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The Approximation of Sum of independent Distributions by the Erlang Distribution Function

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Abstract

The exponential distribution and the Erlang distribution function are been used in numerous areas of mathematics, and specifically in the queueing theory. Such and similar applications emphasize the importance of estimation of error of approximation by the Erlang distribution function. The article gives an analysis and technique of error's estimation of an accuracy of such approximation, especially in some specific cases.

Keywords: accuracy of approximation, Erlang distribution, sum of independent distributions.

1 Introduction and results

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distribution random variables with common distribution function $F(x)$ and characteristic function $f(t)$. Here we consider the accuracy of approximation of the distribution function

$$F_n(x) = P\Big(\sum_{j=1}^{n} \xi_j < x\Big)$$

to the Erlang distribution function.

Let $\eta_1, \eta_2 \dots$ be a sequence of independent and identically distributed random variables with transferred exponential distribution

$$G(x) = \begin{cases} 1 - e^{-\lambda(x-m)}, & \text{when } x > m, \\ 0, & \text{when } x \leq m \end{cases} \tag{1}$$

with the parameters $\lambda > 0$ and $m \in \Re^1$.

The distribution of the sum $Z_n = \sum\limits_{j=1}^{n} \eta_j$ is the shifted Erlang distribution

$$G_n(x) = \begin{cases} 1 - e^{-\lambda(x-m)} \sum\limits_{k=0}^{n-1} \frac{(\lambda(x-nm))^k}{k!}, & \text{when } x > nm, \\ 0, & \text{when } x \leq nm \end{cases} \tag{2}$$

We estimate the accuracy of approximation the distribution function $F_n(x)$ to the Erland distribution function $G_n(x)$

$$\Delta_n = \sup_x |F_n(x) - G_n(x)|.$$

The exponential distribution depend only on one parameter λ. Seeking to identify the second moments $E\xi_1^2 = E\eta_1^2$, we have to impose the condition $E\xi_j^2 = 2\lambda^{-2}$. This is a strong restriction for a distribution, and in general case this condition would not be satisfied.

The transferred exponential distribution depend on two parameters. We choose the values of parameters m and λ in such a way, that two moments of random variables ξ_1 and η_1 would be equal $E\xi_1^s = E\eta_1^s,\ s = 1; 2$.

The moments of the transferred exponential distribution (1) are

$$\alpha_\nu = E\eta^\nu = \nu! \sum_{k=0}^{\nu} \frac{m^k}{\lambda^{\nu-k}}.$$

Write

$$k_\nu = \int\limits_{-\infty}^{\infty} |x|^\nu |d(F{-}G)(x)|, \quad \gamma_\nu = \int\limits_{-\infty}^{\infty} d(F{-}G)(x), \quad \nu = 1; 2; \ldots.$$

Theorem 1. *If* $m = E\xi_1 - \sqrt{D\xi_1}, \lambda = (D\xi_1)^{1/2}$ *and*

1) $\tau \min(\lambda; 0,159 k_3^{-1}\lambda^{-2})$, *then for all* $n > 1$

$$\Delta_n \leq 1,301 n(n{-}1)^{-\frac{3}{2}}\lambda^3 k_3 + 1,818(n{-}1)^{-\frac{1}{2}}\lambda\tau^{-1}; \tag{3}$$

2) $\tau \min(0,5\lambda; 0,308\lambda^{-2}k_3^{-1})$, *then for all* $n > 1$

$$\Delta_n \leq 0,598 n(n{-}1)^{-\frac{3}{2}}\lambda^3 k_3 + 1,818(n{-}1)^{-\frac{1}{2}}\lambda\tau^{-1}. \tag{4}$$

From this theorem we have the following estimates of approximation.

Proposition. If $m = E\xi_1 - \sqrt{D\xi_1}, \lambda = (D\xi_1)^{1/2}, n > 1$ and

1) $k_3 \leq 0,159\lambda^{-3}$, thus

$$\Delta_n \leq 1,301 n(n{-}1)^{-\frac{3}{2}}\lambda^3 k_3 + 1,816(n{-}1)^{-\frac{1}{2}}; \tag{5}$$

2) $0,159\lambda^{-3} < k_3 \leq 0,616\lambda^{-3}$, thus

$$\Delta_n \leq 0,598 n(n{-}1)^{-\frac{3}{2}}\lambda^3 k_3 + 3,636(n{-}1)^{-\frac{1}{2}}; \tag{6}$$

3) $k_3 > 0,616\lambda^{-3}$, thus

$$\Delta_n \leq \lambda^3 k_3(0,598 n(n{-}1)^{-\frac{3}{2}} + 5,903(n{-}1)^{-\frac{1}{2}}. \tag{7}$$

2 Proof of the Results

To prove theorems formulated, we will show, that the assertion of theorem 1 follows from [3, theorem 1]. The simplified form of this theorem will be given, and in addition same definition will be introduced.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common distribution function $F_{\mathrm{x}}(X)$

and characteristic function $f_{\mathrm{x}}(t), G^{*n}(x)$ – the n-th convolution of the same distribution $G(\mathrm{x})$,

$$g(t) = \int\limits_{\Re^1} e^{itx} dG(x); \quad \gamma{-}p = \frac{d^p}{dt^p}(f\mathrm{x}(t){-}g(t))\big|_{t=0}, \quad p = 0; 1; \ldots$$

and k_r – quantity with which the inequality

$$\big|f_{\mathrm{x}}(t){-}g(t){-}\gamma_0{-}\cdots{-}\gamma_s(it)^s/s!\big| \leq k_r|t|^r/r!, \tag{8}$$

is satisfied, when $|t| \leq T_0, s = [r]$ and $s = r - 1$, if r is integer.

For example this inequality is always true when $k_r = \int\limits_{\Re^1} |x|^r|d(F\mathrm{x}-G)(x)| < \infty$.

Theorem 1 [3] *If the following conditions are satisfied:*

1. *This exist such numbers $a > 0$ and $b > 0$, that $|g(t)| \leq e^{-at^2}$, when $|t| < b$;*

2. *$\gamma_0 = \gamma_1 = \gamma_2 = 0$ and $k_3 < \infty$, then for all $n > 1$ and for τ satisfying the system of inequalities*

$$\begin{cases} \tau \leq \min(b; T_0), \\ \tau e^{a\tau^2} \leq 3a(2k_3C(3,2))^{-1}, \end{cases} \tag{9}$$

$$C(p,2) = \max\{2^{-1}p; (2p^{-1})^{\frac{p-2}{2}}\Gamma(p)/\Gamma(2^{-1}p)\},$$

the estimation is true

$$\sup_x \big|P\big(\sum_{j=1}^n X_j < x\big) - G^{*n}(x)\big| \leq 1,73 \times$$

$$\{2\pi^{-1}n\Gamma\big(\frac{3}{2}\big)\Big(a\big(n-1\big)\Big)^{-\frac{3}{2}}k_3/3! + \pi^{-1}R(3,\tau) + 0,81M(\tau)\}.$$

Here

$$R(p,\tau) = \begin{cases} 0, & \textit{when } n \geq 1 + 2a\tau^2/p, \\ (2k_p\tau^p/p!)(pn^{-1}), & \textit{when } n < 1 + 2a\tau^2/p \end{cases}$$

and

$$M(\tau) = 3,25\tau^{-1}\sup_x \big|\frac{d}{dx}G^{*n}(x)\big|.$$

First of all we estimate the characteristic function $h(t) = e^{itm}\lambda(\lambda - it)^{-1}$ of transferred exponential distribution. The Maclaurin series for $|h(t)|$ about $t = 0$ is

$$|h(t)| = 1 - \frac{1!1!t^2}{2!\lambda^2} + \cdots + (-1)^\nu \frac{(2\nu-1)!!t^{2\nu}}{(2\nu)!\lambda^{2\nu}} + \cdots$$

Therefore, for all $|t| \leq \lambda$

$$|h(t) \leq 1 - 0,250t^2\lambda^{-2} \leq e^{-0,250t^2\lambda^{-2}} \tag{10}$$

and at the same time we fix that $a = 0,250\lambda^{-2}$ when $b = \lambda$.

Since $C(3,2) = 1,843$, so the system of inequalities (9) take this form

$$\begin{cases} \tau \leq \min(\lambda; T_0) \\ \tau e^{0.25\lambda^{-2}\tau^2} \leq 0,203\lambda^{-2}k_3^{-1}. \end{cases}$$

The set of solutions in this system of inequalities became narrow, if we put $\tau = \lambda$ in the expression $e^{0,25\lambda^{-2}\tau^2}$ of the second inequality.

We do not pay attention to the quantity T_0, because of the inequality (8) is true for all t, when $r = 3$ and $k_3 < \infty$.

Therefore the set of solutions

$$\tau \leq \min\left\{\lambda; 0,159\lambda^{-2}k_3^{-1}\right\} \tag{11}$$

enter in the set of solutions of the system inequalities (11).

The density of transferred Erlang distribution function

$$\frac{d}{dx}G_n(x) = \begin{cases} \frac{\lambda^n(x-mn)^{m-1}}{(n-1)!}^{n-1} e^{-\lambda(x-mn)}, & \text{when } x > mn, \\ 0 & \text{when } \quad x \leq mn \end{cases}$$

has the maximum at the point $x\big((m+1)n - 1\big)\lambda^{-1}$

$$M(\tau) \leq 3,25\tau^{-1}\lambda(2\pi(n-1))^{-\frac{1}{2}}. \tag{12}$$

Using the cited theorem and (10; 13) we have

$$\Delta_n \leq 1,73\left\{2\pi^{-1}n\Gamma(1,5)\Big(0,25(n-1)\Big)^{-\frac{3}{2}}k_3\lambda^3/3! \right.$$
$$\left. \pi^{-1}R(3,\tau) + 2,633\Big(2\pi(n-1)\Big)^{-\frac{1}{2}}\tau^{-1}\lambda\right\},$$

as $n > 1$ and τ is from (12).

Now the first estimate in the theorem 1 follows from simple calculations.

We get the second estimate when $b = 2^{-1}\lambda$. Then $|h(t)| \leq e^{-0.42\lambda^{-2}t^2}$ and $a = 0,42\lambda^{-2}$.

Proof of the proposition is very simple.

If $k_3 \leq 0,159\lambda^{-3}$, then $\tau = \lambda$ and from (3) we get (5).

If $k_3 \leq 0,616\lambda^{-3}$, then $\tau = 0,5\lambda$ and from (4) we get (6).

When $k_3 > 0,616\lambda^{-3}$, then $\tau = 0,308\lambda^{-2}k_3^{-1}$ and the estimate (7) we obtain from (4).

3 References

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