The Method for Calculation the Hall Effect Parameters

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Abstract. A method for calculating the values of specific resistivity ρ as well as the product $\mu_H B$ of the Hall mobility and magnetic induction on a conductive sample of an arbitrary geometric configuration with two arbitrary fitted current electrodes of nonzero length and has been proposed an grounded. During the experiment, under the constant value U of voltage and in the absence of the magnetic field effect ($\mathbf{B} = 0$) on the sample, the current intensities $I(0),\ I_E(0)$ are measured as well as the mentioned parameters under the effect of magnetic fields $\mathbf{B}_1, \mathbf{B}_2$ ($B_1 \neq B_2$), i.e.: $I_E(\beta^{(i)}), I(\beta^{(i)})$, $i = 1, 2$. It has been proved that under the constant difference of potentials U and sample thickness d, the parameters $I(0), I_E(0)$ and $I_E(\beta^{(i)}), I(\beta^{(i)})$, $i = 1, 2$ uniquely determines the values of the product $\mu_H B$ and specific resistivity ρ of the sample. Basing on the conformal mapping method and Hall's tensor properties, a relation (a system of nonlinear equations) between the above mentioned quantities has been found.

Keywords: mathematical modelling, conductivity, Hall effect.

1 Introduction

When investigating the Hall effect one often has to solve the problem of finding the product $\mu_H B$ of the Hall mobility and magnetic induction as well as specific resistivity ρ of as sample. A widespread way of solving such a problem is the Van der Pauw method [1]: four "point" electrodes are fitted on a plane sample circuit the first two of which are charged with current and the rest two are meant for measuring the emerging difference of potentials. The advantage of this method consists in the fact that the only geometric parameter to be known is thickness d of a sample, while there are no restrictions on the shape of the sample. This method has serious drawback ([2]–[4]) appearing mainly due to the fact that real contacts are not "point like", they are of a certain nonzero length.

Later on (in [5, 6]) there were some attempts to improve the Van der Pauw method when investigating the samples whose form configuration is exactly defined bounded and has second – or fourth order direct or inverse symmetry axes (perpendicular to the plane of the sample). In these works, mappings of the regions with the shape of the sample were applied into regions in which it is easy to calculate the distribution of electric potential and the value of current (similar method were applied in Van der Pauw's work mentioned [1]). The reasons for errors emerging while applying these methods are inaccuracies in the arrangement of a real sample geometric shape and contacts as well as inexact calculation of the integrals that realize the mapping.

The authors of this work have proposed a method [7] for solving the problem considered on a sample of an arbitrary shape with two current electrodes arbitrarily arranged on its contour. To carry out an experiment we need two samples of the same substance but of different geometric shape. By combining them in parallel (Fig. 1) and after measuring current intensities under the effect of the magnetic field (perpendicular to the plane of the sample) and in the absence of, it is possible to calculate the parameters in quest. In applying this method, one has to calculate the integral for many times

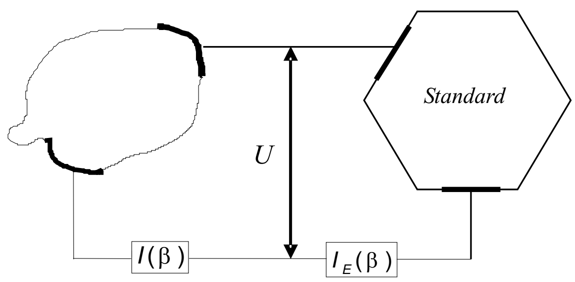

Fig. 1. Parallel connection of the samples.

$$F(\beta, k) = \int_0^1 t^{-\alpha}(1-t)^{\alpha-1}\big(1-(1-k)t\big)^{-\alpha} dt, \tag{1}$$

where $\alpha = \beta/\pi + 0.5,\ \beta = \mathrm{arctg}(\mu_H B)$.

To calculate integral (1), the authors have proposed and grounded an efficient algorithm [8] that does not differ in calculation volume and accuracy from the usual for algorithms calculating the elementary function values.

It should be mentioned that, employing this method during the experiment, the errors of the measured quantities, especially with current electrodes of short length have a tangible influence on the calculated quantities ρ and $\mu_H B$. In order to diminish the influence of measurement errors, we can prepare an experiment by using the standard sample (made from the substance specific resistivity ρ of which is known). In this case, the number of unknown parameters decreases, and therefore the measurement errors have less influence on the calculation results.

2 Determination of the parameters $\mu_H Bp$ and ρ by using the standard sample

Let a standard and a tested sample be of an arbitrary geometric shape and each have two current contacts of any length. During the experiment, under a parallel connection of the samples (Fig. 1), the intensities $\mathbf{B}_1, \mathbf{B}_2$ $(B_1 \neq B_2)$, of currents flowing through the samples are measured under the effect of magnetic fields $I_E(\beta^{(i)}), I(\beta^{(i)}), i = 1, 2$ and $I_E(0), I(0)$, of distinct intensity, and when there is no magnetic field ($\mathbf{B} = 0$). Note that during the experiment the values of B_1, B_2 and that of Hall mobility of the standard sample are unknown.

It has been proved in [7] that the current $I(\beta)$, present in the sample can be calculated:

$$I(\beta) = \frac{Vd}{\rho} \frac{F(\beta, 1-k)}{F(\beta, k)} \cos\beta. \tag{2}$$

Here d is the sample thickness, k is a dimensionless quantity $(0 < k < 1)$, dependent on the geometric shape of a sample, the length of contacts and their arrangement in the contour. Therefore, for the values $I_E(0), I_E(\beta^{(1)}), I_E(\beta^{(2)})$ of currents – $\widehat{I}_E(0), \widehat{I}_E(\beta^{(1)}), \widehat{I}_E(\beta^{(2)})$, obtained during the experiment we can make up a system of nonlinear equations:

$$\begin{cases} \widehat{I}_E(0) = \dfrac{Vd}{\rho_E}\,\dfrac{F(0,1-k_E)}{F(0,k_E)}, \\ \widehat{I}_E(\beta_E^{(1)}) = \dfrac{Vd}{\rho_E}\,\dfrac{F(\beta_E^{(1)},1-k_E)}{F(\beta_E^{(1)},k_E)}\cos\beta_E^{(1)}, \\ \widehat{I}_E(\beta_E^{(2)}) = \dfrac{Vd}{\rho_E}\,\dfrac{F(\beta_E^{(2)},1-k_E)}{F(\beta_E^{(2)},k_E)}\cos\beta_E^{(2)}. \end{cases} \tag{3}$$

The system (3) is solved in such a way. The first equation of the system is uniquely solved with respect to k_E, and the second and third equations (after replacing the unknown k_E in them by the solution $\bar{k}_E$) with respect to $\beta_E^{(1)}$, $\beta_E^{(2)}$ ([9, 10]). Having solved them, i.e., having found $\bar{k}_E$, $\bar{\beta}_E^{(1)}$, and $\bar{\beta}_E^{(2)}$, we calculate the ratio of magnetic inductions:

$$w = \frac{B_2}{B_1} = \frac{\operatorname{tg}\bar{\beta}_E^{(2)}}{\operatorname{tg}\bar{\beta}_E^{(1)}}. \tag{4}$$

After replacing the values of the sample currents $I(0), I(\beta^{(1)}), I(\beta^{(2)})$ by those obtained in the experiment $\widehat{I}(0), \widehat{I}(\beta^{(1)}), \widehat{I}(\beta^{(2)})$, denoting their ratios: $\widehat{s}_1 = \widehat{I}(\beta^{(1)})/\widehat{I}(0), \widehat{s}_2 = \widehat{I}(\beta^{(2)})/\widehat{I}(0)$ and making use of (2), we have

$$\begin{cases} \dfrac{F(\beta^{(1)},1-k)}{F(\beta^{(1)},k)}\cos\beta^{(1)} - \widehat{s}_1\dfrac{F(0,1-k)}{F(0,k)} = 0, \\ \dfrac{F(\beta^{(2)},1-k)}{F(\beta^{(2)},k)}\cos\beta^{(2)} - \widehat{s}_2\dfrac{F(0,1-k)}{F(0,k)} = 0. \end{cases} \tag{5}$$

Each of these equations is uniquely solved with respect to k ($0 < k < 1$) for any fixed value β_i therefore system (5) is equivalent to two functional equations whose solutions are:

$$\beta^{(i)} = \beta^{(i)}(k), \quad 0 < k < 1, \quad i = 1, 2, \tag{6}$$

i.e.,

$$\begin{cases} \dfrac{F\big(\beta^{(1)}(k), 1-k\big)}{F\big(\beta^{(1)}(k), k\big)} \cos\beta^{(1)}(k) - \widehat{s}_1 \dfrac{F(0, 1-k)}{F(0, k)} \equiv 0, \\ \dfrac{F\big(\beta^{(2)}(k), 1-k\big)}{F\big(\beta^{(2)}(k), k\big)} \cos\beta^{(2)}(k) - \widehat{s}_2 \dfrac{F(0, 1-k)}{F(0, k)} \equiv 0. \end{cases}$$

Making use of relation (4), we have the equation for the parameter k:

$$\operatorname{tg}\beta^{(2)}(k) = w \operatorname{tg}\beta^{(1)}(k), \tag{7}$$

having solved it and replacing the parameter k in system (6) by the solution $\bar{k}$, to equation (7), we find that $\bar{\beta}^{(1)} = \beta^{(1)}(\bar{k})$ and $\bar{\beta}^{(2)} = \beta^{(2)}(\bar{k})$. If we know these quantities, we can easily calculate the Hall effect parameter values in quest:

$$\begin{aligned} &\rho = \frac{Vd}{I(0)} \, \frac{F(0, 1-\bar{k})}{F(0, \bar{k})}, \\ &\mu_H B_1 = \operatorname{tg}\bar{\beta}^{(1)}, \; \mu_H B_2 = \operatorname{tg}\bar{\beta}^{(2)}. \end{aligned} \tag{8}$$

3 Iterative procedure for calculating parameters

We present a method that can help to calculate the specific resistivity ρ and quantities $\mu_H B$ of the sample.

In the first stage, by using the current values of the standard sample, we find the ratio of applied fields magnetic inductions B_1, B_2. The current of the standard sample under the inactive magnetic field is equal to

$$\widehat{I}_E(0) = \frac{Vd}{\rho_E} \, \frac{F(0, 1-k_E)}{F(0, k_E)}, \tag{9}$$

where $\widehat{I}_E(0)$ – is the current obtained in the experiment. Let

$$\Phi(\beta, k, I) \equiv I(\beta)\rho_E F(0, k) - VdF(0, 1-k),$$

then the equation $\Phi\big(0, k_E, \widehat{I}_E(0)\big)$ is equivalent to equation (9). By applying an iterative scheme

$$\begin{cases} k_{E,0} = 0.5, \\ k_{E,j} = k_{E,j-1} - 2^{-(j+1)} \operatorname{sgn}\Phi\big(0, k_{E,j-1}, \widehat{I}_E(0)\big), \\ j = 1, 2, \ldots, N \end{cases}$$

we can find the geometric parameter $\bar{k}_E = k_{E,N}$ of this sample. Now, if we know $\bar{k}_E$, from the equations $\Phi\big(\beta_E^{(1)}, \bar{k}_E, \widehat{I}(\beta_E^{(1)})\big)=0$, $\Phi\big(\beta_E^{(2)}, \bar{k}_E, \widehat{I}(\beta_E^{(2)})\big)=0$, by applying a similar iterative scheme (as $i = 1$ and $i = 2$)

$$\begin{cases} \beta_{E,0}^{(i)} = \pi/4, \\ \beta_{E,j}^{(i)} = \beta_{E,j-1}^{(i)} - 2^{-(j+2)}\pi \operatorname{sgn} \Phi\big(\beta_{E,j-1}^{(i)}, \bar{k}_E, \widehat{I}(\beta_E^{(i)})\big), \\ j = 1, 2, \ldots, N, \end{cases}$$

we find the parameters $\bar{\beta}_E^{(1)} = \beta_{E,N}^{(1)}$ and $\bar{\beta}_E^{(2)} = \beta_{E,N}^{(2)}$ of the standard sample as well as the ratio of magnetic inductions

$$\bar{w} = \frac{B_2}{B_1} = \frac{\mu_{H_E} B_2}{\mu_{H_E} B_1} = \frac{\mu_H B_2}{\mu_H B_1} = \frac{\operatorname{tg}(\bar{\beta}_E^{(2)})}{\operatorname{tg}(\bar{\beta}_E^{(1)})} = \frac{\operatorname{tg}(\bar{\beta}^{(2)})}{\operatorname{tg}(\bar{\beta}^{(1)})}. \tag{10}$$

In the second stage, the currents $\widehat{I}(0), \widehat{I}(\beta^{(i)}), i = 1, 2$ of the tested sample, obtained during the experiment, are used. The ratios $\widehat{I}(\beta^{(i)})/\widehat{I}(0)$ of these currents and equation (10), using the notation

$$\Psi(\beta, k, I) \equiv \frac{F(\beta, 1-k)}{F(\beta, k)} \cos\beta - \frac{I(\beta)}{I(0)} \, \frac{F(0, 1-k)}{F(0, k)},$$

are defined by the system

$$\begin{cases} \Psi\big(\beta^{(1)}, k, I(\beta^{(1)})\big) = 0, \\ \Psi\big(\beta^{(2)}, k, I(\beta^{(2)})\big) = 0, \\ \operatorname{tg}\beta^{(2)} = \bar{w} \operatorname{tg}\beta^{(1)} \end{cases}$$

whose solution $\beta^{(1)}, \beta^{(2)}, k$ – are the parameters of the sample. To solve this system, an embedded iterative process (small braces indicate an internal iterative process for calculating the parameters $\beta^{(1)}, \beta^{(2)}$, at each step in of the external iterative process, i.e., as $k = k_m$) has been created:

$$\begin{cases} k_0 = 0.5, \\ \begin{cases} \beta_0^{(1)} = \beta_0^{(2)} = \pi/4, \\ \beta_j^{(1)} = \beta_{j-1}^{(1)} + 2^{-(j+2)}\pi \operatorname{sgn} \Psi\big(\beta_{j-1}^{(1)}, k_m, \widehat{I}(\beta^{(1)})\big), \\ \beta_j^{(2)} = \beta_{j-1}^{(2)} + 2^{-(j+2)}\pi \operatorname{sgn} \Psi\big(\beta_{j-1}^{(2)}, k_m, \widehat{I}(\beta^{(2)})\big), \\ j = 1, 2, \ldots, N, \end{cases} \\ k_{m+1} = k_m + 2^{-(m+2)} \operatorname{sgn}\big(\operatorname{tg}\beta_N^{(2)} - w \operatorname{tg}\beta_N^{(1)}\big) \operatorname{sgn}(1 - w), \\ m = 0, 1, \ldots, N. \end{cases} \tag{11}$$

If we calculate $\bar{\beta}^{(1)} = \beta_N^{(1)}$, $\bar{\beta}^{(2)} = \beta_N^{(2)}$, $\bar{k} = k_N$ we determine the key parameters:

$$\bar{\rho} = \frac{Vd}{\widehat{I}(0)} \frac{F(0, 1-\bar{k})}{F(0, \bar{k})},$$
$$\bar{\mu}_H B_i = \operatorname{tg} \bar{\beta}^{(i)}, \ i = 1, 2.$$

At each step of the iterative procedures described we have to calculate singular integral (1), therefore we present a simple enough but of high accuracy calculation algorithm [5] of integral (1).

Let

$$a_0 = \pi / \cos\beta, \quad a_n = a_{n-1}\Big(1 - \frac{1}{n} + \frac{\alpha - \alpha^2}{n^2}\Big), \quad n = 1, \dots, 3p,$$
$$b_1 = \frac{1}{3p+1}, \quad b_n = b_{n-1}\frac{n-1}{3p+n}, \quad n = 2, \dots, p,$$
$$c_1 = \alpha - \alpha^2, \quad c_n = c_{n-1} - \frac{\alpha - \alpha^2}{n^2 - n}\sum_{i=1}^{n-1} c_i, \quad n = 2, \dots, p,$$
$$d_n = c_n / n, \quad n = 1, \dots, p,$$
$$e_p = d_p, \quad e_n = d_n - u e_{n+1}, \quad n = p-1, \dots, 1, \quad u = k/(1-k),$$

Then, with the absolute error not exceeding 10^{-p} ($1 \leq p \leq 18$), we have

$$F(\beta, k) = \sum_{n=0}^{3p} a_n (1+k)^n - R_p,$$
$$R_p = \begin{cases} (1+ue_1)\Big(\ln k + \sum_{n=1}^{3p} \frac{(1-k)^n}{n}\Big) + (1-k)^{3p} \sum_{n=1}^{p} b_n e_n, & k < 0.5, \\ 0, \quad k \geq 0.5. \end{cases}$$

In conclusion we present particular results calculated by the iterative process (11): Fig's 2 and 3 show the absolute errors of the parameters $\bar{\beta}^{(1)}, \bar{\beta}^{(2)}, \bar{k}$

$$\Delta\beta_j^{(1)} = |\bar{\beta}^{(1)} - \beta_j^{(1)}|, \quad \Delta\beta_j^{(2)} = |\bar{\beta}^{(2)} - \beta_j^{(2)}|, \quad \Delta k_j = |\bar{k} - k_j|,$$

and Table 1 illustrates the parameters values obtained in iterations.

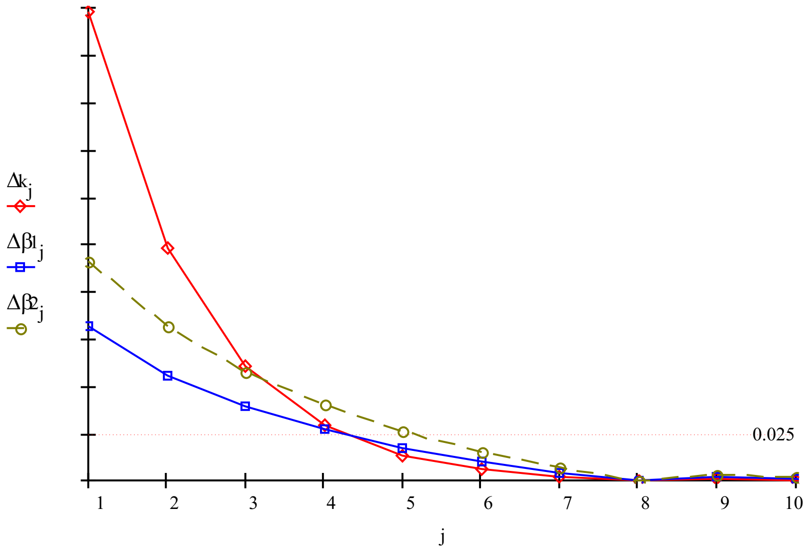

Fig. 2. Absolute errors of the parameters $k, \beta^{(1)}, \beta^{(2)}$ (iteratives process (11), $j \geq 10$).

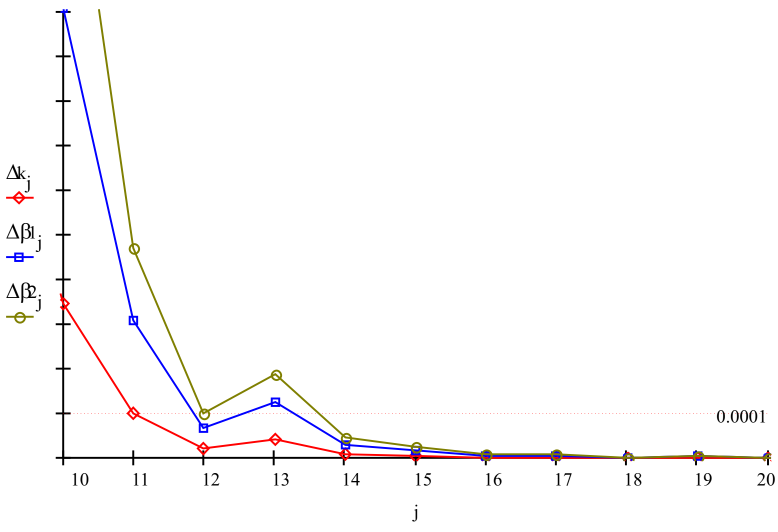

Fig. 3. Absolute errors of the parameters $k, \beta^{(1)}, \beta^{(2)}$ (iteratives process (11), $j \leq 10$).

Table 1. Parameters values obtained in iterations

j	$\beta_j^{(1)}$	$\beta_j^{(2)}$	k_j
1	0.63020009957643	0.98402857310047	0.50000000000000
2	0.58530926374514	0.92751348306746	0.25000000000000
3	0.56002053440066	0.89345085703169	0.12500000000000
4	0.54314810945795	0.86977250728670	0.06250000000000
5	0.53094614931435	0.85215958356268	0.03125000000000
6	0.52167613774475	0.83850157958122	0.01562500000000
7	0.51438714713206	0.82759456126279	0.00781250000000
8	0.50850484604846	0.81868588952346	0.00390625000000
9	0.50365863173432	0.81127596167587	0.00195312500000
10	0.50638417771731	0.81545111205499	0.00292968750000
11	0.50512281798257	0.81352135372481	0.00244140625000
12	0.50442096770147	0.81244574881883	0.00219726562500
13	0.50404814227124	0.81187384975953	0.00207519531250
14	0.50423653315316	0.81216287994797	0.00213623046875
15	0.50414284500550	0.81201915494602	0.00210571289063
25	0.50411308374136	0.81197349393447	0.00209608674049
30	0.50411309816279	0.81197351606102	0.00209609139711
35	0.50411309969507	0.81197351841196	0.00209609189187
40	0.50411309977112	0.81197351852865	0.00209609191643
45	0.50411309976891	0.81197351852527	0.00209609191572
50	0.50411309976887	0.81197351852520	0.00209609191570

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