On the Eigenvalue Problem for One-Dimensional Differential Operator with Nonlocal Integral Condition

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Received: 24.03.2004
Accepted: 08.04.2004

Abstract. The article investigates the eigenvalue problem for ordinary one-dimensional differential operator with nonlocal integral condition. Such a problem is met in the literature quite rarely and is considerably less investigated. Also the conditions for existence of non-positive eigenvalue or multiple eigenvalues are obtained.

Keywords: eigenvalue problem, one-dimensional differential operator, integral nonlocal condition, non-positive eigenvalues, multiple eigenvalues.

1 Introduction

During many years, and especially in the last two decades a lot of attention has been paid to problems of ordinary differential equations with different types of boundary conditions ([1]–[3], see also bibliography quoted in these articles). However the eigenvalue problems for one-dimensional differential operator with nonlocal condition are met quite rare and is considerably less investigated. There are only a few articles [4]–[6] devoted to this problem. This paper is dealing with the eigenvalue problem for one-dimensional operator with nonlocal integral condition. It is worth-while to note, the solution technique for eigenvalue problems with nonlocal condition is closely related to the methods of solution of non-local boundary value problems (see e.g., [5], [7]).

The authors of papers [4], [6] investigated the eigenvalue problem with the nonlocal condition relating to endpoints of the interval. In article [5] the similar problems are investigated for differential operators with respect to nonlocal condition of the type of Bitsadze-Samarski.

This paper while dealing with the eigenvalue problem subject to nonlocal integral condition considers both cases of simple and multiple eigenvalues.

2 The main statement

First of all, we consider the eigenvalue problem for one-dimensional differential operator with given integral condition

$$u'' + \lambda u = 0, \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u(1) = a \int_0^1 u(x)dx. \tag{3}$$

We find the values of λ such, that the problem has the solution $u(x)$ identically not equal to zero. Therefore we formulate three different cases.

Case 1. $\lambda = 0$. In this case, the equation (1) and the boundary condition (2) implies $u(x) = cx$. Putting it into (3), we get

$$c\left(1 - \frac{a}{2}\right) = 0.$$

Now we conclude the following. If $a \neq 2$, then for $\lambda = 0$ there exists only a trivial solution $u(x) \equiv 0$ of the problem. If $a = 2$ then $\lambda = 0$ is the eigenvalue of the problem (1)–(3), along with the corresponding eigenfunction $u = cx$, where c is any number.

Case 2. $\lambda < 0$. By equation (1) and condition (2), it follows that

$$u(x) = c \sinh \alpha x; \tag{4}$$

where $\alpha = \sqrt{-\lambda} > 0.$ Putting (4) into condition (3), after simple rearrangement, we obtain

$$c\left(\cosh\frac{\alpha}{2} - \frac{a}{\alpha}\sinh\frac{\alpha}{2}\right) = 0.$$

Hence, for $\lambda < 0$, solution (4) of the problem (1)–(3) exists for the value of α being a root of the equation

$$\cosh\frac{\alpha}{2} - \frac{a}{\alpha}\sinh\frac{\alpha}{2} = 0 \tag{5}$$

Let us find out the dependency of number of roots of the equation (5) on the value of a. Firstly, we put equation (5) into the form

$$\tanh\frac{\alpha}{2} = \frac{\alpha}{a}. \tag{6}$$

Taking into account the properties of the function $\tanh\alpha/2$ we conclude, that equation (6) has a single root $\alpha = 0$ as $-\infty < a \leq 2.$ For $a > 2$ there exist three roots of the equations (6): $\alpha = 0$, $\bar{\alpha} > 0$ and $-\bar{\alpha} < 0$. For the root $\alpha = 0$ it implies that $\lambda = 0$, while for the roots $\pm\bar{\alpha}$ we have $\bar{\lambda} = -(\pm\bar{\alpha})^2 < 0.$ Thus, for $a > 2$, the eigenvalue $\bar{\lambda} < 0$ of the problem (1)–(3) exists, such that $\sqrt{-\bar{\lambda}} = \bar{\alpha}$ is the only positive root of the equation (6). The corresponding eigenvector is defined by the formula (4), where $\alpha = \sqrt{-\bar{\lambda}}$.

Case 3. $\lambda > 0$. In this case equation (1) and condition (2) imply that

$$u(x) = c\sin\alpha x \quad \alpha = \sqrt{\lambda} > 0. \tag{7}$$

Putting this expresion into nonlocal condition (3), like in Case 2 we obtain:

$$\tan\frac{\alpha}{2} = \frac{\alpha}{a}. \tag{8}$$

Hence, for any value of a there exist infinitely many positive eigenvalues $\lambda_k = \alpha_k^2 > 0$ along with the corresponding eigenvectors of the form (7). The results of the three cases can be joined into the following statement:

For any value of a there exist infinitely many positive eigenvalues λ_k of the problem (1)–(3). *These eigenvalues are the roots of the equation*

$$\tan\frac{\sqrt{\lambda}}{2} = \frac{\sqrt{\lambda}}{a};$$

these correspond to eigenfunctions of the form

$$u_k(x) = \sin\sqrt{\lambda_k}x, \quad k = 1, 2, \dots .$$

Moreover, the following statement is true:

1) if $-\infty < a < 2$, *then there are no other eigenvalues;*

2) if $a = 2$ *then* $\lambda_0 = 0$ *is the eigenvalue of the problem* (1)–(3) *with the corresponding eigenfunction* $u_0(x) = x$.

3) if $2 < a < \infty$, *then one more negative eigenvalue* $\bar{\lambda} = -(\bar{\alpha})^2 < 0$ *exists. It corresponds to the only positive root of equation* (6)*. The corresponding eigenfunction is*

$$\bar{u}(x) = \sin\bar{\alpha}x.$$

3 Multiple eigenvalues

Instead of condition (3) let us consider now another nonlocal condition

$$u(1) = a\int\limits_{a_1}^{b_1} u(x)dx,$$

where $0 < a_1 < b_1 < 1$. Let us take, in particular, $a_1 = 1/4,\ b_1 = 3/4$. Thus, we consider the nonlocal condition

$$u(1) = a\int\limits_{1/4}^{3/4} u(x)dx. \tag{9}$$

As before, we will investigate three cases.

Case 1. $\lambda = 0$. Putting $u(x) = cx$ into (9) we obtain

$$c\left(1 - \frac{a}{4}\right) = 0.$$

Hence, if $a = 4$ then $\lambda = 0$ is the eigenvalue of the problem (1), (2), (9) and $u(x) = x$ is the corresponding eigenfunction.

Case 2. $\lambda < 0$. Putting (4) into (9) and performing simple rearrangement we see that $\alpha = \sqrt{-\lambda} > 0$ should satisfy the following equation:

$$\cosh\frac{\alpha}{2} = \frac{a}{\alpha}\sinh\frac{\alpha}{4}. \tag{10}$$

Rearranging this equation in the form of

$$\tanh\frac{\alpha}{2} = \frac{2\alpha}{a}\cosh\frac{\alpha}{4}$$

we see, that it has a single root $\bar{\alpha}$, which is positive, provided $a > 4$. Thus, if $a > 4$, a single negative eigenvalue $\bar{\lambda} = -(\bar{\alpha})^2$ exists along with the corresponding eigenfunction

$$\bar{u}(x) = \sinh\bar{\alpha}x.$$

Case 3. $\lambda > 0$. Putting (7) into (9), we obtain the following equation:

$$\cos\frac{\alpha}{2} = \frac{a}{\alpha}\sin\frac{\alpha}{4}. \tag{11}$$

It has infinitely many positive roots α_k.

Hence, as before in the case of nonlocal condition (3), the problem (1), (2), (9) has infinitely many eigenvalues $\lambda_k = \alpha_k^2$, along with the corresponding eigenfunctions $u_k(x) = \sin\alpha_k x$.

There is an essential difference between the eigenvalue problem (1)–(3) and analogous problem with nonlocal condition (9). It lies in that all the eigenvalues of the problem (1)–(3) are different, since equation (8) has no multiple roots. This situation is typical for such an eigenvalue problems, where the ordinary differential equation subject to the classical boundary conditions of either the first, the second or the third type is considered. The situation can change if we take the nonlocal condition (9) instead of condition (3). In this case, depending on values of a, equation (11) may have multiple roots.

To illustrate the situation, let us find the least value of a for which the problem (1), (2), (9) has multiple eigenvalue. Solving equation (11), we obtain that it has four positive different roots over the interval $(0, 8\pi)$, provided $0 < a < 4$; it has three possitive different roots and the root $\lambda_0 = 0$, when $a = 4$; it has three positive different roots, provided $4 < a < a^*,\ a^* \approx 18.99$; it has a single root, provided $a^* < a$. For $a = a^*$, in the interval $(0, 8\pi)$ there is one simple root $\alpha_1 \approx 10.94$, and one multiple root $\alpha_2 = \alpha_3 \approx 19.13$.

4 More general problem

Let us consider now more general eigenvalue problem:

$$u'' + \lambda u = 0, \tag{12}$$

$$u(0) = a_1 \int_0^1 u(x)dx, \tag{13}$$

$$u(1) = a_2 \int_0^1 u(x)dx. \tag{14}$$

The methodology of the solution of the problem let be the same as earlier.

Case 1. $\lambda = 0$. Putting the general solution

$$u(x) = c_1 + c_2(x) \tag{15}$$

of equation (12) into expressions (13), (14) and rearranging them we obtain the following system of equations with respect to unknown constants c_1, c_2:

$$\begin{cases} (1 - a_1)c_1 - \dfrac{a_1}{2}c_2 = 0, \\ (1 - a_2)c_1 + \Big(1 - \dfrac{a_2}{2}\Big)c_2 = 0. \end{cases}$$

Thus, there exists the solution (15), not identical zero, if

$$D = \begin{vmatrix} 1 - a_1 & -\dfrac{a_1}{2} \\ 1 - a_2 & 1 - \dfrac{a_2}{2} \end{vmatrix} = 0.$$

This implies that $a_1 + a_2 = 2$. Hence, if $a_1 + a_2 \neq 0$, then $\lambda = 0$ is not an eigenvalue of the problem, since $u(x) \equiv 0$. If $a_1 + a_2 = 2$ then $\lambda = 0$ is an eigenvalue of the problem along with the corresponding eigenfunction:

$$u(x) = 1 + \frac{2(1 - a_1)}{a_1}x.$$

Case 2. $\lambda < 0$. In this case the general solution of equation (12) has a form

$$u(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}, \quad \sqrt{-\lambda} = \alpha > 0.$$

Putting expression for $u(x)$ into both (13) and (14), we get the system of the equations with respect to unknown constants c_1, c_2:

$$\begin{cases} c_1\Big(1 - \dfrac{a_1}{\alpha}(e^{\alpha} - 1)\Big) + c_2\Big(1 + \dfrac{a_1}{\alpha}(e^{-\alpha} - 1)\Big) = 0, \\ c_1\Big(e^{\alpha} - \dfrac{a_2}{\alpha}(e^{\alpha} - 1)\Big) + c_2\Big(e^{-\alpha} + \dfrac{a_2}{\alpha}(e^{-\alpha} - 1)\Big) = 0. \end{cases}$$

Equating to zero the determinant of this system, we get the following condition of existence of the negative eigenvalue

$$\tanh\frac{\alpha}{2} = \frac{\alpha}{a_1 + a_2}. \tag{16}$$

Equation (16) has a single positive root $\bar{\alpha}$, provided $a_1 + a_2 > 2$. In this case, problem (12)–(14) has the negative eigenvalue $\bar{\lambda} = -(\bar{\alpha})^2$.

Case 3. $\lambda > 0$. In this case the general solution of equation (12) is of a form

$$u(x) = c_1 \cos\alpha x + c_2 \sin\alpha x, \quad \sqrt{\lambda} = \alpha > 0. \tag{17}$$

Putting this expression into (13) and (14) we obtain the system of the equations with respect to unknown constants c_1, c_2:

$$\begin{cases} \Big(1 - \dfrac{\sin\alpha}{\alpha}a_1\Big)c_1 + \dfrac{a_1(\cos\alpha - 1)}{\alpha}c_2 = 0, \\ \Big(\cos\alpha - \dfrac{\sin\alpha}{\alpha}a_2\Big)c_1 + \Big(\sin\alpha + \dfrac{a_2(\cos\alpha - 1)}{\alpha}\Big)c_2 = 0. \end{cases}$$

Again, there exists a solution of a form (17), identically not equal to zero, provided $D = 0$. Thus, we obtain a condition of existence of the positive eigenvalue

$$\tan\frac{\alpha}{2} = \frac{\alpha}{a_1 + a_2}. \tag{18}$$

This equation has infinitely many positive roots α_k, and $\lambda_k = \alpha_k^2$. The corresponding eigenfunctions are definited by (18).

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