On Some Extremal Problems on Linearly Invariant Classes

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Received: 13.11.2004
Accepted: 27.12.2004

Abstract. In present paper the definition of linearly invariant class of analytical in the right half-plane is given and some extremal problems on introduced class are solved. For solving we use method based on variational formulas with specially introduced omega-operator, defined on these classes. In case when domain is unit disk similar linearly invariant classes were considered by Ch. Pommerenke, V. Starkov, E.G. Kiriyatzkii.

Keywords: analytic functions, linearly invariant classes, omega-operator, normalizing operator, variational formula, half-plane.

1 Major notational conventions, and definitions and auxiliary statements

Let Π is a half-plane $\mathrm{Re}\, z > 0,\ A_n(\Pi)$ – class of analytical in Π functions $F(z)$ with condition $F^{(n)}(z) \neq 0,\ \forall z \in \Pi$, $\widetilde{A}_n(\Pi)$ – class of analytical in Π functions $F(z)$ from $A_n(\Pi)$, which are normalized by conditions:

$$F(1) = F'(1) = \ldots = F^{(n-1)}(1) = 0, \quad F^{(n)}(1) = n!\,.$$

Obviously, that for any fixed $m \geq 2$ every function $F(z)$ of $\widetilde{A}_n(\Pi)$ can be represented in form

$$F(z) = (z-1)^n + \sum_{k=2}^{m} a_{k,n}(z-1)^{n+k-1} + \Psi_m(z),$$

where $\Psi_m(z)$ – dependent on $F(z)$ analytical in Π function. Number

$$a_{k,n} = \frac{F^{(n+k-1)}(1)}{(n+k-1)!}$$

we call by k-th coefficient of function $F(z)$. Let us introduce the operator

$$N_n[F] = \frac{F(z) - F(1) - F'(1)(z-1) - \ldots - \frac{1}{(n-1)!}F^{(n-1)}(1)(z-1)^{n-1}}{\frac{1}{n!}F^{(n)}(1)},$$

which we call by normalizing operator. This operator transfers any function from $A_n(\Pi)$ to a function of class $\widetilde{A}_n(\Pi)$.

Denote by $A(\Pi)$ class of analytical in domain Π functions. The n-th order divided difference of function $F(z) \in A(\Pi)$ define (see [1, 2]) by formula

$$\big[F(z); z_0, \ldots, z_n\big] = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{F(\xi)d\xi}{(\xi - z_0)\ldots(\xi - z_n)},$$

where Γ is a simple closed contour, located in Π and covering all the points $z_0, \ldots, z_n \in \Pi$. In above formula among the points $z_0, \ldots, z_n \in \Pi$ may occur coincident.

Note that if $P(z)$ is a polynomial of the degree no higher than $n-1$, then

$$\big[P(z); z_0, \ldots, z_n\big] = 0, \quad \forall z_0, \ldots, z_n \in \Pi.$$

Denote by L a set of functions of shape $w = tz$, where $t > 0$. Every function of L univalently maps half-plane Π onto itself.

Let us arbitrarily choose $w \in L$ and introduce omega-operator of n-th order by formula

$$\Omega_n^w[F] = \frac{(z-1)^n \big[F(z); w(z), \overbrace{t, \ldots, t}^{n}\big]}{\dfrac{1}{n!}F^{(n)}(t)}.$$

This operator for any fixed $w = tz$ is defined on class $A_n(\Pi)$ and transfers every function of class $A_n(\Pi)$ to the function of class $\widetilde{A}_n(\Pi)$.

As it will be seen, the linearly invariant classes are defined using operators Ω_n^w, $w \in L$, so we find useful to give without proof some properties of these operators as four theorems formulated below [3].

Theorem 1. *For arbitrarily fixed $w = tz \in L$ and any function $F(z) \in \widetilde{A}_n(\Pi)$, the equation*

$$\Omega_n^w\big[F(z)\big] = N_n\big[F(tz)\big] \tag{1}$$

is valid.

Theorem 2 (On chain). *Let $w_1, w_2 \in \Lambda$ and $F_2 = \Omega_n^{w_1}[F_1]$, $F_3 = \Omega_n^{w_2}[F_2]$. Then $F_3 = \Omega_n^{w_3}[F_1]$, where $w_3 = w_1(w_2) \in L$.*

Theorem 3. *Only function $\Phi_{n,a}(z) = N_n[z^s]$, where $s \neq 0, 1, 2, \ldots, n-1$, $s = (n+1)a + n$ and $a = \frac{\Phi_{n,a}^{(n+1)}(1)}{(n+1)!}$, is a fixed point (fixed function) of operator Ω_n^w for any $w \in L$, i.e., $\Omega_n^w[\Phi_{n,a}] = \Phi_{n,a}$, $\forall w \in L$. This function belongs to class $\widetilde{A}_n(\Pi)$.*

Function $\Phi_{n,a}(z)$ is called by *main* one. Its expansion about point $z = 1$ has a shape

$$\Phi_{n,a}(z) = (z-1)^n + \sum_{k=2}^{\infty} c_{k,n}(z-1)^{n+k-1},$$

where for coefficients $c_{k,n}$, $k = 2, 3, 4, \ldots$ formula

$$c_{k,n} = \frac{n!}{(n+k-1)!}(n+1)a\big((n+1)a-1\big)\ldots\big((n+1)a-(k-2)\big) \tag{2}$$

is valid. In particular, $c_{2,n} = a$.

Let k-th coefficient of some function $F(z) \in \widetilde{A}_n(\Pi)$ is equal to number b_k, where $k \geq 2$. If b_k is the k-th coefficient of function $F(z;t) = \Omega_n^w\big[F(z)\big]$ for any $w \in L$, then k-th coefficient of function $F(z)$ we will call by *invariant coefficient* of this function.

Theorem 4. *Let equation*

$$\frac{n!}{(n+k-1)!}\prod_{m=0}^{k-2}\big((n+1)a-m\big) = b_k$$

with respect to a has $k-1$ of pairwise different roots $a_1, \ldots, a_{k-1}$. Then only functions of form

$$F(z) = \sum_{m=1}^{k-1} c_m \Phi_{n,a_m}(z), \quad c_1 + \ldots + c_{k-1} = 1$$

has number b_k as theirs k-th invariant coefficient.

Let us give the definition of linearly invariant class. Set S of functions $F(z)$ of $\widetilde{A}_n(\Pi)$ we will call by linearly invariant class of n-th order, if from belonging $F(z) \in S$ follows $\Omega_n^w\big[F(z)\big] \in S$ for any $w \in L$ [4].

Let us give some examples of linearly invariant classes of n-th order.

Example 1. $\widetilde{A}_n(\Pi)$ is a linear invariant class. Note that $\widetilde{A}_n(\Pi)$ contains any of linearly invariant classes.

Example 2. Let us fix in $\widetilde{A}_n(\Pi)$ function $F(z)$ and make up the class of functions $\Psi_w(z) = \Omega_n^w\big[F(z)\big]$, where w vary over all set L. Due to Theorem 2 (on chain), such class must be linearly invariant one. We will call this class as *simple* linearly invariant class and denote it by $\widetilde{\mathfrak{R}}_n(\Pi; F)$. Function $F(z)$ we will call by generator of simple class. For simple class we have the following

Property. *If $F_1(z) \in \widetilde{\mathfrak{R}}_n(\Pi; F)$, then $F(z) \in \widetilde{\mathfrak{R}}_n(\Pi; F_1)$ for any $w \in L$. In other words, if function $F(z)$ is the generator of simple class and $F_1(z) \in \widetilde{\mathfrak{R}}_n(\Pi; F)$, then function $F_1(z)$ must be the generator of this simple class too.*

Properties of simple class was investigated in [4].

Example 3. Simple linearly invariant class generated by main function $\Phi_{n,a}(z)$ consists only of this function.

Union of a set of linearly invariant classes of n-th order denote by $\widetilde{\mathfrak{F}}_n(\Pi)$. Denote by $K_n(\Pi)$ class of analytic in Π functions $F(z)$ such, that $\big[F(z); z_0, \ldots, z_n\big] \neq 0$ for any set of pairwise distinct $z_0, \ldots, z_n \in \Pi$. For $n = 1$ one has, as it easily seen, class $K_1(\Pi)$ of all univalent in Π functions, which play a large role in conformal mapping theory and in geometrical theory of analytical functions (see [5–7]).

In class $K_n(\Pi)$ one can extract subclass $\widetilde{K}_n(\Pi)$ normalized functions

$$F(z) = (z-1)^n + a_{2,n}(z-1)^{n+1} + \ldots$$

Example 4. Class $\widetilde{K}_n(\Pi)$ is a linearly invariant class [4].

In case when $n = 1$ and domain is unit disc E linearly invariant classes were considered by Ch. Pommerenke and by V. Starkov (see [8–10]).

2 Some variational formulas

Using the definition of normalizing operator N_n, and denoting $\Omega_n^w\big[F(z)\big]$ by $F(z;t)$, statement (1) of Theorem 1 we can rewrite in the form

$$F(z;t) = \Omega_n^w\big[F(z)\big] = \frac{F(tz) - P(z;t)}{\frac{1}{n!}F^{(n)}(t)t^n},$$

where

$$P(z;t) = F(t) + \frac{F'(t)t}{1!}(z-1) + \cdots + \frac{F^{(n-1)}(t)t^{n-1}}{(n-1)!}(z-1)^{n-1}.$$

Function $F(z;t)$ represent in form

$$F(z;t) = (z-1)^n + \sum_{k=2}^{m} a_{k,n}(t)(z-1)^{n+k-1} + \Psi_m(z;t) \in \widetilde{A}_n(\Pi), \tag{3}$$

where k-th coefficient $a_{k,n}(t)$ in (3) is representable by formula

$$a_{k,n}(t) = \frac{F^{(n+k-1)}(1)t^{k-1}n!}{(n+k-1)!F^{(n)}(t)}. \tag{4}$$

Represent also function $F(z;t)$ using Taylor's formula

$$F(z;t) = F(z;1) + F'_t(z;1)(t-1) + o(z;t-1), \tag{5}$$

where $\frac{o(z;t-1)}{t-1} \to 0$ when $t \to 1$ uniformly whit respect to z inside Π. It is easy to come to the conclusion that

$$F(z;1) = F(z). \tag{6}$$

For derivative with respect to t of function $F(z;t)$ at the point $t = 1$ the formula

$$F'_t(z;1) = zF'(z) - \big((n+1)a_{2,n} + n\big)F(z) - n(z-1)^{n-1} \tag{7}$$

is valid. Formula (5), taking into account (6) and (7) is called by *variational formula for function* $F(z) \in \widetilde{A}_n(\Pi)$. Represent function $a_{k,n}(t)$ using Taylor's formula

$$a_{k,n}(t) = a_{k,n}(1) + a'_{k,n}(1)(t-1) + o(t-1), \tag{8}$$

$\frac{o(t-1)}{t-1} \to 0$ when $t \to 1$. It is easy to see that

$$a_{k,n}(1) = a_{k,n}, \tag{9}$$
$$a'_{k,n}(1) = (n+k)a_{k+1,n} + (k-1)a_{k,n} - (n+1)a_{k,n}a_{2,n}. \tag{10}$$

Formula (8), taking into account (9) and (10) is called by *variational formula for coefficient* $a_{k,n}(t)$.

3 Applications of variational formulas

Using variational formulas we establish several theorems.

Theorem 5. *Let* $F_0(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$ *and at the point* $z_0 \in \Pi$, *where* $z_0 \neq 1$, *the condition*

$$\big|F_0(z_0)\big| \geq \big|F(z_0)\big|, \quad \forall F(z) \in \widetilde{\mathfrak{F}}_n(\Pi) \tag{11}$$

or condition

$$0 < \big|F_0(z_0)\big| \leq \big|F(z_0)\big|, \quad \forall F(z) \in \widetilde{\mathfrak{F}}_n(\Pi) \tag{12}$$

holds. Then in both cases equality

$$\mathrm{Re}\big\{\bar{F}_0(z_0)\big(z_0F'_0(z_0) - \big((n+1)a_{2,n} + n\big)F(z_0) - n(z_0-1)^{n-1}\big)\big\} = 0 \tag{13}$$

is true. Here

$$a_{2,n} = \frac{1}{(n+1)!}F_0^{(n+1)}(1).$$

Proof. Let us consider first case, i.e., when condition (11) holds. Variational formula (5) for function $F_0(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$ at the point z_0, is of following shape:

$$F_0(z_0;t) = F_0(z_0) + {F_0}'_t(z_0;1)(t-1) + o(z_0;t-1) \in \widetilde{\mathfrak{F}}_n(\Pi), \tag{14}$$

for any value of t which is sufficiently close to unit. For such t according to condition (11) of Theorem 1, we have inequality

$$\big|F_0(z_0)\big| \geq \big|F_0(z_0;t)\big|.$$

This inequality due to (14) we can substitute by inequality

$$\big|F_0(z_0)\big|^2 \geq \big|F_0(z) + {F_0}'_t(z_0;t)(t-1) + o(z_0;t-1)\big|^2 \tag{15}$$

for sufficiently small values of $|t-1|$. Carrying out operations in (15) we will get

$$\big|F_0(z_0)\big|^2 \geq \big|F_0(z_0)\big|^2 + 2\mathrm{Re}\big\{\bar{F}_0(z_0){F_0}'_t(z_0;1)\big\}(t-1) + 2\mathrm{Re}\big\{o(z_0;t-1)\big\}$$

or

$$0 \geq \mathrm{Re}\big\{\bar{F}_0(z_0){F_0}'_t(z_0;1)\big\}(t-1) + \mathrm{Re}\big\{o(z_0;t-1)\big\} \tag{16}$$

for sufficiently small values of $|t-1|$, where values of $t-1$ may be of opposite signs. Then, considering (16), come to the conclusion

$$\mathrm{Re}\big\{\bar{F}_0(z_0){F_0}'_t(z_0;1)\big\} = 0.$$

Now, using formula (7) at $z = z_0$, we get (13).

Analogously, if in theorem the condition (12) holds, we come to the equality (13). □

Theorem 6. *Let $F_0(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$ and at the point $z_0 \in \Pi$, where $z_0 \neq 1$, the condition*

$$\mathrm{Re}\big\{F_0(z_0)\big\} \geq \mathrm{Re}\big\{F(z_0)\big\}, \quad \forall F(z) \in \widetilde{\mathfrak{F}}_n(\Pi) \tag{17}$$

or condition

$$\mathrm{Re}\big\{F_0(z_0)\big\} \leq \mathrm{Re}\big\{F(z_0)\big\}, \quad \forall F(z) \in \widetilde{\mathfrak{F}}_n(\Pi) \tag{18}$$

holds. Then in both cases equality

$$\mathrm{Re}\big\{z_0F_0'(z_0) - \big((n+1)a_{2,n} + n\big)F_0(z_0) - n(z_0-1)^{n-1}\big\} = 0 \tag{19}$$

holds. Here

$$a_{2,n} = \frac{F_0^{(n+1)}(1)}{(n+1)!}.$$

Proof. Let us consider first case, i.e., when condition (17) holds. Variational formula (5) for function $F_0(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$ at the point z_0, is of following shape:

$$F_0(z_0;t) = F_0(z_0) + {F_0}'_t(z_0;1)(t-1) + o(z_0;t-1) \in \widetilde{\mathfrak{F}}_n(\Pi), \tag{20}$$

for any value of t which is sufficiently close to unit. For such t according to condition (17) of Theorem 2, we have inequality

$$\mathrm{Re}\big\{F_0(z_0)\big\} \geq \mathrm{Re}\big\{F_0(z_0;t)\big\}.$$

Due to (20), this inequality may be substituted by inequality

$$\mathrm{Re}\big\{F_0(z_0)\big\} \geq \mathrm{Re}\big\{F_0(z_0)\big\} + \mathrm{Re}\big\{{F_0}'_t(z_0;1)\big\}(t-1) + \mathrm{Re}\big\{o(z_0;t-1)\big\},$$

or by inequality

$$0 \geq \mathrm{Re}\big\{{F_0}'_t(z_0;1)\big\}(t-1) + \mathrm{Re}\big\{o(z_0;t-1)\big\} \tag{21}$$

for sufficiently small values of $|t-1|$, where values of $t-1$ may be of opposite signs. Thus, considering (21) come to the conclusion, that

$$\mathrm{Re}\big\{{F_0}'_t(z_0;1)\big\} = 0.$$

Using formula (7), we get (19). Similarly, if in theorem the condition (18) holds, we come to the equality (19). □

Remark 1. *Considering equalities* (13) *and* (19) *one can come to the conclusion, that main function* $\Phi_{n,a}(z)$ *satisfy the differential equation of first order*

$$zF'(z) - \big((n+1)a+n\big)F(z) - n(z-1)^{n-1} = 0.$$

Theorem 7. *Let for some fixed* $k = m \geq 2$ *the coefficient* $a^*_{m,n}$ *of function* $F_*(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$ *has property:*

$$\begin{aligned}|a^*_{m,n}| &= \frac{1}{(n+m-1)!}\big|F_*^{(n+m-1)}(1)\big| \geq |a_{k,n}| \\ &= \frac{1}{(n+m-1)!}\big|F^{(n+m-1)}(1)\big|\end{aligned}$$

for any function $F(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$. *Then equality*

$$\mathrm{Re}\big\{\bar{a}^*_{m,n}\big((n+m)a^*_{m+1,n} + (m-1)a^*_{m,n} - (n+1)a^*_{m,n}a^*_{2,n}\big)\big\} = 0 \tag{22}$$

holds. Here $a^*_{2,n}, a^*_{m,n}, a^*_{m+1,n}$ *are coefficients of function* $F_*(z)$.

Proof. Let us represent variational formula (8) for coefficient $a^*_{m,n}$ of function $F_*(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$:

$$a^*_{m,n}(t) = a^*_{m,n} + {a^*_{m,n}}'(1)(t-1) + o(t-1), \tag{23}$$

where $a^*_{m,n}$ is a coefficient of function $F_*(z;t) \in \widetilde{\mathfrak{F}}_n(\Pi)$. From here and taking into account that $|a^*_{m,n}| \geq \big|a^*_{m,n}(t)\big|$ we get inequality

$$|a^*_{m,n}|^2 \geq \big|a^*_{m,n} + {a^*_{m,n}}'(1) + o(t-1)\big|^2.$$

After several transformations we can reduce it to inequality

$$0 \geq \mathrm{Re}\big\{\bar{a}^*_{m,n}{a^*_{m,n}}'(1)\big\}(t-1)\mathrm{Re}\big\{o(t-1)\big\}, \tag{24}$$

which holds true for sufficiently small values of $|t-1|$ where $t-1$ may be of opposite signs. Then from (24) follows equality

$$\mathrm{Re}\big\{\bar{a}^*_{m,n}{a^*_{m,n}}'(1)\big\} = 0.$$

Using formula (10) we come to (22). ☐

Analogously one can prove

Theorem 8. *Let for some fixed $k = m \geq 2$ the coefficient $a^*_{m,n}$ of function $F_*(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$ has property*

$$\mathrm{Re}\big\{a^*_{m,n}\big\} \geq \mathrm{Re}\big\{a_{m,n}\big\}, \quad \forall F(z) \in \widetilde{\mathfrak{F}}_n(\Pi)$$

or property

$$\mathrm{Re}\big\{a^*_{m,n}\big\} \leq \mathrm{Re}\big\{a_{m,n}\big\}, \quad \forall F(z) \in \widetilde{\mathfrak{F}}_n(\Pi).$$

Then in both cases equality

$$\mathrm{Re}\big\{(n+m)a^*_{m+1,n} + (m-1)a^*_{m,n} - (n+1)a^*_{m,n}a^*_{2,n}\big\} = 0 \tag{25}$$

holds. Here $a^*_{2,n}, a^*_{m,n}, a^*_{m+1,n}$ are coefficients of function $F_*(z)$.

Remark 2. *Considering equalities* (22) *and* (25) *one can come to the conclusion, that coefficients $c_{k,n}, k = 2,3,4,\ldots$ of main function $\Phi_{n,a}(z)$, i.e., coefficients* (2)*, where $c_{2,n} = a$, satisfy the equation*

$$(n+k)c_{k+1,n} - \big((n+1)a - (k-1)\big)c_{k,n} = 0.$$

Thus, the main function $\Phi_{n,a}(z)$ has the property, that in many extremal problems it satisfy the extremal conditions.

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