

## Multiplicative Stochastic Model of the Time Interval between Trades in Financial Markets

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### Abstract

Stock price change in financial market occurs through transactions, in analogy with diffusion in stochastic physical systems. The analysis of price changes in real markets shows that long-range correlations of price fluctuations largely depend on the number of transactions. We introduce the multiplicative stochastic model of time interval between trades and analyze spectral density and correlations of the number of transactions. The model reproduces spectral properties of the real markets and explains the mechanism of power law distribution of trading activity. Our study provides an evidence that statistical properties of financial markets are enclosed in the statistics of the time interval between trades. Multiplicative stochastic diffusion may serve as a consistent model for this statistics.

**Keywords:** statistical analysis, financial markets, stochastic modelling.

## 1 Introduction

Complex collective phenomena usually are responsible for power-laws which are universal and independent of the microscopic details of the

phenomenon. Examples in physics are numerous. Power-laws are intrinsic features of the economic and financial data as well. The aim of this contribution is to analyze a relation between the origin of the power law distribution and the power-law correlations in financial time series. There are numerous studies of power-law probability distributions in various economic systems [1, 2, 3, 4].

The time-correlations in financial time series are studied extensively as well [5, 6, 7]. The random multiplicative process built into the model of wealth distribution yields Pareto style power law [2]. The generalized Lotka Volterra dynamics developed by S.Solomon and P.Richmond is in the use for various systems including financial markets [1] . However, these models generically lead to the non-universal exponents and do not explain the power-law correlations in financial time series [8].

Recently we adopted the model of  $1/f$  noise based on the Brownian motion of time interval between subsequent pulses, proposed by Kaulakys and Meskauskas [9, 10, 11, 12] to model share volume traded in financial markets [13] . The idea to transfer long time correlations into stochastic process of the time interval between trades or time series of trading activity is in consistence with the detailed studies of the empirical financial data [5, 6] and fruitfully reproduces spectral properties of financial time series [13]. However, the investigation of the model revealed, that the simple additive Brownian model of time interval between trades failed to reproduce power-law probability density distribution (pdf) of trading activity.

On the other hand, several authors showed empirically that the fluctuations of various financial time series possess multifractal statistics [14, 15, 16] . Therefore we introduce the stochastic multiplicative model of time interval between trades and analyze the statistical properties of the model trading activity numerically and analytically. We show the consistence of the approach with the results of statistical analyzes of the empirical financial time series. This sheds light on the relation between the power-law probability distribution and the power-law correlations in financial time series.

## 2 Multiplicative time interval model

Following our previous model [13] we consider a signal  $I(t)$  as a sequence of the random correlated pulses

$$I(t) = \sum_k q_k \delta(t - t_k) \quad (1)$$

where  $q_k$  is a contribution to the signal of one pulse at the time moment  $t_k$ , for example, a contribution of one transaction to financial data. Signal (1) was introduced by Kaulakys and Meskauskas for the modelling of  $1/f$  noise and could be used in a large variety of systems with the flow of point objects or subsequent actions. In the statistical data analysis we usually deal with the discrete time series in equal time intervals  $\tau_d$ . Integrating the signal  $I(t)$  in the subsequent time intervals of length  $\tau_d$  we get discrete time series and, in analogy with financial time series, we call it the volume  $V_j$

$$V_j = \int_{t_j}^{t_j + \tau_d} I(t) dt = \sum_{t_j < t_k < t_j + \tau_d} q_k, \quad t_j = j\tau_d. \quad (2)$$

We can define the number of trades  $N_j$  in the time interval  $t_j \div (t_j + \tau_d)$  by the same formula (2), when  $q_k \equiv 1$ ,  $N_j \equiv V_j$ . The power spectral density  $S(f_s)$  of the signal  $V_j$  is defined as

$$S(f_s) = \left\langle \frac{2}{\tau_d n} \left| \sum_{j=1}^n V_j \exp \{ -i2\pi(s-1)(j-1)/n \} \right|^2 \right\rangle \quad (3)$$

where  $f_s = (s-1)/T$ ,  $T = \tau_d n$ . In this paper we investigate statistical properties of the time series  $N_j$ , when the sequence of event time  $t_k$  is generated by the multiplicative stochastic process. There are some arguments for this choice of inter-event time model. First of all the multiplicativity is an essential feature of the processes in economics [1, 2, 3, 4]. Multiplicative stochastic processes yield multifractal intermittency and are able to produce power-law pdf. Pure multiplicative processes are not stationary and artificial diffusion restriction mechanisms have to be introduced. Let us start our analysis

introducing the multiplicative process with the logarithmic diffusion restriction for the large deviations of  $\tau_k$  from the mean value  $\bar{\tau}_k = 1$

$$\begin{aligned} t_{k+1} &= t_k + \tau_{k+1} \\ \tau_{k+1} &= \tau_k(1 - \gamma \ln \tau_k + \sigma \varepsilon_k). \end{aligned} \quad (4)$$

Here the inter-event time  $\tau_k$  fluctuates due to the external random perturbation by a sequence of uncorrelated normally distributed random variables  $\{\varepsilon_k\}$  with zero expectation and unit variance, where  $\sigma$  denotes the standard deviation of the white noise and  $\gamma \ll 1$  is a damping constant. The choice to introduce multiplicative logarithmic diffusion damping  $\gamma \tau_k \ln \tau_k$  retains symmetry in logarithmic scale and multiplicative nature of the process for small values of  $\gamma \ln \tau_k$ . Long time stationary pdf of  $\tau_k$  is a Lognormal distribution

$$P(\tau) = \frac{2}{\sqrt{2\pi\sigma^2/\gamma\tau}} \frac{1}{\tau} \exp \left\{ -\frac{2\gamma}{\sigma^2} \ln^2 \tau \right\}. \quad (5)$$

We derive the distribution (5) from the solution of the generalized Langevine equation [17]

$$\begin{aligned} \dot{x} &= F(x) + G(x)\eta(t), \quad -\frac{\partial V}{\partial x} = \frac{F}{G^2}, \\ p(x) &= C \exp \left[ -\left\{ \frac{V(x)}{\frac{1}{2}\sigma^2} + \ln G(x) \right\} \right] \end{aligned} \quad (6)$$

where  $\langle \eta(t) \rangle = 0$  and  $\langle \eta(t)\eta(t') \rangle = \sigma^2 \delta(t-t')$ . Note that Lognormal distribution is invariant to the change of variables:  $n = 1/\tau$  and in the wide range of argument values  $|\ln \tau| \ll \sigma/\gamma^{1/2}$  is equivalent to the power-law  $\sim 1/\tau$ . Nevertheless, we have to point out that lognormal distribution of  $\tau_k$  is not appropriate to reproduce the power-law distributions of variables in financial markets. We expect that iterative stochastic model (4) producing universal lognormal distribution will be useful for other applications. We base our model on the generic multiplicative process

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \tau_k^\mu \sigma \varepsilon_k, \quad (7)$$

as the most appropriate seeking to reproduce power-law probability density distributions of variables in financial markets. In this paper

we will use a very simple  $\tau_k$  diffusion damping model assuming that  $0 \leq \tau_k \leq 1$  and  $0 < \gamma \ll \sigma$ . More sophisticated damping models can be introduced in further developments of this approach. The most natural choice is to assume  $\mu = 1$ , giving pure multiplicativity of  $\tau_k$ . We decided to keep this parameter in our notation seeking to sustain generic definition. In the  $k$  scale Eq. (7) defines generalized Langevine equation (6) with  $F(\tau) = \gamma\tau_k^{2\mu-1}$  and  $G(\tau) = \tau_k^\mu$ . Note that the solution (6) of the generalized Langevine equation is based on the Stratanovich convention. Our numerical analysis confirms the preference of the Stratanovich convention in comparison with Ito convention regarding discrete iterative equation (7). The stationary pdf of  $\tau_k$  defined by iterative relation (7) can be derived from the solution (6) of the generalized Langevine equation,

$$P_k(\tau) = \frac{C_\tau}{\tau^{\mu-2\gamma/\sigma^2}} \quad (8)$$

where the normalization constant  $C(\tau)$  can be defined from the integral  $\int_0^1 P_k(\tau)d\tau = 1$ . The stationary distribution function for the number of trades  $N_k$  in the time interval  $\tau_d$  can be derived from (8). Simple change of variables  $N = \tau_d/\tau$  in (8) having in mind that  $P_k(\tau)d\tau = P_k(N)dN$ , gives probability distribution function of  $N$

$$P_k(N) = \frac{C_N}{N^{2-\mu+2\gamma/\sigma^2}} \quad (9)$$

where the time interval  $\tau_d$  is included in normalization constant  $C_N$ . Note that variance  $\langle N_k^2 \rangle - \langle N_k \rangle^2$  diverges when  $2\gamma/\sigma^2 < \mu + 1$ . We will see later that the transition from the stationary regime to the non-stationary one is essential for the appearance of  $1/f$  power spectral density. Distribution function (9) can be transformed to the time scale taking into account that every step in the iteration equation (7) is equal to the time interval  $\tau_k$ . In the real time scale the pdf of  $N$  takes the form

$$P_t(N) = \frac{C_{N_t}}{N^{3-\mu+2\gamma/\sigma^2}} \quad (10)$$

Here  $C_{N_t}$  is a new normalization constant.

### 3 Numerical results and discussion

We have already introduced the multiplicative stochastic process as a model for the time interval between trades in financial markets. This yields the power-law distribution for the number of trades per constant time interval  $\tau_d$ , equation (10). The exponent of cumulative power-law distribution  $\alpha$  for pure multiplicative model,  $\mu = 1$ , depends only on the parameter  $\nu = 2\gamma/\sigma^2$ , i.e.,

$$\alpha = 1 + 2\gamma/\sigma^2. \quad (11)$$

Parameter  $\nu$  defines the ratio of damping constant  $\gamma$  to the coefficient of stochastic diffusion  $\frac{1}{2}\sigma^2$ . In this chapter we will analyze spectral properties of the model and related autocorrelation of trading activity  $N$ . The model presented can be easily calculated numerically using equations (1), (2), (3) and (7). We smooth the power spectral density with standard moving average procedure. We present the numerically calculated power spectral density  $S(f)$  of  $N$  for  $\nu = 2.4; 2.0$  and  $1.5$  on the Fig. 1. (a), (b) and (c) respectively with  $\sigma = 0.1$  and  $\tau_d = 10$ . For the values of parameter  $\nu > 2$ , we observe the power spectral density  $S(f) \sim 1/f^{1.5}$ . For  $\nu \simeq 2$  we obtain  $S(f) \sim 1/f$  and we get the slope of  $S(f) \sim 1/f^{0.5}$  when  $\nu < 2$ .

Numerical results confirm that  $1/f$  noise occurs when stochastic process defining the trading activity  $N_k$  experience the transfer from the stationary to the non-stationary regime. This general rule is probably applicable to the various models of the time intervals between pulses and is in agreement with the theory of self-affine fractals, see [18]. It should be noted that transition from slope  $\beta = 1.5$  to  $\beta = 0.5$  of the power spectral density is very sharp with the change of parameters  $\nu$  or  $\gamma$  less then twice.

Calculations with different values of parameters confirm a universal nature of the relation between the slope of the power spectral density  $\beta$  and parameter of multiplicative diffusion  $\nu$ . This relation is independent of  $\tau_d$  in wide range of values.

We present our numerical results of power spectral density for different sets of parameters in Fig. 2. The slopes  $\beta = 1.5; 1.0$  and  $0.5$ , of power spectral density are related with the corresponding values of parameter  $\nu = 2.4; 2.0$  and  $1.6$ , as in Fig. 1. Numerous calculations

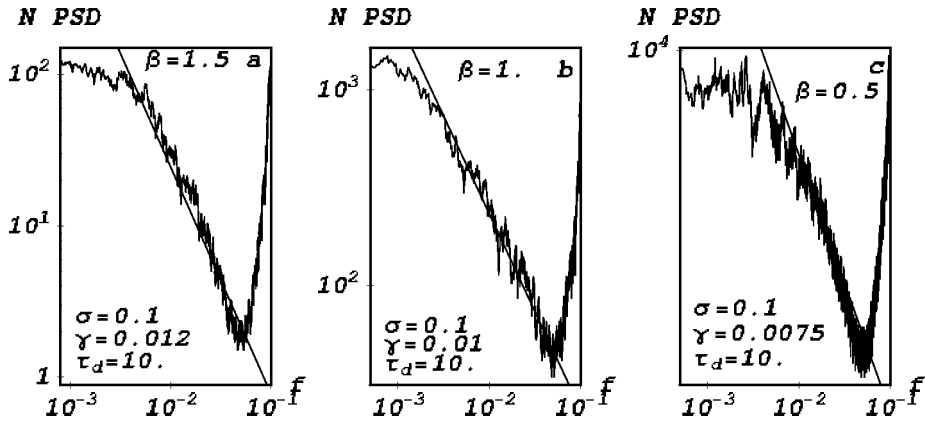


Figure 1: Power spectral density of trading activity  $N$  (N PSD) versus frequency  $f$  calculated from the model described by Eqs. (1), (2), (3) and (7) with  $\mu = 1$ . The sinuous curves represent results of the numerical simulations smoothed with standard moving average procedure over 43 points with  $\sigma = 0.1$  and  $\tau_d = 10$ , while the straight lines represent power-law power spectral density  $S(f) \sim 1/f^\beta$  with the various slopes  $\beta$ . For  $\gamma = 0.012$ ,  $\beta = 1.5$  (a); for  $\gamma = 0.01$ ,  $\beta = 1$  (b) and for  $\gamma = 0.0075$ ,  $\beta = 0.5$  (c).

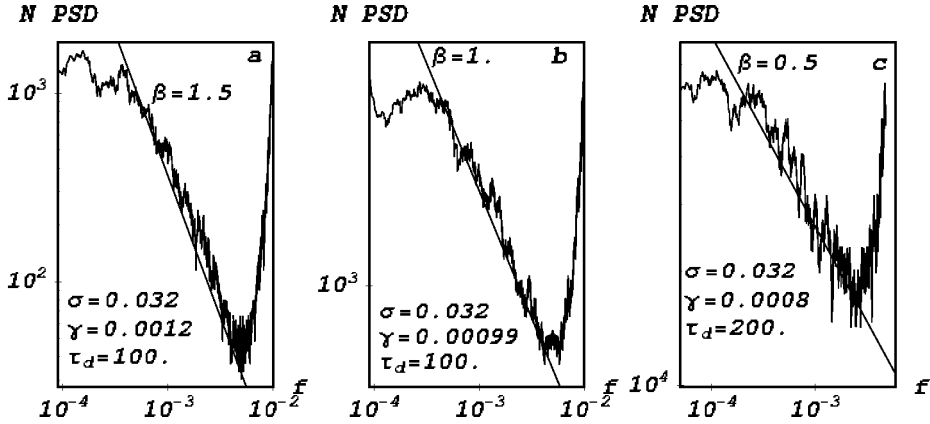


Figure 2: The same as in fig. 1 but with parameters  $\sigma = 0.032$  and  $\tau_d = 100$  or  $200$ . The straight lines represent power-law power spectral density  $S(f) \sim 1/f^\beta$  with the various slopes  $\beta$ . For  $\gamma = 0.0012$ ,  $\beta = 1.5$  (a); for  $\gamma = 0.001$ ,  $\beta = 1$  (b); for  $\gamma = 0.0008$ ,  $\beta = 0.5$  (c).

confirm that the slope of power spectral density  $\beta$  is defined by  $\nu$ , exhibiting sharp transition from  $\beta = 1.5$  to  $\beta = 0.5$ , when  $\nu \simeq 2.0$ . This transition is related with divergence of  $N_k$  variance  $\langle N_k^2 \rangle - \langle N_k \rangle^2$  defined from (9). In the previous section we derived the power-law probability distribution function of  $N$ , Eq. (10). Let us compare the exponent of cumulative distribution from (10),  $\alpha = 1 + \nu$ , with numerical results in the interval of  $\nu$  values  $1.5 \div 3.5$  related with the slope  $\beta$  change. In Fig. 3. we present numerically calculated  $\alpha$  as a function of  $\nu$  for various values of  $\sigma, \gamma$ , and  $\tau_d$ . The least square linear approximation of numerical results, straight line, yields that  $\alpha = 0.96 + 0.97\nu$ .

It shows that the probability distribution function (10) derived from the generalized Langevine equation (6) describes the power-law distribution of  $N$  very well and serves as an evidence that the Stratanovich convention is appropriate for the multiplicative stochastic model (7). For  $\nu \simeq 2$ , corresponding to  $\beta = 1.$ , we get the value of  $\alpha \simeq 3$ . This is in good agreement with empirical data of power-law distribution of trading activity  $N$  in the financial markets  $\alpha \simeq 3.4$  [5].



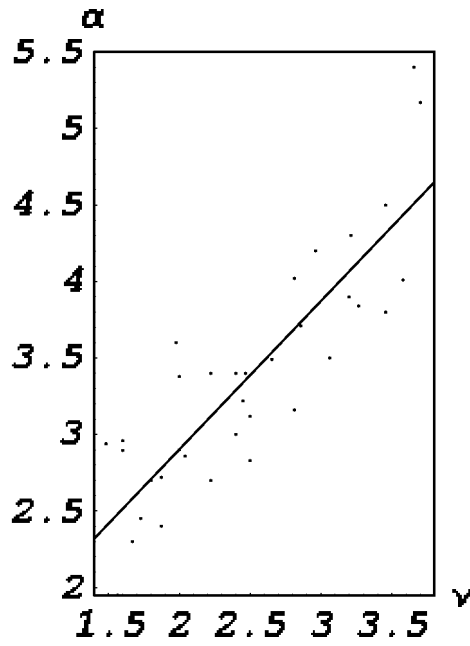


Figure 3: The exponent  $\alpha$  of power-law distribution function for  $N$  versus multiplicative diffusion parameter  $\nu$  calculated numerically from the model described by Eqs. (1), (2), (3) and (7). Points represent various realizations with the parameters  $\sigma = 0.05 \div 0.1$ ,  $\gamma = 0.001 \div 0.015$ ,  $\tau_d = 10 \div 100$ . Straight line represents the least square approximation to the numerical data  $\alpha = 0.96 + 0.97\nu$ .

Numerical results confirm that multiplicative stochastic model of the time intervals between trades in the financial markets is able to reproduce the main statistical properties of trading activity  $N$ . First of all, even in a very simple multiplicative diffusion restriction  $0 \leq \tau_k \leq 1$ , the model reproduces the power-law distribution function with  $\alpha \simeq 3$ . The power-law exponent is related with the parameter  $\nu = 2\gamma/\sigma^2$ , which defines the distribution functions of time interval between trades  $\tau$ , and of trading activity  $N$  as well as the slope  $\beta$  of its' power spectral density  $S(f) \sim 1/f^\beta$ . This sheds light on the origin of universal nature of the power-law distributions of such variables as trading activity in financial markets  $N$  and its' power spectral density  $S(f)$ .

## 4 Conclusions

We have introduced multiplicative stochastic model of the time interval  $\tau$  between trades in the financial markets Eq. (7) with the diffusion restriction in the interval  $0 \leq \tau_k \leq 1$ . This defines stochastic fluctuations of trading activity  $N$ , Eq. (2), and reproduces its' power-law distribution of probability and the slope of power spectral density  $S(f) \sim 1/f^\beta$ .

The comparison of the empirical data from the financial market analysis with the model proposed implies that markets function in the area of parameters, where convergence of  $N_k$  variance disappears and  $1/f$  noise of  $N$  occurs. This rather a sharp transition to the non-stationary regime of  $N_k$  fluctuations probably is the indispensable feature of the financial markets. This relates the exponent of the power-law distribution to the slope  $\beta$  of power spectral density into universal interdependence. It implies that the main statistical properties of the financial markets have to be defined by the fluctuations of time interval between the trades accumulating the power-laws and the long time correlations. Further empirical analysis of  $\tau$  statistics and the adjusted specification of the model is desirable. We do expect that multiplicative model of time interval between the trades with more specific diffusion restriction conditions and more precisely adjusted  $\mu$  lies in the background of the financial market statistics and can be

very useful in financial time series analysis.

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