Features and Partial Derivatives of Bertalanffy-Richards Growth Model in Forestry*

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Abstract. The Bertalanffy-Richards growth model is employed more than any other models for forest growth and yield modelling. However, its features have not completely been recognised. As a result, misunderstanding of the model still appears in some papers published in forest journals. A study by [1] is cited here as an evidence of the misunderstanding. This paper tries to explain different features of the Bertalanffy-Richards growth model based on the different conditions of the allometric parameter and introduces an assessment software to easily get the partial derivatives with respect to each parameter when more complex techniques (e.g., the Marquardt method) are employed to estimate parameters of any nonlinear models. This paper indicates that [1] study appears some unreasonable evidences of nonlinear growth models from a forestry perspective.

Keywords: feature, Bertalanffy-Richards, nonlinear analysis, growth model, forestry.

1 Introduction

Many nonlinear theoretical models (e.g., the logistic, the Gompertz, the Bertalanffy-Richards and the Schnute models) rather than empirical models (e.g.,

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polynomial model) have been used to model forest growth and yield and tree height-diameter relationships (e.g. [2]–[6]) because theoretical models have an underlying hypothesis associated with cause or function of the phenomenon described by the response variable [7]. However, empirical models such as polynomial equations were not considered as modeling nonlinear growth and yield in forestry because they are devoid of any biological interpretation and do not have meaningful parameters from a forestry perspective. In theoretical models, the Bertalanffy-Richards (or Chapman-Richards) growth model has been commonly used historically for modeling forest growth and yield. This model, valued for its accuracy, has been employed more than any other functions in studies of tree and stand growth [5]. So far, about 90% of the literature consulted has utilized this model in forest growth and yield [8]. However, the mathematical features and the growth performance of the Bertalanffy-Richards growth function have been not fully understood and there have still existed some unclear conceptions for the growth function. For instance, whether the Bertalanffy-Richards function has a point of inflection or not and appears sigmoid or concave curve shape. The study by [1] is a useful contribution to nonlinear growth models. Their study was based on deriving the partial derivatives of the nine well-known nonlinear growth models because the Marquardt iterative method [9] was employed to fit the parameters of the growth models, and gave the method of parameter estimation using experimental height growth data and the features of the nonlinear models. However, we believe that there are some limitations that need to be discussed, in particular, the features of the Bertalanffy-Richards growth model and the partial derivative of nonlinear growth models. In this paper, we discuss the features of the Bertalanffy-Richards growth model and examine the evidences and arguments of the Bertalannffy-Richards growth model presented by [1].

2 Features of Bertalanffy-Richards growth model

From a forestry perspective, [1] indicated that the negative exponential, monolocular and the Mithcherlich growth models have no points of inflection and are not sigmoid shape, while the Gompertz, logistic, Chapman-Richards (or Bertalanffy-Richards), Richard's and the von Bertalanffy growth models have points of inflection and are sigmoid. This is a misunderstanding of the Chapman-Richards growth model because whether the growth model possesses a point of inflection or not mainly depends on the allometric parameter m. In other word, whether the growth model demonstrates an sigmoid or concave curve shape is based on the different conditions of the allometric parameter m. Some features of the Bertalanffy-Richards growth model will be discussed on the basis of the conditions of allometric parameter m in order to clarify whether the growth model possesses a point of inflection or not (i.e., sigmoid or concave curve).

According to [10] and [11], the simplest assumption leading to limited growth is that the growth rate is proportional to the current size (y), that is:

$$\frac{dy}{dt} = k(\alpha - y) \tag{1}$$

where y is any variable, t is time, k and α are constants.

Much greater flexibility is obtained by substituting a power transformation, y^v, for y:

$$\frac{dy^v}{dt} = k(\alpha^v - y^v). \tag{2}$$

If the derivative on the left-hand side (LHS) of equation (2) is calculated, equation (2) can be rewritten as follows:

$$vy^{v-1}\frac{dy}{dt} = k(\alpha^v - y^v),$$
$$\frac{dy}{dt} = \frac{k}{v}y\Big[\Big(\frac{\alpha}{y}\Big)^v - 1\Big]$$

or

$$\frac{dy}{dt} = \eta y^m - ry \tag{3}$$

where $\eta = \dfrac{k\alpha^v}{v},\ m = 1 - v,\ r = k/v.$

Equation (3) is the Bertalanffy-Richards growth rate (differential) equation. The integral forms of equation (3) describe the size as an explicit function of age and can provide additional information about growth patterns. Moreover, we found that there are many different solutions and features from equation (3), which depend on the parameters m, η and r. Under the initial condition $y = y_0$ at $t = 0$ in equation (3), the solutions and features are respectively.

For $m > 1,\ \eta$ and $r < 0;\ 0 < m < 1,\ \eta$ and $r > 0$ and $m < 0,\ \eta$ and $r > 0$, the integral form of equation (3) is:

$$y = A\big[1 - B\exp(-kt)\big]^{(1/(1-m))} \tag{4}$$

where A is an asymptote value of the response y, B is a biological constant, k is related to proportional of y and m is a shape parameter of the growth curve (or an allometric constant), respectively. The relations between them are $A = (\eta/r)^{\left(1/(1-m)\right)},\ B = (\eta/r - y_0^{1-m})/(\eta/r)$ and $k = (1-m)r$.

For $m < 0,\ \eta > 0$ and $r < 0$, and making $r' = -r$, the integral form of equation (3) is:

$$y = A'(B'e^{k't} - 1)^{(1/(1-m))} \tag{5}$$

where $A' = (\eta/r')^{(1/(1-m))},\ B' = 1 + \left(\frac{y_0}{A'}\right)^{(1-m)},\ k' = r'(1-m)$.

When $m < 1,\ \eta$ and $r > 0$, the integral equation (4) possesses a sigmoid curve with an upper asymptote A (or y_∞) and an inflection point (t_δ, y_δ) which is obtained by $d^2y/dt^2 = 0$, and intersect the time axis at age t_0. This curve represents the classical growth situation which is widespread in biology and forest growth modelling (e.g. [2], p. 6–8).

When $m > 1,\ \eta$ and $r < 0$, equation (4) has an S-shaped curve again. Unlike the above growth situation, however, the curve has a lower asymptote (y_0) as $t \to 0$ besides having an upper asymptote A (or y_∞) and an inflection point (t_δ, y_δ). This curve is often seen in forest growth modelling. The above two types of curves start at a fixed point ($(t_0, 0)$ or $(0, y_0)$) and increase their instantaneous growth rates monotonically until an inflection point is reached; after this the growth rates decrease to approach asymptotically some final value as determined by the genetic nature of the living organism and the carrying capacity of the environment.

When $m < 0,\ \eta$ and $r > 0$, equation (4) possesses an upper asymptote A (or y_∞) and cross the time axis, but it has no inflection point (t_δ, y_δ). The case of $m < 0$ contradicts earlier papers (e.g. [12], p. 1989), which concluded that

$m > 0$ in equation (4) defines a subset of realistic solutions satisfying basic requirements for growth curves. Actually, a curve for the case m<0 can be used to simulate growth for fast-growing trees. This curve form has been widely used to fit fast-growing young eucalypt trees in forest growth (e.g. [13], p. 44) and to describe the law of diminishing returns in agriculture and economics [5]. The curve is rapid at an initial period and the instantaneous growth rate is monotonically decreasing to approach asymptotically some final value.

When $m < 0$, $\eta > 0$ and $r < 0$, equation (5) does not possess an asymptote, but an inflection point (t_δ, y_δ) is present and the curve does intersect the t-axis. In this case, an initial period of decelerated growth starts at t_0 and then continues later with an indefinite period of accelerated growth. Such a curve may be uncommon in forest growth modelling, but may be seen when competition inducing mortality occurs to the extent that dbh growth for residual trees accelerates (e.g. [14], p. 792–793). This case describes unlimited growth as age increases. Generally speaking, it contradicts tree or stand growth, which tends to a certain finite value as age increases. However, there also is strong evidence that growth volume per hectare in even-aged stands is not asymptotic [15].

It can be seen from the above analysis that the parameter m should be greater than zero when modelling forest growth. It is possible to model forest growth using the Beterlanffy-Richards function with $m < 0$, but in this case r should be greater than zero.

To derive special cases of the Bertalanffy-Richards growth model, equation (2) can also be expressed as follows:

$$\frac{dw}{dt} = -kw \tag{6}$$

where the power transformation is used:

$$w = \begin{cases} (y^v - \alpha^v)/v & \text{for} \quad v \neq 0, \\ \ln(y/\alpha) & \text{for} \quad v = 0. \end{cases} \tag{7}$$

This continuous family depends on a single parameter v (or m). When $v = 0$ (or $m = 1$), the Gompertz model can be obtained from equation (7):

$$y = \alpha \exp\big[-C\exp(-kt)\big] \tag{8}$$

where C is constant of integration or biological constant, α is the value of the asymptote and k is a growth rate related parameter.

When $v = -1$ $(m = 2)$ and $v = 1$ $(m = 0)$, the logistic model and the monomolecular model can also be obtained from equation (7). They are as follows, respectively:

$$y = \alpha / \big[1 + C \exp(-kt)\big], \tag{9}$$

$$y = \alpha + C \exp(-kt) \tag{10}$$

where $C,\ \alpha$ and k are as previously defined. The Gompertz and the logistic models have an asymptote, intersect y axis, and their inflection points are α/e and $\alpha/2$, respectively. The monomolecular model possesses only an asymptote and intersects the time axis. The Bertalanffy-Richards growth model has the features of flexibility and versatility based on different allometric parameter m. Thus, it can demonstrate different curves and the special cases some growth models can not demonstrate.

However, [1] concluded that "the Gompertz, logistic, Chapman-Richards, Richard's, and the von Bertalanffy growth models have points of inflection and are sigmoid. These models are suitable for quantifying a growth phenomenon that exhibits a sigmoid pattern over time". As analysed above, that the Bertalanffy-Richards function possesses the sigmoid or concave shape curve depends on the allometric parameter m, and therefore the function may demonstrate the sigmoid with inflection point or the concave curve shape. In fact, from many publications the Bertalanffy-Richards model is suitable not only for quantifying a growth phenomenon that exhibits a sigmoid pattern over time, but also for quantifying a growth phenomenon that exhibit a concave pattern over time.

3 Partial derivatives of nonlinear growth models

Estimates of nonlinear models are more difficult than that of linear models and the solutions are determined iteratively. [16] and [17] gave the detailed discussion of nonlinear growth models. The simple method of iterative estimation, the Gauss-Newton method, can be employed and the resulting parameter estimates are unbiased, normally distributed, minimum variance estimators. If the model does not behave in a near to linear fashion, the parameter estimates will not have these desirable properties and more complex estimation techniques such as Marquardt method may be necessary [7, 16]. The Marquardt iterative method requires specification of the names and starting values of the parameters to be estimated, expressions for the model, and the partial derivatives of the model with respect to each parameter [18]. The method is a compromised approach between the linearization (e.g., Gauss-Newton) method and the steepest decent method and appears to combine the best features of both while avoiding their most serious limitations [19]. In such cases, the use of partial derivative rather than computational approximations usually results in more efficient and more precise parameter estimation. Therefore, [1] derived and provided the partial derivative of the nine nonlinear growth models for estimating the parameters of these models using SAS program from a forestry perspective. As [5] mentioned, there are many equations that can describe plant growth. Therefore, many partial derivatives with respect to parameters of different models should be required to estimate nonlinear models in forestry science. Apparently, the study of [1] has not met this demand of a large of models where one needs comparing and selecting the best one to be used in forest growth and yield modelling. In fact, however, JMP statistic software [20] can provide parameter partial derivatives of any nonlinear models with precise and convenient for foresters. The software easily accesses partial derivative formulas of each parameter and can satisfy the requirement of partial derivatives with respect to each parameter when fitting nonlinear growth models using SAS syntax. Please refer to the Nonlinear Fit of [20] concerning the software application. That means, it is not necessary to develop the parameter partial derivatives for estimating nonlinear models.

4 Conclusions

The Bertalanffy-Richards growth model has been widely employed in forest growth and yield modelling for long time. However, some foresters have not completely recognised the features of the model from a theoretical point of view so that the model would be considered to demonstrate only S-curve shape with an inflection point. We have shown that the Bertalanffy-Richards growth model possesses not only the S-curve but also the concave curve in which depends on the allometric parameter m. Better understanding of the Bertalanffy-Richards growth model features is very important for modelling forest growth and yield and thus foresters can use the model correctly and effectively to fit S-curve with an inflection point or concave curve. We have also indicated that the partial derivatives with respect to each parameter should be used to estimate parameters of nonlinear models when more complex techniques are employed in which the techniques can deliver more efficient and more precise parameter estimation. The parameter partial derivatives of different nonlinear models can be obtained in JMP software to fit any nonlinear growth models. Thus, it seems to be not necessary to develop partial derivatives of any nonlinear growth models for estimating parameters of the models when using more complicated methods such as the Marquardt.

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