

Stability Analysis of a Stochastic Model for Prey-Predator System with Disease in the Prey*

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Abstract. In this paper we consider a prey-predator system where the prey population is infected by a microparasite. Local as well as global stability properties of the interior equilibrium point are discussed. The stochastic stability properties of the model are investigated, suggesting that the deterministic model is robust with respect to stochastic perturbations.

Keywords: prey-predator, microparasite, Lyapunov function, stochastic stability.

1 Introduction

There has been growing interest in the study of diseases in a prey-predator system. It is observed in nature, species does not exist alone. While species spreads the disease, it also competes with the other species for space or food, or is predated by other species. Therefore it is more of biological significance to consider the effect of interacting species when we study the dynamical behaviour of epidemiological models so an appropriate mathematical model is essential to study the effect of disease on interacting species. Freedman [1] studied a predator-prey system in which some members of the prey population and all predators are subjected to infection by parasites and derived conditions

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for persistence of all the populations and global stability criterion of the interior equilibrium. Anderson and May [2] showed that invasion of a resident predator-prey or host parasite system by a new strain of parasites. Haderl and Freedman [3] observed a similar phenomena. Mukherjee [4] analyzed a generalized prey-predator system with parasitic infection and obtained conditions for persistence and impermanence. Recently some works have been done in this area (see [5]–[7]). In this paper we incorporate the predator based on the basic epidemiological model, namely the $S - I$ model in order to investigate how the predation process influences the epidemics. We consider the case where the predator eats infected prey only. This is in accordance with the fact that the infected individuals are less active and be caught more easily, or the behaviour of the prey is modified such that they live in parts of the habitat which are accessible to the predator (fish and aquatic snails staying close to water surface, snails staying on the top of the vegetation rather than under plant cover [8].) Peterson and Page [9] have indicated wolf attacks on moose are more often successful if the moose is heavily infected by “*Echinococcus granulosis*”. Thus we present the following model:

$$\begin{aligned}\frac{dS}{dt} &= S\left\{r\left(1 - \frac{S + I}{K}\right) - \beta I\right\}, \\ \frac{dI}{dt} &= I\{\beta S - c - pY - aI\}, \\ \frac{dY}{dt} &= Y\{-d + qpI - bY\}\end{aligned}\tag{1}$$

where $S(t)$, $I(t)$, $Y(t)$ are the population density of the susceptible prey, infected prey and predator respectively at a give time t . Here r is the intrinsic birth rate. K denotes the carrying capacity of the environment. β is the transmission coefficient, c and d are the death rate of infected prey and predator respectively. a and b denote the intraspecific competition coefficient the infected prey and predator respectively. The coefficient in conversing prey into predator is q ($0 < q < 1$). p represents the predation coefficient.

In this model we have considered the effect of intraspecific competition between infected prey as well as on predator which are not considered in [4]–[7]. Also the stochastic stability properties of the model are not studied in earlier papers.

The topological type of a differential equation in a neighbourhood of a generic singular point is determined by the linearization of the field at the point (see the Grobman-Hartman Theorem [10].)

In this paper we consider the problem of the robustness of the model (1) with respect to white noise stochastic perturbations around its positive endemic equilibrium. This paper is organized as follows. In Section 2, we discuss boundedness of solutions and dynamical behaviour of boundary as well as interior equilibrium point of the deterministic model. In Section 3, we introduce the stochastic model and in Section 4 we carry out an analysis of its stability properties by means of Lyapunov functions methods. We conclude a short discussion in Section 5.

2 Boundedness, boundary equilibria and stability

In this section, we first show that solutions of system (1) are bounded.

Theorem 1. *System (1) is dissipative.*

Proof. Let $(S(t), I(t), Y(t))$ be any solution with positive initial conditions (S_0, I_0, Y_0) . Since

$$\frac{dS}{dt} \leq Sr \left(1 - \frac{S}{K}\right)$$

by a standard comparison theorem we have

$$\limsup_{t \rightarrow \infty} S(t) \leq M \quad \text{where} \quad M = \max \{S(0), K\}.$$

Consider the function

$$W = S + I + Y.$$

The time derivative along a solution of (1) is

$$\begin{aligned} \frac{dW}{dt} &= S \left\{ r \left(1 - \frac{S+I}{K}\right) - \beta I \right\} \\ &\quad + I \{ \beta S - C - pY - aI \} + Y \{ -d + qpI - bY \} \\ &\leq S(r+1) - S - cI - dY \\ &\leq M(r+1) - mW \end{aligned}$$

where $m = \min\{1, c, d\}$.

Thus $\frac{dW}{dt} + mW \leq M(r+1)$.

Applying a theorem in differential inequalities [11] we obtain

$$0 \leq W(S, I, Y) \leq \frac{M(r+1)}{m} + W(S(0), I(0), Y(0))/e^{mt}$$

and for $t \rightarrow \infty$, $0 \leq W \leq \frac{M(r+1)}{m}$. Therefore all solutions of system (1) enter into the region.

$$B = \left\{ (S, I, Y) \in R_+^3 : W \leq \frac{M(r+1)}{m} + \varepsilon, \text{ for any } \varepsilon > 0 \right\}.$$

This completes the proof. \square

Now we discuss the boundary and interior equilibrium point. The model equation (1) has the following non-negative equilibria namely

$$E_0 = (0, 0, 0), \quad E_1 = (K, 0, 0), \quad E_{12} = (\bar{S}, \bar{I}, 0)$$

where

$$\bar{S} = \frac{c + a\bar{I}}{\beta}, \quad \bar{I} = \frac{r(K\beta - c)}{ra + r\beta + K\beta^2}.$$

The interior equilibrium point $E^* = (S^*, I^*, Y^*)$ where

$$\begin{aligned} S^* &= \frac{rK(q\beta^2 + ab) + (r + K\beta)(bc - dp)}{r(q\beta^2 + ab) + (r + K\beta)\beta b}, \\ I^* &= \frac{\beta S^* b - bc + dp}{qp^2 + ab}, \\ Y^* &= \frac{-d + qpI^*}{b}. \end{aligned}$$

Theorem 2. *If $K\beta > c$ then E_{12} is feasible. If $d < \frac{r(K\beta - c)qp}{ra + r\beta + K\beta^2}$ then E^* is feasible.*

It can be easily shown that E_0 is unstable. E_1 is unstable if $\beta K > c$. E_{12} is globally asymptotically stable in the $S - I$ plane. E^* is locally asymptotically stable. We now show that E^* is globally asymptotically stable whenever it exists.

Theorem 3. *If E^* is feasible then it is globally asymptotically stable.*

Proof. Define a Lyapunov function $V(S, I, Y)$ such that

$$V(S, I, Y) = C_1(S - S^* - S^* \ln S/S^*) + C_2(I - I^* - I^* \ln I/I^*) \\ + C_3(Y - Y^* - Y^* \ln Y/Y^*)$$

where $C_i, i = 1, 2, 3$ are positive constants to be chosen later. Evidently V is a positive definite function in the region B except at E^* where it is zero.

Calculating the rate of change of V along the solutions of system (1), we get

$$\begin{aligned} \frac{dV}{dt} &= C_1(S - S^*) \frac{\dot{S}}{S} + C_2(I - I^*) \frac{\dot{I}}{I} + C_3(Y - Y^*) \frac{\dot{Y}}{Y} \\ &= C_1(S - S^*) \left\{ r \left(1 - \frac{S+I}{K} \right) - \beta I \right\} \\ &\quad + C_2(I - I^*) \{ \beta S - c - pY - aI \} \\ &\quad + C_3(Y - Y^*) (-d + qpI - bY) \\ &= C_1(S - S^*) \left\{ -\frac{r}{K}(S - S^*) - \left(\frac{r}{K} + \beta \right) (I - I^*) \right\} \\ &\quad + C_2(I - I^*) \{ \beta(S - S^*) - p(Y - Y^*) - a(I - I^*) \} \\ &\quad + C_3(Y - Y^*) \{ qp(I - I^*) - b(Y - Y^*) \}. \end{aligned}$$

Choosing $C_2\beta - C_1\left(\frac{r}{K} + \beta\right) = 0$, $C_3q - C_2 = 0$. It follows that

$$\frac{dV}{dt} = -C_1(S - S^*)^2 - C_2a(I - I^*)^2 - C_3b(Y - Y^*)^2$$

and hence \dot{V} is negative. So the largest invariant set at which $\dot{V} = 0$ is the equilibrium point and by LaSalle's invariance principle, E^* is globally asymptotically stable. \square

3 The stochastic model

Stochastic perturbations were introduced in some of the main parameters involved in the model equations.

In this paper, instead we allow stochastic perturbations of the variables S, I, Y around their values at the positive equilibrium E^* , in the case when it

is feasible and locally asymptotically stable. Local stability of E^* is implied by the existence condition of E^* . So, in model (1) we assume that stochastic perturbations of the variables around their values at E^* are of white noise type, which are proportional to the distances of S, I, Y from values S^*, I^*, Y^* . So system (1) results

$$\begin{aligned} dS &= S \left[r \left(1 - \frac{S+I}{K} \right) - \beta I \right] dt + \sigma_1 (S - S^*) d\xi_t^1, \\ dI &= I [\beta S - c - pY - aI] dt + \sigma_2 (I - I^*) d\xi_t^2, \\ dY &= Y [-d + qpI - bY] dt + \sigma_3 (Y - Y^*) d\xi_t^3 \end{aligned} \quad (2)$$

where $\sigma_i, i = 1, 2, 3$ are real constants, $\xi_t^i = \xi_i(t), i = 1, 2, 3$ are independent from each other standard Wiener processes [12]. We wonder whether the dynamical behaviour of model (1) is robust with respect to such a kind of stochasticity by investigating the asymptotic stability behaviour of the equilibrium E^* for (2) and comparing the results with those obtained for (1).

We will consider (2) as the Ito stochastic differential system.

4 Stochastic stability of the positive equilibrium

The stochastic differential system (2) can be centred at its positive equilibrium E^* by the change of variables

$$u_1 = S - S^*, \quad u_2 = I - I^*, \quad u_3 = Y - Y^*. \quad (3)$$

The linearized SDEs around E^* take the form

$$du(t) = f(u(t))dt + g(u(t))d\xi(t) \quad (4)$$

where $u(t) = \text{col}(u_1(t), u_2(t), u_3(t))$ and

$$f(u(t)) = \begin{pmatrix} -\frac{2rS^*}{K} & -\frac{S^*(\beta K + r)}{K} & 0 \\ \beta I^* & -2aI^* & -pI^* \\ 0 & qpY^* & -2bY^* \end{pmatrix} u(t), \quad (5)$$

$$g(u) = \begin{pmatrix} \sigma_1 u_1 & 0 & 0 \\ 0 & \sigma_2 u_2 & 0 \\ 0 & 0 & \sigma_3 u_3 \end{pmatrix}. \quad (6)$$

Of course in (4) the positive equilibrium E^* corresponds to the trivial solution $u(t) = 0$.

Let U be the set $U = (t \geq t_0) \times R^n$, $t_0 \in R^+$. Hence $V \in C_2^0(U)$ is a twice continuously differentiable function with respect to u and a continuous functions with respect to t .

With reference to the book by Afanas'ev et al. [13], the following theorem holds.

Note that, with reference to (4)

$$LV(t, u) = \frac{\partial V(t, u)}{\partial t} + f^T(u) \frac{\partial V(t, u)}{\partial u} + \frac{1}{2} \text{Tr} \left[g^T(u) \frac{\partial^2 V(t, u)}{\partial u^2} g(u) \right]$$

where

$$\begin{aligned} \frac{\partial V}{\partial u} &= \text{Col} \left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_2}, \frac{\partial V}{\partial u_3} \right), \\ \frac{\partial^2 V(t, u)}{\partial u^2} &= \left(\frac{\partial^2 V}{\partial u_j \partial u_i} \right), \quad i, j = 1, 2, 3 \end{aligned}$$

and 'T' means transposition.

Theorem 4. *Suppose there exists a function $V(t, u) \in C_2^0(U)$ satisfying the inequalities*

$$K_1|u|^p \leq V(t, u) \leq K_2|u|^p, \quad (7)$$

$$LV(t, u) \leq -K_3|u|^p, \quad K_i > 0, \quad p > 0. \quad (8)$$

Then the trivial solution of (4) is exponentially p -stable for $t \geq 0$.

Note that, if in (7), (8), $p = 2$, then the trivial solution of (4) is globally asymptotically stable in probability. For definitions of stability again we refer to [13].

Theorem 5. *Suppose that $\sigma_1^2 < \frac{4rS^*}{K}$, $\sigma_2^2 < 4aI^*$, $\sigma_3^2 < 4bY^*$. Then the zero solution of (4) is asymptotically mean square stable.*

Proof. Let us consider the Lyapunov function

$$V(u) = \frac{1}{2} [w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2] \quad (9)$$

where w_i are real positive constants to be chosen in the following. It is easy to check that inequalities (7) hold true with $p = 2$.

Furthermore

$$\begin{aligned}
LV(u) = & w_1 \left(-\frac{2rS^*}{K}u_1 - \left(\frac{r}{K} + \beta \right) S^*u_2 \right) u_1 \\
& + w_2 (\beta I^*u_1 - 2aI^*u_2 - pI^*u_3) u_2 \\
& + w_3 (qpY^*u_2 - 2bY^*u_3) u_3 \\
& + \frac{1}{2} \text{Tr} \left[g^T(u) \frac{\partial^2 V}{\partial u^2} g(u) \right].
\end{aligned} \tag{10}$$

Now remark that

$$\frac{\partial^2 V}{\partial u^2} = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix}$$

and hence

$$g^T(u) \frac{\partial^2 V}{\partial u^2} g(u) = \begin{pmatrix} w_1 \sigma_1^2 u_1^2 & 0 & 0 \\ 0 & w_2 \sigma_2^2 u_2^2 & 0 \\ 0 & 0 & w_3 \sigma_3^2 u_3^2 \end{pmatrix}$$

with

$$\frac{1}{2} \text{Tr} \left[g^T(u) \frac{\partial^2 V}{\partial u^2} g(u) \right] = \frac{1}{2} [w_1 \sigma_1^2 u_1^2 + w_2 \sigma_2^2 u_2^2 + w_3 \sigma_3^2 u_3^2]. \tag{11}$$

If in (10) we choose

$$w_1 \left(\frac{r}{K} + \beta \right) = w_2 \beta I^* \quad \text{and} \quad w_2 I^* = w_3 q Y^*,$$

from (11) it is easy to check that

$$\begin{aligned}
LV(u) = & - \left(\frac{2r}{K} S^* - \frac{1}{2} \sigma_1^2 \right) w_1 u_1^2 - \left(2aI^* - \frac{1}{2} \sigma_2^2 \right) w_2 u_2^2 \\
& - \left(2bY^* - \frac{1}{2} \sigma_3^2 \right) w_3 u_3^2.
\end{aligned} \tag{12}$$

According to Theorem 4 the proof is completed. \square

5 Discussion

In this paper we have analyzed a prey-predator system where the prey population is divided into two groups, infected and non-infected. Intraspecific competition of the infected prey and predator are also incorporated in the model system. The threshold parameter $\rho = \frac{K\beta}{C}$ controls the dynamics of the system. It is observed that if $\rho > 1$, the boundary equilibrium point E_{12} is feasible. If the death rate of the predator remains a certain threshold value then the positive equilibrium is feasible. Moreover all the solutions coverage to the positive equilibrium. We observed that deterministic model is robust with respect to stochastic perturbations. It is to be noted that when S^*, I^*, Y^* increases the asymptotic mean square stability property is achieved.

References

1. Freedman H.I. "A model of predator-prey dynamics as modified by the action of a parasite", *Math. Biosci.*, **99**, p. 143–155, 1990
2. Anderson R.M., May R.M. "The invasion and spread of infectious diseases within animal and plant communities", *Philos. Trans. R. Soc., Lond B*, **314**, p. 533–570, 1986
3. Haderer K.P., Freedman H.I. "Predator-prey populations with parasite infection", *J. Math. Biol.*, **27**, p. 609–631, 1989
4. Mukherjee D. "Uniform persistence in a generalized prey-predator system with parasite infection", *Biosystems*, **47**, p. 149–155, 1998
5. Mukherjee D. "Persistence in a prey-predator system with disease in the prey", *J. Biol. Systems*, **11**, p. 101–112, 2003
6. Xiao Y., Chen L. "Modeling and analysis of a predator-prey model with disease in the prey", *Math. Biosci.*, **171**, p. 59–82, 2001
7. Xiao Y., Chen L. "Analysis of a three species Eco-Epidemiological model", *J. Math. Anal. Appl.*, **258**, p. 733–754, 2001
8. Holmes J.C., Bethel W.M. "Modification of intermediate host behaviour by parasite", In: Cuning E.V., Wright C.A (Eds.) *Behavioural Aspect of Parasite Transmission*, Suppl. No.1 to the *Zool. J. Linnean Soc.*, **51**, p. 123–149, 1972

9. Peterson R.O., Page R.E. "Wolf density as a predator of predation rate", *Swedish Wildlife Research*, Suppl. 1, 1987
10. Anosov D.V., Aranson S.Kh., Arnold V.I., Bronshtein I.U., Grines V.Z., Il'yashenko Yu.S. *Ordinary Differential Equations and Smooth Dynamical Systems*, Springer-Verlag, Berlin, Heidelberg, 1997
11. Birkhoff G., Rota G.C. *Ordinary Differential Equations*, Ginn, 1982
12. Gikhman I.I., Skorokhod A.V. *The Theory of Stochastic processes, I*, Springer, Berlin, 1974
Gikhman I.I., Skorokhod A.V. *The Theory of Stochastic processes, II*, Springer, Berlin, 1975
Gikhman I.I., Skorokhod A.V. *The Theory of Stochastic processes, III*, Springer, Berlin, 1979
13. Afanas'ev V.N., Kolmanowski V.B., Nosov V.R. *Mathematical Theory of Global Systems Design*, Kluwer Academic, Dordrecht, 1996