-1}$, because the Haar measure m_H is invariant with respect to translations by points from Ω. The theorem is proved. $\square$

Note that in [1] an another proof of Theorem 3 based on the study of a finite-dimensional torus has been given.

4 Limit theorems for absolutely convergent Dirichlet series

In this section we construct a general Dirichlet series related to series (1) which converges absolutely for $\sigma > \sigma_1$. We also correct some inaccuracies of [1].

We take $\sigma_2 > \sigma_a - \sigma_1 > 0$ and define, for $\sigma > \sigma_1$,

$$g_n(s) = \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} f(s+z) l_n(z) \frac{dz}{z},$$

where

$$l_n(s) = \frac{s}{\sigma_2} \Gamma\Bigl(\frac{s}{\sigma_2}\Bigr) e^{\lambda_n s}.$$

Using Mellin's inversion formula and the definition of $l_n(s)$, we find that

$$\begin{aligned} g_n(s) &= \sum_{m=1}^{\infty} \frac{a_m e^{-\lambda_m s}}{2\pi i \sigma_2} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \Gamma\Bigl(\frac{z}{\sigma_2}\Bigr) e^{-(\lambda_m - \lambda_n)z} dz \\ &= \sum_{m=1}^{\infty} a_m \exp\{-e^{(\lambda_m - \lambda_n)\sigma_2}\} e^{-\lambda_m s} = \sum_{m=1}^{\infty} a_m v(m,n) e^{-\lambda_m s}. \end{aligned} \tag{8}$$

It remains to prove that the later series converges absolutely for $\sigma > \sigma_1$. Clearly,

$$g_n(s) = \sum_{m=1}^{\infty} a_m a_n(m) e^{-\lambda_m s}, \tag{9}$$

where

$$a_n(m) = \frac{1}{2\pi i \sigma_2} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \Gamma\Big(\frac{z}{\sigma_2}\Big) e^{-(\lambda_m - \lambda_n)s} ds \ll_n e^{-\lambda_m \sigma_2}.$$

This and (9) yield the absolute convergence of series (8) for $\sigma > \sigma_1$.

Define, for $\omega \in \Omega$,

$$g_n(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) v(m, n) e^{-\lambda_m s}.$$

The aim of this section is to obtain limit theorems for probability measures

$$P_{T,n,\sigma}(A) = \nu_T\big(g_n(\sigma + it) \in A\big)$$

and

$$\widehat{P}_{T,n,\sigma}(A) = \nu_T\big(g_n(\sigma + it, \omega) \in A\big),$$

where $A \in \mathcal{B}(\mathbb{C})$.

Theorem 4. *Let $\sigma > \sigma_1$. Then on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ there exists a probability measure $P_{n,\sigma}$ such that the probability measures $P_{T,n,\sigma}$ and $\widehat{P}_{T,n,\sigma}$ both converge weakly to $P_{n,\sigma}$ as $T \to \infty$.*

Proof. We will give a shortened proof, because it only in some details differs from that given in [1].

By Theorem 3 the probability measures $P_{T,N,n,\sigma}$ and $\widehat{P}_{T,N,n,\sigma}$ both converge weakly to the measure $P_{N,n,\sigma}$ as $T \to \infty$. We will prove that the family of probability measures $\{P_{N,n,\sigma}\}$ is tight. By the Chebyshev inequality, for any positive M,

$$\begin{aligned} P_{T,N,n,\sigma}\big(\{z \in \mathbb{C}\colon |z| > M\}\big) &= \nu_T\big(|g_{N,n}(\sigma + it)| > M\big) \\ &\le \frac{1}{MT} \int_0^T |g_{N,n}(\sigma + it)| dt. \end{aligned}$$

Since the series for $g_n(s)$ converges absolutely for $\sigma > \sigma_1$, there exists a constant $C > 0$ such that

$$\sup_{N\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \big|g_{N,n}(\sigma+it)\big| dt \le C.$$

For arbitrary $\varepsilon > 0$, let $M = C/\varepsilon$. Then we deduce from the last two inequalities that

$$\limsup_{T\to\infty} P_{T,N,n,\sigma}\big(\{z \in \mathbb{C}\colon |z| > M\}\big) \le \varepsilon. \tag{10}$$

The function $h\colon \mathbb{C} \to \mathbb{R},\ z \to |z|$, is continuous and so, by Theorem 3, the probability measure

$$\nu_T\big(|g_{N,n}(\sigma+it)| \in A\big), \quad A \in \mathcal{B}(\mathbb{R}),$$

converges weakly to $P_{N,n,\sigma}h^{-1}$ as $T \to \infty$. This, the properties of the weak convergence, and (10) imply

$$\begin{aligned} P_{N,n,\sigma}\big(\{z \in \mathbb{C}\colon |z| > M\}\big) &\le \liminf_{T\to\infty} P_{T,N,n,\sigma}\big(\{z \in \mathbb{C}\colon |z| > M\}\big) \\ &\le \limsup_{T\to\infty} P_{T,N,n,\sigma}\big(\{z \in \mathbb{C}\colon |z| > M\}\big) \le \varepsilon. \end{aligned}$$

Define $K_\varepsilon = \{z \in \mathbb{C}\colon |z| \le M\}$. Then K_ε is a compact set, and

$$P_{N,n,\sigma}(K_\varepsilon) \ge 1 - \varepsilon$$

for all $N \in \mathbb{N}$. This shows the tightness of the family $\{P_{N,n,\sigma}\}$. Hence by the Prokhorov theorem, see, for example, [4], the family $\{P_{N,n,\sigma}\}$ is relatively compact.

Now let a random variable θ_T be defined on a certain probability space $\big(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P}\big)$ and uniformly distributed on $[0, T]$. We put

$$X_{T,N,n}(\sigma) = g_{N,n}(\sigma + i\theta_T).$$

Then, by Theorem 3,

$$X_{T,N,n}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} X_{N,n}(\sigma), \tag{11}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $X_{N,n}(\sigma)$ is a complex-valued random variable with the distribution $P_{N,n,\sigma}$. Moreover, the relative compactness implies the existence of $\{P_{N_1,n,\sigma}\} \subset \{P_{N,n,\sigma}\}$ such that $P_{N_1,n,\sigma}$ converges weakly to some measure $P_{n,\sigma}$ as $N_1 \to \infty$. Then

$$X_{N_1,n}(\sigma) \xrightarrow[N_1\to\infty]{\mathcal{D}} P_{n,\sigma}. \tag{12}$$

This, (11), the relation

$$\lim_{N\to\infty} \limsup_{T\to\infty} \nu_T\big(|g_{N,n}(\sigma+it) - g_n(\sigma+it)| \geq \varepsilon\big) = 0$$

and Theorem 4.2 of [4] show that

$$X_{T,n}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} P_{n,\sigma}, \tag{13}$$

where

$$X_{T,n}(\sigma) = g_n(\sigma + i\theta_T).$$

Hence the measure $P_{T,n,\sigma}$ converges weakly to $P_{n,\sigma}$ as $T \to \infty$. By (13) the measure $P_{n,\sigma}$ does not depends on the choice of N_1, and we have that

$$X_{N,n}(\sigma) \xrightarrow[N\to\infty]{\mathcal{D}} P_{n,\sigma}. \tag{14}$$

To complete the proof of the theorem it remains to repeat the above arguments for random variables

$$\widehat{X}_{T,N,n}(\sigma) = g_{N,n}(\sigma + i\theta_T, \omega)$$

and

$$\widehat{X}_{T,n}(\sigma) = g_n(\sigma + i\theta_T, \omega),$$

and to use (14). □

5 Proof of Theorem 1

First we observe that the method of the contour integration shows that the function $f(s)$ is approximated in the mean by the function $g_n(s)$. More precisely, we have, for $\sigma > \sigma_1$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \big|f(\sigma+it) - f_n(\sigma+it)\big| dt = 0. \tag{15}$$

An analogous assertion is also valid for the function $f(s,\omega)$, namely, for $\sigma > \sigma_1$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \big|f(\sigma+it,\omega) - f_n(\sigma+it,\omega)\big| dt = 0. \qquad (16)$$

The details of the proof of (15) and (16) can be found in [1].

We introduce one more probability measure on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$. Let

$$\widehat{P}_{T,\sigma}(A) = \nu_T\big(f(\sigma+it,\omega) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 5. *Let $\sigma > \sigma_1$. Then on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ there exists a probability measure P_σ as $T \to \infty$.*

Proof. We argue similarly to the proof of Theorem 4. Repeating the proof of Theorem 4 with $f_n(\sigma+it)$ and $f_n(\sigma+it,\omega)$ in place of $f_{N,n}(\sigma+it)$ and $f_{N,n}(\sigma+it,\omega)$, respectively, and with $f(\sigma+it)$ and $f(\sigma+it,\omega)$ in place of $f_n(\sigma+it)$ and $f_n(\sigma+it,\omega)$, respectively, and using (15), (16) and Theorem 4.2 of [4], we obtain the theorem. □

Proof of Theorem 1. Define the one parameter group $\{\varphi_t\colon t \in \mathbb{R}\}$ of measurable measure preserving transformations on Ω by $\varphi_t(\omega) = a_t\omega, \omega \in \Omega$, where $a_t = \{e^{-i\lambda_m t}\colon m \in \mathbb{N}\}$. Then the group $\{\varphi_t\colon t \in \mathbb{R}\}$ is ergodic [1]. Let A be a continuity set of the measure P_σ in Theorem 5. Then by Theorem 5

$$\lim_{T\to\infty} \widehat{P}_{T,\sigma}(A) = P_\sigma(A). \qquad (17)$$

Taking

$$\theta(\omega) = \begin{cases} 1, & \text{if} \quad f(\sigma,\omega) \in A, \\ 0, & \text{if} \quad f(\sigma,\omega) \neq A, \end{cases}$$

we have that θ is a random variable on $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$, and

$$\mathbb{E}(\theta) = \int_\Omega \theta dm_H = m_H\big(\omega \in \Omega\colon f(\sigma,\omega) \in A\big) = P_f(A), \qquad (18)$$

where P_f is the distribution of $f(\sigma, \omega)$. Moreover, the process $\theta\big(\varphi_t(\sigma)\big)$ is ergodic, therefore by the classical Birkhoff-Khinchine theorem

$$\mathbb{E}(\theta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \theta\big(\varphi_T(\omega)\big) dt \tag{19}$$

for almost all ω with respect to the measure m_H. On the other hand,

$$\frac{1}{T} \int_0^T \theta\big(\varphi_t(\omega)\big) dt = \widehat{P}_{T,\sigma}(A).$$

This, (18) and (19) show that

$$\lim_{T\to\infty} \widehat{P}_{T,\sigma}(A) = P_f(A),$$

and in view of (17) we conclude that $P_\sigma(A) = P_f(A)$ for all continuity sets A of P_σ. Hence P_σ coincides with P_f, and the theorem is proved. □

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