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Semiparametric Estimates and Goodness-of-Fit Tests for Tire Wear and Failure Time Data

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Abstract

The goodness-of-fit test for tire wear and failure time data with multiple failure modes is proposed. Parametric estimators of traumatic event cumulative intensities and semiparametric estimates of various reliability characteristics are given and their large sample properties are investigated. Real tire wear and failure time data are analyzed.

Keywords: estimation, failure time, reliability, goodness-of-fit.

1 Introduction

Suppose that a tire fails because of the natural cause (the wear attains critical level z_0) or because of traumatic events of one of s possible types. Let Z(t) be the wear of tire protector at the "moment" t (i.e. at the moment when tire ran reaches t thousands of kilometres). Suppose that the wear process Z(t) is modeled by the linear path model (see [5])

$$Z(t) = \frac{t}{A},\ t \geq 0 \tag{1}$$

where A is a positive random variable with the distribution function π.

Denote by $T^{(0)}$ the time of non-traumatic failure and by $T^{(k)}$, $(k = 1, \cdots, s)$ the failure time corresponding to the k-th traumatic failure mode. As in [2], suppose that the random variables $T^{(1)}, \cdots, T^{(s)}$ are conditionally independent (given $A = a$) and have the intensities $\lambda^{(k)}(z)$, $k = 1, \cdots, s$ depending only on the wear level. It means that the conditional survival function of $T^{(k)}$ is

$$\begin{aligned} S^{(k)}(t \mid a) &= \mathbf{P}(T^{(k)} > t | A = a) = \exp\Big(-\int_0^t \lambda^{(k)}(s/a)ds\Big) \\ &= \exp\big(-a\Lambda^{(k)}(t/a)\big), \end{aligned}$$

$$\text{where} \quad \Lambda^{(k)}(z) = \int_0^z \lambda^{(k)}(y)dy. \tag{2}$$

are the cumulative intensities. The failure time of a tire is the random variable

$$T = \min(T^{(0)}, T^{(1)}, \cdots, T^{(s)}). \tag{3}$$

$$\text{Set} \quad \Lambda^{(\cdot)}(z) = \sum_{k=1}^{s} \Lambda^{(k)}(z), \tag{4}$$

$$V = \begin{cases} 0, & \text{if} \quad T = T^{(0)}, \\ 1, & \text{if} \quad T = T^{(1)}, \\ \cdots & \qquad \cdots \\ s, & \text{if} \quad T = T^{(s)} \end{cases} \tag{5}$$

The random variable V is the indicator of the failure type.

2 Reliability Characteristics of Tires

The survival function and the mean of the random variable T are

$$S(t) = \mathbf{P}(T > t) = \int_{t/z_0}^{\infty} e^{-a\Lambda^{(\cdot)}(t/a)} d\pi(a), \tag{6}$$

$$e = z_0 \int_0^{\infty} a d\pi(a) - \int_0^{\infty} \int_0^{z_0} (z_0 - z) a^2 e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(\cdot)}(z) d\pi(a). \tag{7}$$

The probability $P^{(k)}(t)$ of the failure of the kth mode in the interval $[0, t]$ and its limit value $P^{(k)} = P^{(k)}(\infty)$ are

$$P^{(k)}(t) = \int_0^\infty \int_0^{z_0 \wedge (t/a)} a e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(k)}(z) d\pi(a), \tag{8}$$

$$P^{(k)} = \int_0^\infty \int_0^{z_0} a e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(k)}(z) d\pi(a) \tag{9}$$

where $a \wedge b = min\,(a, b)$. The probability $P^{(tr)}(t)$ of a traumatic failure in the interval $[0, t]$ and its limit value $P^{(tr)} = P^{(tr)}(\infty)$ are

$$P^{(tr)}(t) = 1 - \int_0^\infty e^{-a\Lambda^{(\cdot)}(z_0 \wedge (t/a))} d\pi(a), \tag{10}$$

$$P^{(tr)} = 1 - \int_0^\infty e^{-a\Lambda^{(\cdot)}(z_0)} d\pi(a). \tag{11}$$

The probability of the natural failure in the interval $[0, t]$ is

$$P^{(0)}(t) = 1 - S(t) - P^{(tr)}(t) = \int_0^{t/z_0} e^{-a\Lambda^{(\cdot)}(z_0)} d\pi(a) \tag{12}$$

and

$$P^{(0)} = \int_0^\infty e^{-a\Lambda^{(\cdot)}(z_0)} d\pi(a). \tag{13}$$

Suppose that at the moment t the wear value is measured to be z. The following (conditional) reliability characteristics are important to estimate: the probability to fail in the interval $(t, t+\Delta]$, the probability of a failure of the kth mode, the probability of a traumatic failure in the same interval, and the mean residual life of an unit. We denote them by $Q(\Delta; t, z)$, $Q^{(k)}(\Delta; t, z)$, $Q^{(tr)}(\Delta; t, z)$, and $e(t, z)$

$$Q^{(k)}(\Delta; t, z) = \mathbf{P}(T = T^{(k)} \le t + \Delta \mid T > t, A = t/z), \tag{14}$$

if $k = 1, \cdots, s$ then

$$Q^{(k)}(\Delta; t, z) = \frac{t}{z} \int_z^{z_0 \wedge \frac{z(t+\Delta)}{t}} \exp\left\{-\frac{t}{z}\left(\Lambda^{(\cdot)}(y) - \Lambda^{(\cdot)}(z)\right)\right\} d\Lambda^{(k)}(y), \tag{15}$$

$$Q^{(tr)}(\Delta; t, z) = 1 - \exp\left\{-\frac{t}{z}\left[\Lambda^{(\cdot)}\left(z_0 \wedge \frac{z(t+\Delta)}{t}\right) - \Lambda^{(\cdot)}(z)\right]\right\}, \tag{16}$$

$$Q^{(0)}(\Delta; t, z) = \begin{cases} 0 & \text{for} \quad \Delta < \tau; \\ \exp\left\{-\frac{t}{z}\left(\Lambda^{(\cdot)}(z_0) - \Lambda^{(\cdot)}(z)\right)\right\} & \text{for} \quad \Delta \geq \tau; \end{cases} \tag{17}$$

$$Q(\Delta; t, z) = \begin{cases} 1 - \exp\left\{-\frac{t}{z}\left(\Lambda^{(\cdot)}(z(1+\Delta/t)) - \Lambda^{(\cdot)}(z)\right)\right\} & \text{for } \Delta < \tau; \\ 1 & \text{for } \Delta \geq \tau; \end{cases} \tag{18}$$

where $\tau = t(z_0/z - 1)$ The conditional mean is

$$\begin{aligned} e(t, z) &= \mathbf{E}\{T - t \mid T > t, A = t/z\} = \int_0^\infty \Delta\, Q(d\Delta; t, z) \\ &= \left(\frac{t}{z}\right)^2 \int_z^{z_0} (y - z)\, e^{-\frac{t}{z}\left(\Lambda^{(\cdot)}(y) - \Lambda^{(\cdot)}(z)\right)}\, d\Lambda^{(\cdot)}(y) \\ &+ \ t(z_0/z - 1)e^{-\frac{t}{z}\left(\Lambda^{(\cdot)}(z_0) - \Lambda^{(\cdot)}(z)\right)}. \end{aligned} \tag{19}$$

3 Semiparametric Estimators of Reliability Characteristics

Suppose that n tires are on test and the failure moments T_i, the indicators of the failure modes V_i and the wear values

$$Z_i = \frac{T_i}{A_i} \tag{20}$$

at the failure moments T_i are observed. Thus, the data are:

$$(T_1, Z_1, V_1), \cdots, (T_n, Z_n, V_n).$$

$$\text{Set} \quad N_n^{(k)}(z) = \sum_{i=1}^n \mathbf{1}_{\{Z_i \leq z, V_i = k\}}, \quad k = 1, \cdots, s,\ 0 \leq z \leq z_0. \tag{21}$$

It is the number of units having a failure of the kth type before the wear attains the level z.

$$\text{Set} \quad Y_n(z) = \sum_{i=1}^n A_i \mathbf{1}_{\{Z_i \geq z\}} = \sum_{Z_i \geq z} A_i, \tag{22}$$

$$M_n^{(k)}(z) = N_n^{(k)}(z) - \int_0^z \lambda^{(k)}(u) Y_n(u) du. \tag{23}$$

Let $\mathcal{F}_z$ be the σ-algebra generated by the random variables $A_1, \cdots, A_n$ and $N_n^{(1)}(u), \cdots, N_n^{(s)}(u)$, $u \leq z$ and $EA < \infty$.

Then (see [2]) $(M_n^{(k)}(z), 0 \le z \le z_0)$ is a martingale with respect to the filtration $(\mathcal{F}_z, 0 \le z \le z_0)$ and the optimal non-parametric estimators of the cumulative intensities $\Lambda^{(k)}(z)$ are of Nelson-Aalen type:

$$\hat{\Lambda}^{(k)}(z) = \int_0^z \frac{1}{Y_n(y)}\, dN_n^{(k)}(y) = \sum_{Z_i \le z, V_i = k} \frac{1}{Y_n(Z_i)} \tag{24}$$

$$\text{and} \quad \frac{Y_n(z)}{n} \xrightarrow{P} b(z), \quad n \to \infty, \quad b(z) = E\left(Ae^{-A\,\Lambda(z)}\right). \tag{25}$$

The non-parametric estimator of the distribution function π is the empirical distribution function

$$\hat{\pi}(a) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{A_i \le a\}}. \tag{26}$$

If the function π is continuous then (see [5]) the random function $\sqrt{n}(\hat{\pi} - \pi)$ tends in distribution in the Skorokhod space $D[0, \infty]$ to a zero mean Gaussian process $W^{(0)}$ with the covariance function

$$\sigma_\pi^2(a, a') = \mathbf{E}\{W^{(0)}(a) W^{(0)}(a')\} = \pi(a \wedge a') - \pi(a)\,\pi(a'). \tag{27}$$

We are interested in semiparametric and parametric estimates of reliability characteristics and their large sample properties.

Suppose that the intensities $\lambda^{(k)}(z)$ belong to a parametric class

$$\lambda^{(k)}(z) = \lambda^{(k)}(z, \gamma_k),\ 0 \le z \le z_0,\ k = 1, \cdots, s \tag{28}$$

where γ_k are a possibly multi-dimensional parametres. The purpose is to estimate parametres γ_k, $k = 1, \cdots, s$ and derive large sample properties of estimators.

Logarithm of the likelihood function equals

$$\ln L = \sum_{k=1}^{s} \sum_{V_i = k} \ln\left(\lambda^{(k)}(Z_i)\right) - \sum_{i=1}^{n}\left(A_i\, \Lambda^{(\cdot)}(Z_i) + \ln f(A_i)\right) \tag{29}$$

where $f(u)$ is the density function of A.

Estimators $\hat{\gamma}_k$, $k = 1, \cdots, s$ verify the equations

$$\frac{\partial \ln L(\hat{\gamma}_k)}{\partial \gamma_k} = \sum_{V_i = k} \frac{\partial}{\partial \gamma_k} \ln\left(\lambda^{(k)}(Z_i, \hat{\gamma}_k)\right) - \sum_{i=1}^{n} A_i \frac{\partial}{\partial \gamma_k} \Lambda^{(k)}(Z_i, \hat{\gamma}_k) = 0, \tag{30}$$

Then the parametric estimators of cumulative intensities are

$$\hat{\Lambda}^{(k)}(z) = \Lambda^{(k)}(z, \hat{\gamma}_k), \; k = 1, \cdots, s. \tag{31}$$

Semiparametric estimators of all reliability characteristics defined by (6)-(19) are obtained changing $\Lambda^{(k)}(z), \; k = 1, \cdots, s$ by their parametric estimators defined by (31), $\Lambda^{(\cdot)}(z)$ by $\hat{\Lambda}^{(\cdot)}(z) = \sum_{k=1}^{s} \hat{\Lambda}^{(k)}(z) = \sum_{k=1}^{s} \Lambda^{(k)}(z, \hat{\gamma}_k)$ and $\pi(a)$ by its nonparametric estimator $\hat{\pi}(a)$ defined by (26).

4 Large sample Properties of Estimators

4.1 Large sample properties of estimators $\hat{\gamma}_{\mathbf{k}}$

By (21), (22) we may (30) rewrite

$$\frac{\partial \ln L(\hat{\gamma}_k)}{\partial \gamma_k} = \int_0^{z_0} \frac{\partial}{\partial \gamma_k} \ln(\lambda^{(k)}(u, \hat{\gamma}_k)) dN_n^{(k)}(u) - \int_0^{z_0} Y_n(u) \frac{\partial}{\partial \gamma_k} \lambda^{(k)}(u, \hat{\gamma}_k) du$$

Thus, estimators $\hat{\gamma}_k$ verify the equations

$$\int_0^{z_0} \frac{\partial}{\partial \gamma_k} \ln(\lambda^{(k)}(u, \hat{\gamma}_k)) dM_n^{(k)}(u, \hat{\gamma}_k) = 0, \; k = 1, \cdots, s \tag{32}$$

$$\text{where} \quad dM_n^{(k)}(u, \gamma_k) = dN_n^{(k)}(u) - Y_n(u) \lambda^{(k)}(u, \gamma_k) du. \tag{33}$$

$$\text{Set} \quad U^{(k)}(z, \gamma_k) = \int_0^{z} \frac{\partial}{\partial \gamma_k} \ln(\lambda^{(k)}(u, \gamma_k)) dM_n^{(k)}(u, \gamma_k), \; k = 1, \cdots, s. \tag{34}$$

We derive the large sample properties of estimator $\hat{\gamma} = (\hat{\gamma}_1, \cdots, \hat{\gamma}_s)$. Denote by $\gamma^{(0)} = \left(\gamma_1^{(0)}, \cdots, \gamma_s^{(0)}\right)$ the true value of the parameter γ. Let $\gamma_k = (\gamma_{k1}, \cdots, \gamma_{kq_k}), \; k = 1, \cdots, s$, where q_k is the dimension of parameter γ_k. Then the score statistics are

$$\begin{aligned} U_j^{(k)}(z, \gamma) &= U_j^{(k)}(z, \gamma_k) = \int_0^{z} \frac{\partial}{\partial \gamma_{kj}} \ln \lambda^{(k)}(u, \gamma_k) dN_n^{(k)}(u) \\ &- \int_0^{z} Y_n(u) \frac{\partial}{\partial \gamma_{kj}} \lambda^{(k)}(u, \gamma_k) du, \quad k = 1, \cdots, s \end{aligned} \tag{35}$$

where $\lambda^{(k)}(z, \gamma) = \lambda^{(k)}(z, \gamma_k) = \lambda^{(k)}(z, \gamma_{k1}, \cdots, \gamma_{kq_k}), \; j = 1, \cdots, q_k$.

For $l, j = 1, \cdots, q_k$ and $k = 1, \cdots, s$ set

$$I_{jl}^{(kk)}(z_0, \gamma) = I_{jl}^{(k)}(z_0, \gamma_k) = \int_0^{z_0} Y_n(u) \frac{\partial^2}{\partial \gamma_{kj} \partial \gamma_{kl}} \lambda^{(k)}(u, \gamma_k) du - \int_0^{z_0} \frac{\partial^2}{\partial \gamma_{kj} \partial \gamma_{kl}} \ln \lambda^{(k)}(u, \gamma_k) dN_n^{(k)}(u), \quad (36)$$

$$I_{jl}^{(km)}(z_0, \gamma) = 0, \quad k, m = 1, \cdots, s, \ k \neq m. \quad (37)$$

Assumption A.

1) There exists a neighborhood $\Gamma^{(0)}$ of $\gamma^{(0)}$ such that for all $\gamma = (\gamma_1, \cdots, \gamma_k)$ in $\Gamma^{(0)}$ and almost all $z \in [0, z_0]$ the partial derivatives of $\lambda^{(k)}(z, \gamma)$ and $\ln \lambda^{(k)}(z, \gamma)$ of first, second and third order with respect to γ exist and are continuous in γ for $\gamma \in \Gamma^{(0)}$. The log-likelihood function (29) may be differentiated three times with respect to $\gamma \in \Gamma^{(0)}$ by interchanging the order of integration and differentiation.

2) Let (25) hold and for all $k = 1, \cdots, s$ and $j, l = 1, \cdots, q_k$

$$\sigma_{jl}^{(k)}(\gamma_k^{(0)}) = \int_0^{z_0} \phi_{kj}^{(k)}(u, \gamma_k^{(0)}) \phi_{kl}^{(k)}(u, \gamma_k^{(0)}) b(u) \lambda^{(k)}(u, \gamma_k^{(0)}) du < \infty \quad (38)$$

where b(u) is defined by (25) and

$$\phi_{kj}^{(k)}(u, \gamma_k^{(0)}) = \frac{\partial}{\partial \gamma_{kj}} \ln \lambda^{(k)}(u, \gamma_k^{(0)}). \quad (39)$$

3) The matrix $\Sigma = \{\sigma_{jl}^{(k)}(\gamma_k^{(0)}), \ k = 1, \cdots, s, \ j, l = 1, \cdots, q_k\}$ with $\sigma_{jl}^{(k)}(\gamma_k^{(0)})$ defined by (38) is positive defined.

4) For all $k = 1, \cdots, s$ and $j, l, m = 1, \cdots, q_k$ there exist functions η and ρ such that

$$\sup_{\gamma_k \in \Gamma^{(0)}} \left| \frac{\partial^3}{\partial \gamma_{kj} \partial \gamma_{kl} \partial \gamma_{km}} \lambda^{(k)}(u, \gamma_k) \right| \leq \eta(u), \quad (40)$$

$$\sup_{\gamma_k \in \Gamma^{(0)}} \left| \frac{\partial^3}{\partial \gamma_{kj} \partial \gamma_{kl} \partial \gamma_{km}} \ln \lambda^{(k)}(u, \gamma_k) \right| \leq \rho(u), \quad (41)$$

where $0 \leq u \leq z_0$ and

$$\int_0^{z_0} \rho(u) b(u) \lambda^{(k)}(u, \gamma_k^{(0)}) du < \infty, \quad (42)$$

$$\int_0^{z_0} \left(\frac{\partial^2}{\partial\gamma_{kj}\partial\gamma_{kl}} \ln \lambda^{(k)}(u,\gamma_k) \right)^2 b(u)\lambda^{(k)}(u,\gamma_k^{(0)})du < \infty. \qquad (43)$$

Theorem 1. Assume that Condition A holds. Then

1) with a probability tending to one the equation $U^{(k)}(z,\gamma_k) = 0$ has a solution $\hat{\gamma}_k$ and $\hat{\gamma}_k \xrightarrow{P} \gamma_k^{(0)}$ *as* $n \to \infty$, $k = 1, \cdots, s$.

2) $\sqrt{n}(\hat{\gamma}_k - \gamma_k^{(0)}) \xrightarrow{\mathcal{D}} N(0, \Sigma_k^{-1})$ where $\Sigma_k = \{\sigma_{jl}^{(k)}(\gamma_k^{(0)}),\ j,l = 1,\cdots,q_k\}$ may be estimated consistently by $\frac{1}{n}I^{(k)}(z,\hat{\gamma}_k),\ k = 1,\cdots,s$ defined by (36).

Proof.

By (25) and (38) we have

$$\frac{1}{n}\int_0^{z_0} \phi_{kj}^{(k)}(u,\gamma_k^{(0)})\phi_{kl}^{(k)}(u,\gamma_k^{(0)})Y_n(u)\lambda^{(k)}(u,\gamma_k^{(0)})du \xrightarrow{P} \sigma_{jl}^{(k)}(\gamma_k^{(0)}) < \infty$$

where $\phi_{kj}^{(k)}(u,\gamma_k^{(0)})$ defined by (39) and

$$\frac{1}{n}\int_0^{z_0} \left(\phi_{kj}^{(k)}(u,\gamma_k^{(0)})\right)^2 I\left(\mid \frac{1}{\sqrt{n}}\phi_{kj}^{(k)}(u,\gamma_k^{(0)}) \mid > \epsilon \right) *$$

$$* \ Y_n(u)\lambda^{(k)}(u,\gamma_k^{(0)})du \xrightarrow{P} 0$$

as $n \to \infty$, $k = 1,\cdots,s,\ j,l = 1,\cdots,q_k$. Condition 4) and (25) give

$$\frac{1}{n}\int_0^{z_0} \rho(u)Y_n(u)\lambda^{(k)}(u,\gamma_k)du \xrightarrow{P} \int_0^{z_0} \rho(u)b(u)\lambda^{(k)}(u,\gamma_k)du < \infty,$$

$$\frac{1}{n}\int_0^{z_0} h^2(u,\gamma_k)Y_n(u)\lambda^{(k)}(u,\gamma_k)du \xrightarrow{P}$$

$$\xrightarrow{P} \int_0^{z_0} h^2(u,\gamma_k)b(u)\lambda^{(k)}(u,\gamma_k)du < \infty,$$

where $h(u,\gamma_k) = \frac{\partial^2}{\partial\gamma_{kj}\partial\gamma_{kl}} \ln \lambda^{(k)}(u,\gamma_k)$ and

$$\frac{1}{n}\int_0^{z_0} \rho(u)I\left(\frac{1}{\sqrt{n}}\sqrt{\rho(u)} > \epsilon\right) Y_n(u)\lambda^{(k)}(u,\gamma_k)du \xrightarrow{P} 0$$

as $n \to \infty$, $k = 1,\cdots,s,\ j,l = 1,\cdots,q_k$. Then by ([1]), T.VI.1.1. and T.VI.1.2. Theorem 1 holds.

4.2 Large Sample Properties of the Reliability Characteristics Estimators

Assume that Condition A holds. Then by Theorem 1 we have

$$\sqrt{n}(\hat{\gamma}_k - \gamma_k^{(0)}) \xrightarrow{d} X_k \quad n \to \infty, \quad k = 1, \cdots, s \tag{44}$$

where X_k is normally distributed with zero mean and the covariance matrix Σ_k^{-1} defined in Theorem 1. By (31), (44) and the delta method we get

$$\sqrt{n}(\hat{\Lambda}^{(k)}(z) - \Lambda^{(k)}(z)) \xrightarrow{d} W^{(k)}(z) \quad as\ n \to \infty, \quad k = 1, \cdots, s \tag{45}$$

where $W^{(k)}(z) = \left(\frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k}\right)^T X_k$ is normally distributed with zero mean and the variance

$$Var(W^{(k)}(z)) = \left(\frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k}\right)^T \Sigma_k^{-1} \left(\frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k}\right) \tag{46}$$

where $k = 1, \cdots, s$. By (4) we get

$$\hat{\Lambda}^{(\cdot)}(z) = \sum_{k=1}^{s} \hat{\Lambda}^{(k)}(z) = \sum_{k=1}^{s} \Lambda^{(k)}(z, \hat{\gamma}_k). \tag{47}$$

Then

$$\sqrt{n}(\hat{\Lambda}^{(\cdot)}(z) - \Lambda^{(\cdot)}(z)) \xrightarrow{d} \sum_{k=1}^{s} W^{(k)}(z) = W^{(\cdot)}(z) \quad as \quad n \to \infty \tag{48}$$

where $W^{(\cdot)}(z)$ is normally distributed with zero mean and the variance

$$Var(W^{(\cdot)}(z)) = \sum_{k=1}^{s} \left(\frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k}\right)^T \Sigma_k^{-1} \left(\frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k}\right). \tag{49}$$

The first equation in (49) is get by using fact that $W^{(k)}(z)$, $k = 1, \cdots, s$ are independent. Furthermore

$$\sqrt{n}(\hat{\Lambda}^{(1)} - \Lambda^{(1)}, \cdots, \hat{\Lambda}^{(s)} - \Lambda^{(s)}) \xrightarrow{d} (W^{(1)}, \cdots, W^{(s)}) \tag{50}$$

where $W^{(k)}$ are independent normally distributed with zero mean and the covariance

$$\sigma_k^2(z,z') = \mathbf{E}\{W^{(k)}(z)W^{(k)}(z')\} = \left(\tilde{\Lambda}^{(k)}(z)\right)^T \Sigma_k^{-1}\left(\tilde{\Lambda}^{(k)}(z')\right) \quad (51)$$

$$\text{where} \quad \tilde{\Lambda}^{(k)}(z) = \tilde{\Lambda}^{(k)}(z,\gamma_k^{(0)}) = \frac{\partial \Lambda^{(k)}(z,\gamma_k^{(0)})}{\partial \gamma_k}. \quad (52)$$

$$\text{Set} \quad \sigma^2(z,z') = \sum_{k=1}^{s} \sigma_k^2(z,z'). \quad (53)$$

For notational simplicity denote $\Lambda^{(k)}(z) = \Lambda^{(k)}(z,\gamma_k^{(0)})$ and $\Lambda^{(\cdot)}(z) = \Lambda^{(\cdot)}(z,\gamma_k^{(0)})$

Lemma 1. Suppose that μ is a finite measure on $[0,z_0]$ and $B = \int_0^{z_0} W^{(\cdot)}(z)d\mu(z)$ where $W^{(\cdot)}(z)$ is defined by (48). Then $\mathbf{E}(B)=0$ and $\mathbf{E}(B^2) = L = \sum_{k=1}^{s}\left(L^{(k)}\right)^T \Sigma_k^{-1} L^{(k)}$ where Σ_k^{-1} is defined in Theorem 1 and $L^{(k)} = \int_0^{z_0} \frac{\partial \Lambda^{(k)}(z,\gamma_k^{(0)})}{\partial \gamma_k} d\mu(z)$.

Proof. $\mathbf{E}(B) = \int_0^{z_0} \mathbf{E}(W^{(\cdot)}(z))d\mu(z) = 0$. By (51), (53) we get

$$\begin{aligned}
\mathbf{E}(B^2) &= \int_0^{z_0} d\mu(z) \int_0^{z_0} \sigma^2(z,z')d\mu(z') \\
&= \sum_{k=1}^{s}\left(\int_0^{z_0} \frac{\partial \Lambda^{(k)}(z)}{\partial \gamma_k} d\mu(z)\right)^T \Sigma_k^{-1} \left(\int_0^{z_0} \frac{\partial \Lambda^{(k)}(z')}{\partial \gamma_k} d\mu(z')\right).
\end{aligned}$$

Theorem 2. Let g be any of reliability characteristics defined by (6)-(13). Suppose that Assumption A holds, $\mathbf{E}(A^3) < \infty$ and the distribution function π is continuous then the distribution of the random variable $\sqrt{n}(\hat{g}-g)$ tends to the normal law with zero-mean and the variance $V(\hat{g})$, where

$$V(\hat{S}(t)) = \sum_{k=1}^{s}\left(L_1^{(k)}(t)\right)^T \Sigma_k^{-1} L_1^{(k)}(t) + \int_{t/z_0}^{\infty} e^{-2a\Lambda^{(\cdot)}(t/a)}d\pi(a) - S^2(t) \quad (54)$$

where Σ_k^{-1} is defined in Theorem 1 and

$$L_1^{(k)}(t) = \int_{t/z_0}^{\infty} a\, \frac{\partial \Lambda^{(k)}(t/a,\gamma_k^{(0)})}{\partial \gamma_k} e^{-2a\Lambda^{(\cdot)}(t/a)}d\pi(a),$$

$$V(\hat{e}) = \sum_{k=1}^{s} \left(L_2^{(k)}\right)^T \Sigma_k^{-1} L_2^{(k)} - e^2 + \int_0^\infty \left(az_0 - a^2 \int_0^{z_0} (z_0 - z) e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(\cdot)}(z)\right)^2 d\pi(a) \quad (55)$$

where

$$L_2^{(k)} = -\int_0^{z_0} \frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k} \left(\int_0^\infty a^2 e^{-a\Lambda^{(\cdot)}(z)} d\pi(a)\right) dz,$$

$$V(\hat{P}^{(k)}(t)) = \sum_{i=1}^{s} \left(L_3^{(i)}(t)\right)^T \Sigma_i^{-1} L_3^{(i)}(t) + \sum_{l=1, l\neq k}^{s} \left(L_4^{(l)}(t)\right)^T \Sigma_l^{-1} L_4^{(l)}(t) - \left(P^{(k)}(t)\right)^2 + \int_0^\infty a^2 \left(\int_0^{z_1} e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(k)}(z)\right)^2 d\pi(a) \quad (56)$$

where $z_1 = z_0 \wedge (t/a)$,

$$L_3^{(k)}(t) = \int_0^\infty a e^{-a\Lambda^{(\cdot)}(z_1)} \frac{\partial \Lambda^{(k)}(z_1, \gamma_k^{(0)})}{\partial \gamma_k} d\pi(a) + \int_0^\infty \int_0^{z_1} a^2 e^{-a\Lambda^{(\cdot)}(z)} \frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k} d\Lambda^{(-k)}(z) d\pi(a),$$

where $\Lambda^{(-k)}(z) = \sum_{l\neq k} \Lambda^l(z)$,

$$L_4^{(l)}(t) = \int_0^\infty \int_0^{z_1} a^2 e^{-a\Lambda^{(\cdot)}(z)} \frac{\partial \Lambda^{(l)}(z, \gamma_l^{(0)})}{\partial \gamma_l} d\Lambda^{(k)}(z) d\pi(a),$$

$$V(\hat{P}^{(k)}) = V(\hat{P}^{(k)}(\infty)), \quad (57)$$

$$V(\hat{P}^{(tr)}(t)) = \sum_{k=1}^{s} \left(L_5^{(k)}(t)\right)^T \Sigma_k^{-1} L_5^{(k)}(t) - \left(P^{(tr)}(t)\right)^2 + \int_0^\infty \left(1 - e^{-a\Lambda^{(\cdot)}(z_0 \wedge (t/a))}\right)^2 d\pi(a) \quad (58)$$

where

$$L_5^{(k)}(t) = \int_0^\infty a e^{-a\Lambda^{(\cdot)}(z_0 \wedge (t/a))} \frac{\partial \Lambda^{(k)}(z_0 \wedge (t/a), \gamma_k^{(0)})}{\partial \gamma_k} d\pi(a),$$

$$V(\hat{P}^{(tr)}) = V(\hat{P}^{(tr)}(\infty)), \tag{59}$$

$$\begin{aligned} V(\hat{P}^{(0)}(t)) &= \sigma^2(z_0, z_0)\left(\int_0^{t/z_0} a e^{-a\Lambda^{(\cdot)}(z_0)} d\pi(a)\right)^2 \\ &+ \sigma_\pi^2(t/z_0, t/z_0)\, e^{-\frac{2t}{z_0}\Lambda^{(\cdot)}(z_0)} \\ &+ \left(\Lambda^{(\cdot)}(z_0)\right)^2 \int_0^{t/z_0}\int_0^{t/z_0} e^{-(a+a')\Lambda^{(\cdot)}(z_0)} \sigma_\pi^2(a,a') da da' \\ &+ \Lambda^{(\cdot)}(z_0) e^{-\frac{t}{z_0}\Lambda^{(\cdot)}(z_0)} \int_0^{t/z_0} e^{-a\Lambda^{(\cdot)}(z_0)} \sigma_\pi^2(t/z_0, a) da \end{aligned} \tag{60}$$

where $\sigma^2(z,z)$ is defined by (53) and $\sigma_\pi^2(a,a')$ is defined by (27).

$$\begin{aligned} V(\hat{P}^{(0)}) &= V(\hat{P}^{(0)}(\infty)) = V(\hat{P}^{(tr)}) = \int_0^\infty e^{-2a\Lambda^{(\cdot)}(z_0)} d\pi(a) \\ &- (P^{(0)})^2 + \sigma^2(z_0,z_0)\left(\int_0^\infty a e^{-a\Lambda^{(\cdot)}(z_0)} d\pi(a)\right)^2. \end{aligned} \tag{61}$$

Approximate $(1-\alpha)$-confidence interval for g has the form

$$\hat{g} \pm z_{1-\alpha/2}\sqrt{\hat{V}(\hat{g})/n} \tag{62}$$

where $\hat{V}(\hat{g})$ is obtained replacing the unknown quantities $\pi(a)$, $\Lambda^{(k)}(z)$, $\Lambda^{(\cdot)}(z)$ by $\hat{\pi}(a)$, $\hat{\Lambda}^{(k)}(z) = \Lambda^{(k)}(z, \hat{\gamma}_k)$, $\hat{\Lambda}^{(\cdot)}(z) = \Lambda^{(\cdot)}(z, \hat{\gamma}_k)$ respectively.

Proof.

1. *Estimator* $\hat{S}(t)$. Set $\varphi(\Lambda, a) = e^{-a\Lambda^{(\cdot)}(t/a)}\mathbf{1}_{(t/z_0;\infty)}(a)$. Then by (48) and delta method we get

$$\sqrt{n}(\varphi(\hat{\Lambda}, a) - \varphi(\Lambda, a)) \xrightarrow{d} -a e^{-a\Lambda^{(\cdot)}(t/a)}\mathbf{1}_{(t/z_0,\infty)}(a) W^{(\cdot)}(t/a).$$

The inequality $\mid e^{-u_1} - e^{-u_2} \mid \le \mid u_1 - u_2 \mid$, $u_1, u_2 \ge 0$ implies $\sqrt{n} \mid \varphi(\hat{\Lambda}, a) - \varphi(\Lambda, a) \mid \le a \sup_{0\le z\le z_0} \mid W^{(\cdot)}(z) \mid$. Then (see [2])
$\sqrt{n}\{\hat{S}(t) - S(t)\} \xrightarrow{d} \xi_1 + \xi_2$, where

$$\xi_1 = -\int_{t/z_0}^\infty a e^{-a\Lambda^{(\cdot)}(t/a)} W^{(\cdot)}(t/a) d\pi(a) = -\int_0^{z_0} W^{(\cdot)}(z) d\mu(z),$$

$d\mu(z) = \frac{t}{z} e^{-\frac{t}{z}\Lambda^{(\cdot)}(z)} d(1 - \pi(t/z))$, $W^{(\cdot)}(z) = \sum_{k=1}^s W^{(k)}(z)$ and ξ_2 is independent of $W^{(\cdot)}$ normally distributed random variable with zero-mean and the variance

$$\mathbf{Var}(\xi_2) = \int_{t/z_0}^\infty e^{-2a\Lambda^{(\cdot)}(t/a)} d\pi(a) - S^2(t). \tag{63}$$

By Lemma 1 we get $\mathbf{Var}(\xi_1) = \sum_{k=1}^{s} \left(L^{(k)}(t)\right)^T \Sigma_k^{-1} L^{(k)}(t)$ where

$$L^{(k)}(t) = \int_{t/z_0}^{\infty} a \frac{\partial \Lambda^{(k)}(t/a, \gamma_k^{(0)})}{\partial \gamma_k} e^{-a\Lambda^{(\cdot)}(t/a)} d\pi(a).$$

2.*Estimator* $\hat{e}$. Set $\varphi(\Lambda, a) = az_0 - a^2 \int_0^{z_0} (z_0 - z) e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(\cdot)}(z)$, $f(z) = -a^2(z_0 - z)e^{-a\Lambda^{(\cdot)}(z)}$, $h(z) = \Lambda^{(\cdot)}(z)$. Then $\sqrt{n}(\hat{f}(z) - f(z)) \xrightarrow{d} a^3(z_0 - z)e^{-a\Lambda^{(\cdot)}(z)} W^{(\cdot)}(z)$, $\sqrt{n}(\hat{h}(z) - h(z)) \xrightarrow{d} W^{(\cdot)}(z)$. Using these facts and delta method we get
$\sqrt{n}(\varphi(\hat{\Lambda}, a) - \varphi(\Lambda, a)) \xrightarrow{d} -a^2 \int_0^{z_0} e^{-a\Lambda^{(\cdot)}(z)} W^{(\cdot)}(z) dz$. Then (see [2]) $\sqrt{n}(\hat{e} - e) \xrightarrow{d} \xi_1 + \xi_2$ where

$$\xi_1 = -\int_0^{\infty} a^2 \int_0^{z_0} e^{-a\Lambda^{(\cdot)}(z)} W^{(\cdot)}(z) dz d\pi(a)$$

and ξ_2 is independent of $W^{(\cdot)}$ normally distributed random variable with zero-mean and the variance

$$\mathbf{Var}(\xi_2) = \int_0^{\infty} \left(az_0 - a^2 \int_0^{z_0} (z_0 - z) e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(\cdot)}(z)\right)^2 d\pi(a) - e^2 \tag{64}$$

and $V(\hat{e}) = \mathbf{Var}(\xi_1) + \mathbf{Var}(\xi_2)$. It remains to find the variance of $\xi_1 = \int_0^{z_0} W^{(\cdot)}(z) \tilde{f}(z) dz$ where $\tilde{f}(z) = -\int_0^{\infty} a^2 e^{-a\Lambda^{(\cdot)}(z)} d\pi(a)$. Then using Lemma 1 we get

$$\mathbf{Var}(\xi_1) = \sum_{k=1}^{s} \left(L^{(k)}\right)^T \Sigma_k^{-1} L^{(k)} \tag{65}$$

where $L^{(k)} = -\int_0^{z_0} \frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k} \left(\int_0^{\infty} a^2 e^{-a\Lambda^{(\cdot)}(z)} d\pi(a)\right) dz$. Then (64), (65) give (55).

3.*Estimator* $\hat{P}^{(k)}(t)$, $k = 1, \cdots, s$. Set $z_1 = z_0 \wedge (t/a)$ and $\varphi(\Lambda, a) = a \int_0^{z_1} e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(k)}(z)$. By delta method $\sqrt{n}\left(ae^{-a\hat{\Lambda}^{(\cdot)}(z)} - ae^{-a\Lambda^{(\cdot)}(z)}\right) \xrightarrow{d} -a^2 e^{-a\Lambda^{(\cdot)}(z)} W^{(\cdot)}(z)$. Then

$$\sqrt{n}(\varphi(\hat{\Lambda}, a) - \varphi(\Lambda, a)) \xrightarrow{d} -a^2 \int_0^{z_1} e^{-a\Lambda^{(\cdot)}(z)} W^{(\cdot)}(z) d\Lambda^{(k)}(z)$$

$$+ae^{-a\Lambda^{(\cdot)}(z_1)} W^{(k)}(z_1) + a^2 \int_0^{z_1} e^{-a\Lambda^{(\cdot)}(z)} W^{(k)}(z) d\Lambda^{(\cdot)}(z). \tag{66}$$

Similarly as above we get $\sqrt{n}(\hat{P}^{(k)}(t) - P^{(k)}(t)) \xrightarrow{d} \xi_1 + \xi_2$ where

$$\xi_1 = \int_0^\infty \left(-a^2 \int_0^{z_1} e^{-a\Lambda^{(\cdot)}(z)} W^{(\cdot)}(z) d\Lambda^{(k)}(z) + a e^{-a\Lambda^{(\cdot)}(z_1)} W^{(k)}(z_1) + \right.$$
$$\left. + \int_0^{z_1} a^2 e^{-a\Lambda^{(\cdot)}(z)} W^{(k)}(z) d\Lambda^{(\cdot)}(z) \right) d\pi(a)$$

and ξ_2 is independent of $W^{(\cdot)}$, $W^{(k)}$ normally distributed random variable with zero-mean and the variance

$$\begin{aligned}
\mathbf{Var}(\xi_2) &= \int_0^\infty \varphi^2(\Lambda, a) d\pi(a) - \left(\int_0^\infty \varphi(\Lambda, a) d\pi(a) \right)^2 \\
&= \int_0^\infty \left(a \int_0^{z_1} e^{-a\Lambda^{(\cdot)}(z)} d\Lambda^{(k)}(z) \right)^2 d\pi(a) - (P^{(k)}(t))^2.
\end{aligned}$$

The random variable ξ_1 may be written in the form

$$\begin{aligned}
\xi_1 &= \int_0^\infty a e^{-a\Lambda^{(\cdot)}(z_1)} W^{(k)}(z_1) d\pi(a) \\
&+ \int_0^\infty \int_0^{z_1} a^2 e^{-a\Lambda^{(\cdot)}(z)} W^{(k)}(z) d\Lambda^{(-k)}(z) d\pi(a) \\
&- \int_0^\infty \int_0^{z_1} a^2 e^{-a\Lambda^{(\cdot)}(z)} W^{(-k)}(z) d\Lambda^{(k)}(z) d\pi(a) \qquad (67)
\end{aligned}$$

where $W^{(-k)}(z) = \sum_{l \neq k} W^{(l)}(z)$, $\Lambda^{(-k)}(z) = \sum_{l \neq k} \Lambda^{(l)}(z)$.

Denote by ξ_1' the sum of the first two terms and by ξ_1'' - the third term. Since $W^{(-k)}$ and $W^{(k)}$ are independent we have $\mathbf{Var}(\xi_1) = \mathbf{Var}(\xi_1') + \mathbf{Var}(\xi_1'')$. Changing integration order write ξ_1' in the form $\xi_1' = \int_0^{z_0} W^{(k)}(z) d\mu(z)$ with $d\mu(z) = \int_0^{t/z} a^2 e^{-a\Lambda^{(\cdot)}(z)} d\pi(a) d\Lambda^{(-k)}(z) + \frac{t}{z} e^{-\frac{t}{z}\Lambda^{(\cdot)}(z)} d(1 - \pi(t/z)) + \int_0^{t/z_0} a e^{-a\Lambda^{(\cdot)}(z_0)} d\pi(a) d\delta_{z_0}(z)$, where δ_{z_0} denotes the probability measure concentrated at the point z_0. Then Lemma 1 implies

$$\mathbf{Var}(\xi_1') = \sum_{k=1}^{s} \left(L^{(k)}(t) \right)^T \Sigma_k^{-1} L^{(k)}(t) \qquad (68)$$

$$\text{where} \quad L^{(k)}(t) = \int_0^{z_0} \frac{\partial \Lambda^{(k)}(z, \gamma_k^{(0)})}{\partial \gamma_k} d\mu(z) \qquad (69)$$

$$\text{Analogously} \quad \mathbf{Var}(\xi_1'') = \sum_{l=1, l \neq k}^{s} \left(L^{(l)}(t) \right)^T \Sigma_l^{-1} L^{(l)}(t) \qquad (70)$$

where

$$L^{(l)}(t)=\int_0^\infty\int_0^{z_0\wedge(t/a)}a^2e^{-a\Lambda^{(\cdot)}(z)}\frac{\partial\Lambda^{(l)}(z,\gamma_l^{(0)})}{\partial\gamma_l}d\Lambda^{(k)}(z)d\pi(a). \tag{71}$$

Then (67), (68), (70) give (56).

4.*Estimator* $\hat{P}^{(k)}$, $k=1,\cdots,s$. Using the fact that $\hat{P}^{(k)}=\hat{P}^{(k)}(\infty)$ we get $V(\hat{P}^{(k)})=V(\hat{P}^{(k)}(\infty))$.

5.*Estimators* $\hat{P}^{(tr)}(t)$, $\hat{P}^{(0)}(t)$, . Asymptotic normality of these estimators is proved analogously.

Theorem 3. If g denotes $Q=Q(\Delta;t,z)$, $Q^{(k)}=Q^{(k)}(\Delta;t,z)$, $Q^{(tr)}=Q^{(tr)}(\Delta;t,z)$ or $e(t,z)$ then the distribution of the random variable $\sqrt{n}(\hat{g}-g)$ tends to the normal law with zero-mean and the variance $V(\hat{g})$. The asymptotic variances are:

$$\begin{aligned}V(\hat{Q}^{(k)}) &= \left(\frac{t}{z}\right)^4\int_z^{z_1}\int_z^{z_1}\varphi(u)\varphi(v)\tilde{\sigma}^2_{-k}(u,v,z)d\Lambda^{(k)}(u)d\Lambda^{(k)}(v)\\ &+ \left(\frac{t}{z}\right)^4\int_z^{z_1}\int_z^{z_1}\varphi(u)\varphi(v)\tilde{\sigma}^2_{(k)}(u,v,z)d\Lambda^{(-k)}(u)d\Lambda^{(-k)}(v)\\ &+ \left(\frac{t}{z}\right)^2\varphi(z_1)\tilde{\tilde{\sigma}}^2_k(z_1,z)\\ &+ \left(\frac{t}{z}\right)^3\int_z^{z_1}\varphi(u)\varphi(z_1)\tilde{\sigma}^2_k(u,z_1,z)d\Lambda^{(-k)}(u),\end{aligned} \tag{72}$$

$$V(\hat{Q}^{(tr)})=\left(\frac{t}{z}\right)^2(\varphi(z_1))^2\tilde{\tilde{\sigma}}^2(z_1,z), \tag{73}$$

$$V(\hat{e}(t,z))=\left(\frac{t}{z}\right)^4\int_z^{z_0}\int_z^{z_0}\varphi(u)\varphi(v)\tilde{\sigma}^2(u,v,z)dudv, \tag{74}$$

$$V(\hat{Q}^{(0)})=V(\hat{Q}^{(tr)})\quad\text{for}\quad\Delta\geq t(z_0/z-1), \tag{75}$$

$$V(\hat{Q})=V(\hat{Q}^{(tr)})\quad\text{for}\quad\Delta<t(z_0/z-1) \tag{76}$$

where $z_1=z_0\wedge\frac{z(t+\Delta)}{t}$ and

$$\varphi(u)=\varphi(u,t,z)=e^{-\frac{t}{z}(\Lambda^{(\cdot)}(u)-\Lambda^{(\cdot)}(z)}, \tag{77}$$

$$\tilde{\sigma}^2(u,v,z)=\sum_{k=1}^{s}\tilde{\sigma}^2_k(u,v,z)=$$

$$= \sum_{k=1}^{s} (\sigma_k^2(u,v) - \sigma_k^2(u,z) - \sigma_k^2(z,v) + \sigma_k^2(z,z)), \tag{78}$$

$$\tilde{\tilde{\sigma}}^2(z_1, z) = \sum_{k=1}^{s} \tilde{\tilde{\sigma}}_k^2(z_1, z) = \sigma^2(z_1, z_1) - 2\sigma^2(z_1, z) + \sigma^2(z, z), \tag{79}$$

$$\tilde{\sigma}^2_{(-k)}(u,v,z) = \sum_{j=1, j\neq k}^{s} (\sigma_j^2(u,v) - \sigma_j^2(u,z) - \sigma_j^2(z,v) + \sigma_j^2(z,z)) \tag{80}$$

and $\sigma^2(z, z')$, $\sigma_k^2(z, z')$ are defined by (51),(53).

Approximate $(1-\alpha)$-confidence interval for g has the form $\hat{g} \pm z_{1-\alpha/2}\sqrt{\hat{V}(\hat{g})/n}$ where $\hat{V}(\hat{g})$ is obtained replacing the unknown quantities $\Lambda^{(k)}(z)$, $\Lambda^{(\cdot)}(z)$ by $\hat{\Lambda}^{(k)}(z) = \Lambda^{(k)}(z, \hat{\gamma}_k)$, $\hat{\Lambda}^{(\cdot)}(z) = \Lambda^{(\cdot)}(z, \hat{\gamma}_k)$ respectively.

Proof.

From (49)-(51) it follows

$$\begin{aligned} \tilde{\sigma}^2(u,v,z) &= \sum_{k=1}^{s} \tilde{\sigma}_k^2(u,v,z) = \mathbf{cov}(W^{(\cdot)}(u) - W^{(\cdot)}(z),\ W^{(\cdot)}(v) \\ &- W^{(\cdot)}(z)) = \sum_{k=1}^{s} (\sigma_k^2(u,v) - \sigma_k^2(u,z) - \sigma_k^2(z,v) + \sigma_k^2(z,z)), \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{\sigma}}^2(z_1,z) &= \sum_{k=1}^{s} \tilde{\tilde{\sigma}}_k^2(z_1,z) = \sum_{k=1}^{s} \mathbf{cov}(W^{(k)}(z_1) - W^{(k)}(z),\ W^{(k)}(z_1) \\ &- W^{(k)}(z)) = \sigma^2(z_1,z_1) - 2\sigma^2(z_1,z) + \sigma^2(z,z), \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}^2_{(-k)}(u,v,z) &= \mathbf{cov}(W^{(-k)}(u) - W^{(-k)}(z),\ W^{(-k)}(v) - W^{(-k)}(z)) \\ &= \sum_{j=1, j\neq k}^{s} (\sigma_j^2(u,v) - \sigma_j^2(u,z) - \sigma_j^2(z,v) + \sigma_j^2(z,z)). \end{aligned}$$

Set $z_1 = z_0 \wedge \frac{z(t+\Delta)}{t}$.

1. Estimator $\hat{Q}^{(k)}$, $k = 1, \cdots, s$. Set $f(y) = \frac{t}{z} e^{-\frac{t}{z}(\Lambda^{(\cdot)}(y) - \Lambda^{(\cdot)}(z))}$. Then $\sqrt{n}(\hat{f}(y) - f(y)) \xrightarrow{d} -\left(\frac{t}{z}\right)^2 e^{-\frac{t}{z}(\Lambda^{(\cdot)}(y) - \Lambda^{(\cdot)}(z))}(W^{(\cdot)}(y) - W^{(\cdot)}(z))$.

Using this fact and (45) we get $\sqrt{n}(\hat{Q}^{(k)} - Q^{(k)}) \xrightarrow{d} \xi = \xi_1 + \xi_2 + \xi_3$ where

$$\xi_1 = -\left(\frac{t}{z}\right)^2 \int_z^{z_1} \varphi(y,t,z)(W^{(-k)}(y) - W^{(-k)}(z))d\Lambda^{(k)}(y)$$

$$\xi_2 = \frac{t}{z}\varphi(z_1,t,z)(W^{(k)}(z_1) - W^{(k)}(z))$$

$$\xi_3 = \left(\frac{t}{z}\right)^2 \int_z^{z_1} \varphi(y,t,z)(W^{(k)}(y) - W^{(k)}(z))d\Lambda^{(-k)}(y)$$

where $\Lambda^{(-k)}(y) = \sum_{j=1,j\neq k}^{s} \Lambda^{(j)}(y)$ and $W^{(-k)}(y) = \sum_{j=1,j\neq k}^{s} W^{(j)}(y)$. Since $W^{(-k)}$ and $W^{(k)}$ are independent we get $V(\hat{Q}^{(k)}) = \mathbf{Var}(\xi_1) + \mathbf{Var}(\xi_2) + \mathbf{Var}(\xi_3) + \mathbf{cov}(\xi_2, \xi_3)$ where

$$\mathbf{Var}(\xi_1) =$$

$$= \left(\frac{t}{z}\right)^4 \int_z^{z_1}\int_z^{z_1} \varphi(u,t,z)\varphi(v,t,z)\tilde{\sigma}^2_{-k}(u,v,z)d\Lambda^{(k)}(u)d\Lambda^{(k)}(v),$$

$$\mathbf{Var}(\xi_2) =$$

$$= \left(\frac{t}{z}\right)^4 \int_z^{z_1}\int_z^{z_1} \varphi(u,t,z)\varphi(v,t,z)\tilde{\sigma}^2_{k}(u,v,z)d\Lambda^{(-k)}(u)d\Lambda^{(-k)}(v),$$

$$\mathbf{Var}(\xi_3) = \left(\frac{t}{z}\right)^2 \varphi(z_1,t,z)\tilde{\tilde{\sigma}}^2_k(z_1,z), \quad (81)$$

$$\mathbf{cov}(\xi_2,\xi_3) = \left(\frac{t}{z}\right)^3 \int_z^{z_1} \varphi(u,t,z)\varphi(z_1,t,z)\tilde{\sigma}^2_k(u,z_1,z)d\Lambda^{(-k)}(u).$$

Asymptotic normality of other estimators is proved analogously.

5 Goodness-of-Fit Test

The purpose is to derive a goodness-of-fit test for the hypothesis

$$H_0: \lambda^{(k)}(z) = \lambda^{(k)}(z,\gamma_k),\ \gamma_k = (\gamma_{k1},\cdots,\gamma_{kq_k}),\ k = 1,\cdots,s. \quad (82)$$

The construction of the goodness-of-fit test is based on comparison of non-parametric and parametric estimators of traumatic event intensities. For $k = 1, \cdots, s$ set

$$B^{(k)}(z) = \hat{\Lambda}^{(k)}(z) - \Lambda^{*(k)}(z) = \int_0^z \frac{J(u)dN_n^{(k)}(u)}{Y_n(u)} - \int_0^z \lambda^{(k)}(u, \hat{\gamma}_k)du$$
$$= \int_0^z \frac{J(u)}{Y_n(u)} \left(dN_n^{(k)}(u) - Y_n(u)\lambda^{(k)}(u, \hat{\gamma}_k)du\right) \quad (83)$$

where $Y_n(u)$ is defined by (25) and $J(u) = I\left(Y_n(u) > 0\right)$. Suppose that Assumption A and (25) hold. Then from (83) by a Taylor series expansion around $\gamma_k^{(0)}$, $k = 1, \cdots, s$ we get

$$B^{(k)}(z) = \int_0^z \frac{J(u)}{Y_n(u)} dM_n^{(k)}(u, \gamma_k^{(0)}) + o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$- \sum_{j=1}^{q_k} \left(\left(\hat{\gamma}_{kj} - \gamma_{kj}^{(0)}\right) \int_0^z \frac{\partial}{\partial \gamma_{kj}} \lambda^{(k)}(u, \gamma_k^{(0)})du\right) \quad (84)$$

where $k = 1, \cdots, s$ and $M_n^{(k)}(u, \gamma_k^{(0)})$ is defined by (33). By Theorem 1 for $k = 1, \cdots, s$ we get

$$\sqrt{n}(\hat{\gamma}_k - \gamma_k^{(0)}) = \Sigma_k^{-1} \frac{1}{\sqrt{n}} \int_0^{z_0} X_k(u, \gamma_k^{(0)}) dM_n^{(k)}(u, \gamma_k^{(0)}) + o_p(1) \quad (85)$$

where Σ_k is defined in condition 2) of Theorem 1, $o_p(1) \xrightarrow{P} 0 \quad as$ $n \to \infty$ and

$$X_k(u) = X_k(u, \gamma_k^{(0)}) =$$
$$[= \left(\frac{\partial}{\partial \gamma_{k1}} \ln \lambda^{(k)}(u, \gamma_k^{(0)}), \cdots, \frac{\partial}{\partial \gamma_{kq_k}} \ln \lambda^{(k)}(u, \gamma_k^{(0)})\right)^T. \quad (86)$$

$$\text{Set} \quad \Psi^{(k)}(z) = \int_0^z \left(X_k(u, \gamma_k^{(0)})\right)^T \lambda^{(k)}(u, \gamma_k^{(0)})du. \quad (87)$$

Then (84) for $k = 1, \cdots, s$ may be rewritten as

$$\sqrt{n}B^{(k)}(z) = \sqrt{n} \int_0^z \frac{J(u)}{Y_n(u)} dM_n^{(k)}(u, \gamma_k^{(0)}) -$$
$$- \frac{1}{\sqrt{n}} \left(\Psi^{(k)}(z, \gamma_k^{(0)})\right)^T \Sigma_k^{-1} \int_0^{z_0} X_k(u, \gamma_k^{(0)}) dM_n^{(k)}(u, \gamma_k^{(0)}) + o_p(1). \quad (88)$$

Then for $k = 1, \cdots, s$

$$\begin{aligned}
\tilde{\tilde{B}} &= <\sqrt{n}B^{(k)}(z),\ \sqrt{n}B^{(k)}(t)>= n\int_0^{z\wedge t}\frac{J(u)}{Y_n(u)}\lambda^{(k)}(u,\gamma_k^{(0)})du \\
&- 2\left(\Psi^{(k)}(z)\right)^T\Sigma_k^{-1}\Psi^{(k)}(t) \\
&+ \frac{1}{n}\left(\Psi^{(k)}(z)\right)^T\Sigma_k^{-1}\zeta(z_0,\gamma_k^{(0)})\Sigma_k^{-1}\Psi^{(k)}(t)+o_p(1). \qquad (89)
\end{aligned}$$

where $\zeta(z_0,\gamma_k^{(0)}) = \int_0^{z_0} X_k(u)\left(X_k(u)\right)^T Y_n(u)\lambda^{(k)}(u,\gamma_k^{(0)})du$. By Rebolledo's limit theorem (see [1]) it follows that $\sqrt{n}B^{(k)}(z),\ 0\le z\le z_0$, $k=1,\cdots,s$ converges weakly to the process

$$W^{(k)}(z) = \int_0^z \frac{d\tilde{W}^{(k)}(u)}{b(u)} - \left(\Psi^{(k)}(z)\right)^T\Sigma_k^{-1}\int_0^{z_0} X_k(u,)d\tilde{W}^{(k)}(u) \qquad (90)$$

where $\tilde{W}^{(k)},\ k=1,\cdots,s$ are Gaussian martingales with $\tilde{W}^{(k)}(0)=0$ and $cov(\tilde{W}^{(k)}(z),\tilde{W}^{(k)}(t)) = \int_0^{z\wedge t}\lambda^{(k)}(u,\gamma_k^{(0)})b(u)du$ with $b(u)$ defined by (25). By (86) and (38) it follows

$$\Sigma_k = \int_0^{z_0} X_k(u,\gamma_k^{(0)})\left(X_k(u,\gamma_k^{(0)})\right)^T b(u)\lambda^{(k)}(u,\gamma_k^{(0)})du. \qquad (91)$$

Then for $k = 1, \cdots, s$

$$\begin{aligned}
<W^{(k)}(z),W^{(k)}(t))> &= \int_0^{z\wedge t}\frac{d\Lambda^{(k)}(u,\gamma_k^{(0)})}{b(u)} \\
&- \left(\Psi^{(k)}(t,\gamma_k^{(0)})\right)^T\Sigma_k^{-1}\Psi^{(k)}(z,\gamma_k^{(0)}). \qquad (92)
\end{aligned}$$

By (89) and (92) $as\ n\to\infty,\ k=1,\cdots,s$ we get

$$\begin{aligned}
\tilde{\tilde{B}} &= <\sqrt{n}B^{(k)}(z),\ \sqrt{n}B^{(k)}(t)>\xrightarrow{P}\xi^{(k)}(z,t) =<W^{(k)}(z),W^{(k)}(t))> \\
&= \int_0^{z\wedge t}\frac{d\Lambda^{(k)}(u,\gamma_k^{(0)})}{b(u)} - \left(\Psi^{(k)}(t,\gamma_k^{(0)})\right)^T\Sigma_k^{-1}\Psi^{(k)}(z,\gamma_k^{(0)}) \qquad (93)
\end{aligned}$$

We shall approximate $B^{(k)}(z)$ by a process $\hat{B}^{(k)}(z)$ which distribution can be easily generated through simulation:

$$\sqrt{n}\hat{B}^{(k)}(z) = \sqrt{n}\sum_{i=1}^n\int_0^z\frac{J(u)}{Y_n(u)}G_i^{(k)}dN_i^{(k)}(u) -$$

$$-\frac{1}{\sqrt{n}}\left(\Psi(z,\hat{\gamma}_k)\right)^T\hat{\Sigma}_k^{-1}\sum_{i=1}^{n}\int_0^{z_0}X_k(u,\hat{\gamma}_k)G_i^{(k)}dN_i^{(k)}(u) \tag{94}$$

where $\hat{\Sigma}_k=\frac{1}{n}I^{(k)}(z,\hat{\gamma}_k)$ with $I^{(k)}(z,\hat{\gamma}_k)$ defined by (36),

$$N_i^{(k)}(z)=\mathbf{1}_{\{Z_i\le z,V_i=k\}},\quad k=1,\cdots,s,\ 0\le z\le z_0, \tag{95}$$

and $\{G_i^{(k)},\ i=1,\cdots,n,\ k=1\cdots,s\}$ are independent standard normal variables which are independent of $N_i^{(k)}(.)$. When approximating the distribution of $\sqrt{n}B^{(k)}(.)$ we regard $\{G_i^{(k)},\ i=1,\cdots,n,\ k=1\cdots,s\}$ as random and $N_i^{(k)}(.)$ as fixed in (94). It will be shown later that $\sqrt{n}B^{(k)}(.)$ and $\sqrt{n}\hat{B}^{(k)}(.)$ have the same limiting distribution. The supremum test statistic is

$$Q^{(k)}=\sup_{z\in[0,z_0]}\mid\sqrt{n}B^{(k)}(z)\mid,\quad k=1,\cdots,s. \tag{96}$$

Let

$$\hat{Q}^{(k)}=\sup_{z\in[0,z_0]}\mid\sqrt{n}\hat{B}^{(k)}(z)\mid,\quad k=1,\cdots,s. \tag{97}$$

Suppose $q^{(k)}$ is the observed value of $Q^{(k)}$. Then the P-value, $P\{Q^{(k)}>q^{(k)}\}$ can be approximated by $P\{\hat{Q}^{(k)}>q^{(k)}\}$ through simulation.

Theorem 4. Let Assumption A and (25) hold. Then conditional distribution of the process $\hat{B}^{(k)}(\cdot)$ defined by (94) given $N_i^{(k)}(\cdot),\ k=1,\cdots,s,\ i=1,\cdots,n$ is the same in the limit as the unconditional distribution of $B^{(k)}(\cdot)$ defined by (88).

Proof.

Conditional on $N_i^{(k)}(z)$, $\hat{B}^{(k)}(z)$ is a sum of n independent zero-mean random variables for every z. Then the finite-dimensional distribution of $B^{(k)}(\cdot)$ is asymptotically zero-mean normal under Assumption A.

Define $\mathcal{F}_z^{(k)},\ k=1,\cdots,s$ be the σ-algebra generated by the random variables $N_i^{(k)}(u),\ i=1,\cdots,n,\ u\le z$. Let us write

$$\begin{aligned}\sqrt{n}\hat{B}^{(k)}(z) &= \hat{B}_1^{(k)}(z)-\hat{B}_2^{(k)}(z)=\sqrt{n}\sum_{i=1}^{n}\int_0^{z}\frac{J(u)}{Y_n(u)}G_i^{(k)}dN_i^{(k)}(u)\\ &- \frac{1}{\sqrt{n}}\left(\Psi(z,\hat{\gamma}_k)\right)^T\Sigma_k^{-1}\sum_{i=1}^{n}\int_0^{z_0}X_k(u,\hat{\gamma}_k)G_i^{(k)}dN_i^{(k)}(u).\end{aligned}$$

Let $0 \leq x \leq y \leq z \leq z_0$. Then $\mathbf{E}((\hat{B}_1^{(k)}(y) - \hat{B}_1^{(k)}(x))(\hat{B}_1^{(k)}(z) - \hat{B}_1^{(k)}(y)) \mid \mathcal{F}_z^{(k)}) = \sqrt{n}\sum_{i=1}^n \int_x^y \frac{J(u)}{Y_n(u)} dN_i^{(k)}(u) \int_y^z \frac{J(u)}{Y_n(u)} dN_i^{(k)}(u) = 0$.

The last equality is obtained using (95) (if $N_i^{(k)}$ has a jump in interval $(x, y]$ (or $(y, z]$) then in the interval $(y, z]$ (or $(x, y]$) it has no jump).

By (81),(95) and Assumption A we get $\mid \frac{\partial}{\partial \gamma_{kj}} \lambda^{(k)}(u, \hat{\gamma}_k) \mid \leq c_1$, $\mid \frac{\partial}{\partial \gamma_{kj}} \ln \lambda^{(k)}(u, \hat{\gamma}_k) \mid \leq c_2$, $k = 1, \cdots, s$, $j = 1, \cdots, q_k$ $N_i(z_0) \leq 1$, $i = 1, \cdots, n$. Then $\mathbf{E}((\hat{B}_2^{(k)}(y) - \hat{B}_2^{(k)}(x))(\hat{B}_2^{(k)}(z) - \hat{B}_1^{(k)}(y)) \mid \mathcal{F}_z^{(k)}) \leq c_1 \frac{1}{n} \sum_{i=1}^n C^T \hat{\Sigma}_k^{-1} \hat{\Sigma}_k^{-1} C (y-x)(z-y) \leq K(y-x)(z-y)$ where C$=\int_0^{z_0} X_k(u, \hat{\gamma}_k) dN_i^{(k)}(u)$ and $0 < K < \infty$ is some constant. Then the limiting process of $\sqrt{n}\hat{B}^{(k)}(z)$ $(k = 1, \cdots, s,\ 0 \leq z \leq z_0)$ for fixed $N_i^{(k)}(\cdot)$ is a zero-mean Gaussian process (see [5]).

It remains to show that covariances of this limiting process are the same as covariances of limiting process of $\sqrt{n}B^{(k)}(z)$, $k = 1, \cdots, s$, $0 \leq z \leq z_0$). By Lenglart's inequality we get

$\mathbf{E}(\hat{B}_1^{(k)}(y)\hat{B}_1^{(k)}(z) \mid \mathcal{F}_z^{(k)}) \overset{a.s.}{\to} \int_0^{z \wedge y} \frac{d\Lambda^{(k)}(u, \gamma_k^{(0)})}{b(u)}$, $\mathbf{E}(\hat{B}_2^{(k)}(y)\hat{B}_2^{(k)}(z) \mid \mathcal{F}_z^{(k)}) \overset{a.s.}{\to} \left(\Psi^{(k)}(y, \gamma_k^{(0)})\right)^T \Sigma_k^{-1} \Psi^{(k)}(z, \gamma_k^{(0)})$, Then

$$\mathbf{E}(\sqrt{n}\hat{B}^{(k)}(y)\hat{B}^{(k)}(z) \mid \mathcal{F}_z^{(k)}) \overset{a.s.}{\to} \xi^{(k)}(y, z)$$

where $\xi^{(k)}(y, z)$ $k = 1, \cdots, s$ is the limiting covariance of $\sqrt{n}B^{(k)}(\cdot)$ given by (93). The proof of the theorem is completed.

6 The Analysis of Real Data

Let us consider the failure time and wear data of 101 bus tires 01-73B manufaktured at the Omsk tire plant . The critical tire wear value is $z_0 = 15$ mm . The tires were used in the first quarter of 2000 year at the Tashkent bus park N7 on the buses DEU-BS-106 made in South Korea. Traumatic failures of seven types were observed, but taking into consideration small size of the data, traumatic failures were grouped only in two groups related with the protector and the side. The data are of the form (T_i, Z_i, V_i), $i = 1, \cdots, n$, $n = 101$. Analysis of the failure time and wear data by non-parametric methods shows (see [2]) that the intensities $\lambda^{(k)}(z)$ typically have one of the following forms: $(\alpha_k z)^{\nu_k}$, $\alpha_k(z-u_k)^{\nu_k}$ (production defects and defects caused by fatigue of tire components) or $\beta_k + \alpha_k z^{\nu_k}$ (failures caused by the mechanical damages).

6.1 The Estimators of Parameters and the Goodness-of-Fit Test

Model 1.

Let $\lambda^{(k)}(z,\gamma_k) = \lambda^{(k)}(z,\alpha_k,\nu_k) = (\alpha_k z)^{\nu_k}$, $n_k = \sum_{i=1}^{n} I\left(V_i = k\right)$, $k = 1,2$. From the tire data we get $n_1 = 31$, $n_2 = 22$. Set $\mu_k = \alpha_k^{\nu_k}$. Then $\lambda^{(k)}(z,\gamma_k) = \mu_k z^{\nu_k}$, $k = 1,2$. Then by (30) we get that estimators $\hat{\nu}_k$ and $\hat{\mu}_k$ verify the equations

$$\hat{\mu}_k = \frac{n_k(\hat{\nu}_k+1)}{\sum_{i=1}^{n} T_i Z_i^{\hat{\nu}_k}},$$

$$\frac{1}{n_k}\sum_{V_i=k} \ln Z_i - \frac{\sum_{i=1}^{n} T_i Z_i^{\hat{\nu}_k}\ln Z_i}{\sum_{i=1}^{n} T_i Z_i^{\hat{\nu}_k}} + \frac{1}{\hat{\nu}_k+1} = 0, \quad k = 1,2.$$

Then the estimators are $\hat{\alpha}_1 = 0.043$, $\hat{\nu}_1 = 6.883$, $\hat{\alpha}_2 = 0.05$, $\hat{\nu}_2 = 10.116$. From (21), (22), (83) we get that observed value of $Q^{(k)}$ with tire data is $q^{(1)} = 0.296564$, $q^{(2)} = 0.248509$.

Model 2.

Let

$$\lambda^{(k)}(z,\gamma_k) = \lambda^{(k)}(z,\alpha_k,\nu_k) = \begin{cases} (\alpha_k(z-z^*))^{\nu_k}, & \text{if} \quad z > z^*, \\ 0, & \text{if} \quad z \le z^*, \end{cases}$$

where $k = 1,2$, z^* is a known constant. Set $\mu_k = \alpha_k^{\nu_k}$. From tire data we get $z^* = 9.4$. Then by (30) we get that estimators are (taking $z^* =$ 9.4mm) $\hat{\alpha}_1 = 0.00692$, $\hat{\nu}_1 = 1.04423$, $\hat{\alpha}_2 = 0.05446$, $\hat{\nu}_2 = 2.57231$. From (21), (22), (83) we get that observed value of $Q^{(k)}$ with tire data is $q^{(1)} = 0.21522$, $q^{(2)} = 0.22707$.

There were $N = 10000$ realizations of the $\hat{Q}^{(k)}$, $k = 1,2$ process generated to calculate the P-value for the supremum test. The P-values $(\mathbf{P}(\hat{Q}^{(k)} > q))$ are:

$Model\ 1: \quad 0.0847 \quad if \quad k = 1 \quad and \quad 0.0911 \quad if \quad k = 2;$

$Model\ 2: \quad 0.2636 \quad if \quad k = 1 \quad and \quad 0.1252 \quad if \quad k = 2.$

The hypothesis H_0 is rejected if the simulated P-value is less than 0.05 (or 0.01). The results above show that the hypothesis H_0 is not rejected. Model 2 fits better than Model 1 but it must be noted that in reality the value z^* is unknown and must be estimated. A natural estimator for z^* is $\hat{z}^* = \min\{Z_1, \ldots, Z_n\}$. But in the case of the model with unknown z^* Assumptions A are not satisfied and we can not apply the proposed test.

6.2 The Estimators of Reliability Characteristics

Fig.1 and Fig.2 gives the graphs of non-parametric (solid line) and parametric (Model 1) - dotted line, Model 2) - dashed line) estimators of cumulative intensities $\Lambda^{(1)}(z)$ and $\Lambda^{(2)}(z)$ respectively.

Fig.3 and Fig.4 gives the graphs of parametric estimators of the intensities of the traumatic failures $\lambda^{(1)}(z)$ (solid line) and $\lambda^{(2)}(z)$ (dotted line) for Model 1 and Model 2 respectively.

Consider Model 2.

Fig.5 gives the graphs of semi-parametric estimator (solid line) and its approximate $(1-\alpha)$ confidence intervals with $\alpha = 0.05$ (dotted lines) of failure time survival function $S(t)$. The estimator of the mean run of tires is $\hat{e} = \hat{\mathbf{E}}T = 64.518$. The estimator of variance of $\hat{e}$ is $\hat{V}(\hat{e}) = 75.988$ and the $(1-\alpha)$ confidence interval for $\hat{e}$ with $\alpha = 0.05$ is [62.818; 66.218].

Fig.6 gives the graphs of semi-parametric estimator (solid line) of the probability of a traumatic failure $P^{(tr)}(t)$ and its approximate $(1-\alpha)$ confidence interval with $\alpha = 0.05$ (dotted lines). When all tires are failed we get $\hat{P}^{(tr)} = 0.531$, the variance of $\hat{P}^{(tr)}$ is $\hat{V}(\hat{P}^{(tr)}) = 0.003$ and approximate $(1-\alpha)$ confidence interval with $\alpha = 0.05$ of $\hat{P}^{(tr)}$ is [0.521; 0.541].

Fig.7 gives the graphs of semi-parametric estimator (solid line) of the probability of a natural failure $P^{(0)}(t)$ and its approximate $(1-\alpha)$ confidence interval with $\alpha = 0.05$ (dotted lines). When all tires are failed we get $\hat{P}^{(0)} = 1 - \hat{P}^{(tr)} = 0.469$, the variance of $\hat{P}^{(0)}$ is $\hat{V}(\hat{P}^{(0)}) = \hat{V}(\hat{P}^{(tr)}) = 0.003$ and approximate $(1-\alpha)$ confidence interval with $\alpha = 0.05$ of $P^{(0)}$ is [0.435; 0.503].

Fig.8 gives dynamics of the proportions $\hat{P}^{(k)}(t)$ of the tires having failures of 0, 1, 2 mode ($P^{(0)}(t)$ (solid line), $P^{(1)}(t)$ (dotted line) and $P^{(2)}(t)$ (dashed line)). When all tires are failed we get $\hat{P}^{(0)} = 0.469$, $\hat{P}^{(1)} = 0.309$, $\hat{P}^{(2)} = 0.223$.

7 References

1. Andersen, P.K., Borgan,O., Gill.R.D. & N. Keiding, N. (1993). Statistical models based on counting processes. *Springer,* New York.

2. Bagdonavičius, V., Bikelis,A., Kazakevičius,V. (2001). Statistical analysis of linear degradation and failure time data with multiple failure modes. *Lifetime Data Analysis* (to appear) .

3. Fleming, Th.R., Harrington, D.P. (1991). Counting processes and survival analysis, John Wiley & Sons, Inc., USA.

4. Lin,D.Y., Spiekerman,C.F., (1996). Model checking techniques for parametric regression with censored data, Scandinavian journal of statistics, Vol.23: 157-177.

5. Meeker, W.Q. and Escobar, L.A. (1998). Statistical methods for reliability data, Wiley: New York.

6. Shorack,G.R., Wellner, J.A. (1986). Empirical processes with applications to statistics. Wiley, New York.

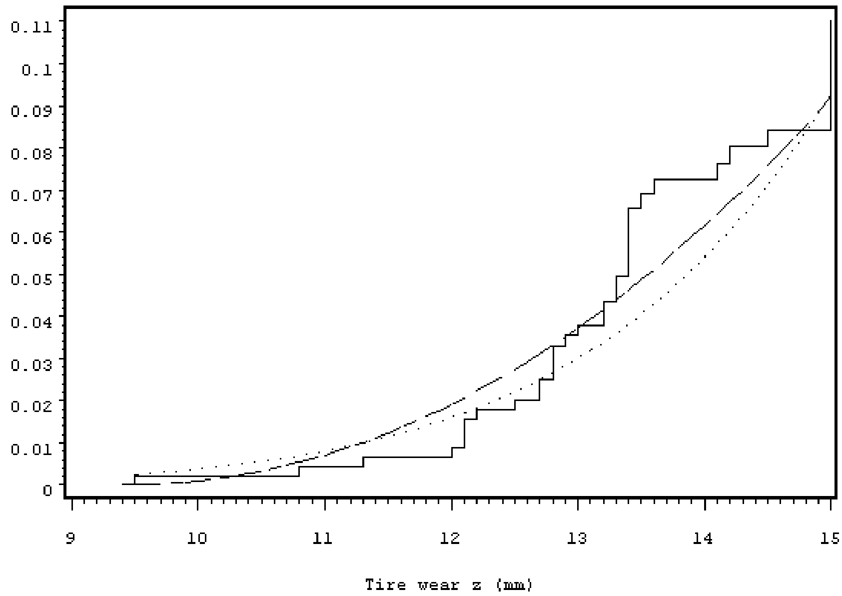

Figure 1: Nonparametric (solid line) and parametric (Model 1 - dotted line, Model 2 - dashed line estimators of $\Lambda^{(1)}(z)$)

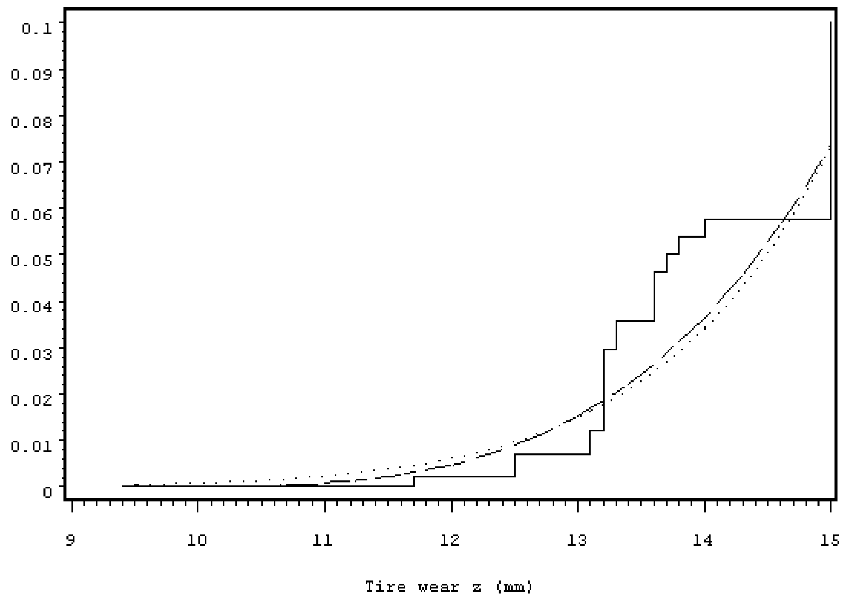

Figure 2: Nonparametric (solid line) and parametric (Model 1 - dotted line, Model 2 - dashed line estimators of $\Lambda^{(2)}(z)$)

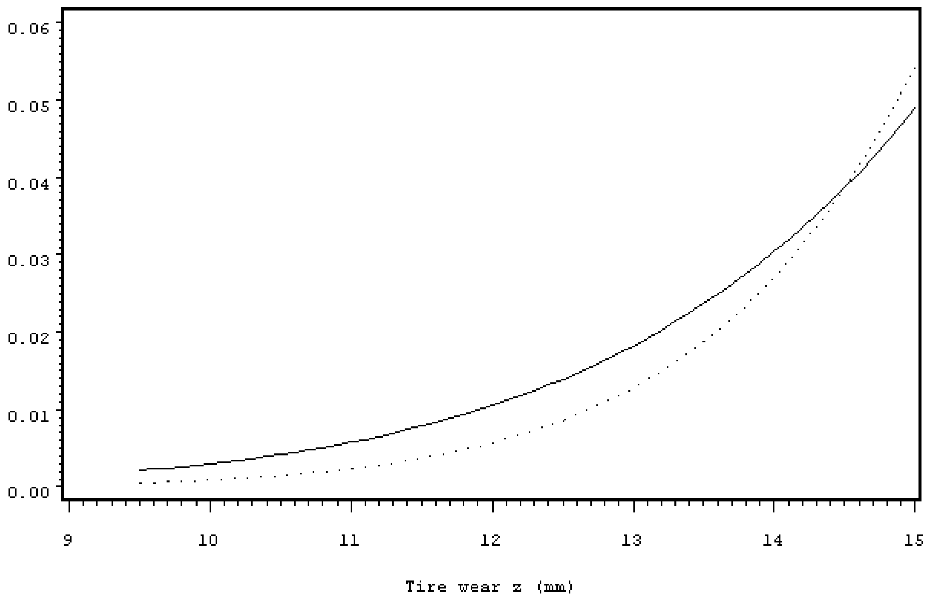

Figure 3: Parametric estimators of $\lambda^{(1)}(z)$ (solid line) and $\lambda^{(2)}(z)$ (dotted line) for Model 1

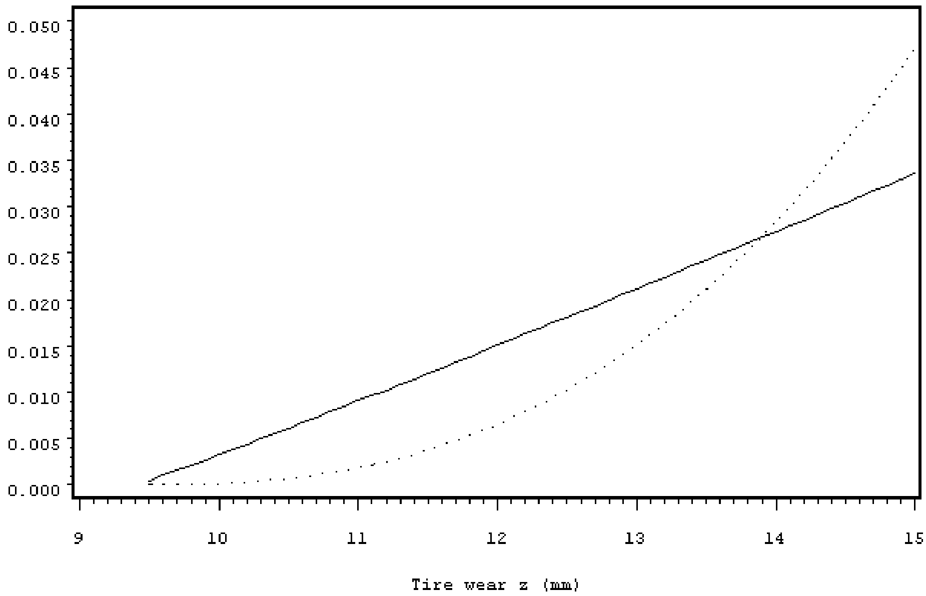

Figure 4: Parametric estimators of $\lambda^{(1)}(z)$ (solid line) and $\lambda^{(2)}(z)$ (dotted line) for Model 2

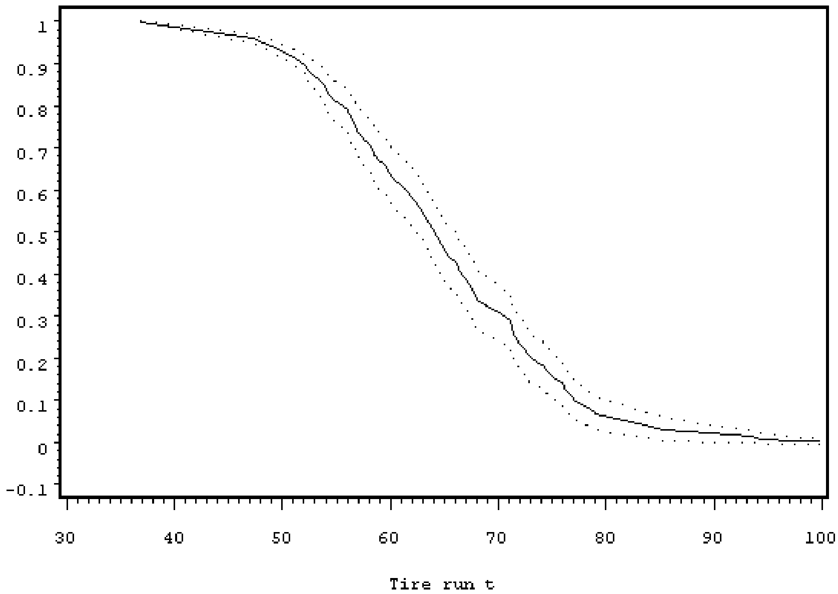

Figure 5: Semi-parametric estimator of $S(t)$ (solid line) and its 95 % confidence interval (dotted line)

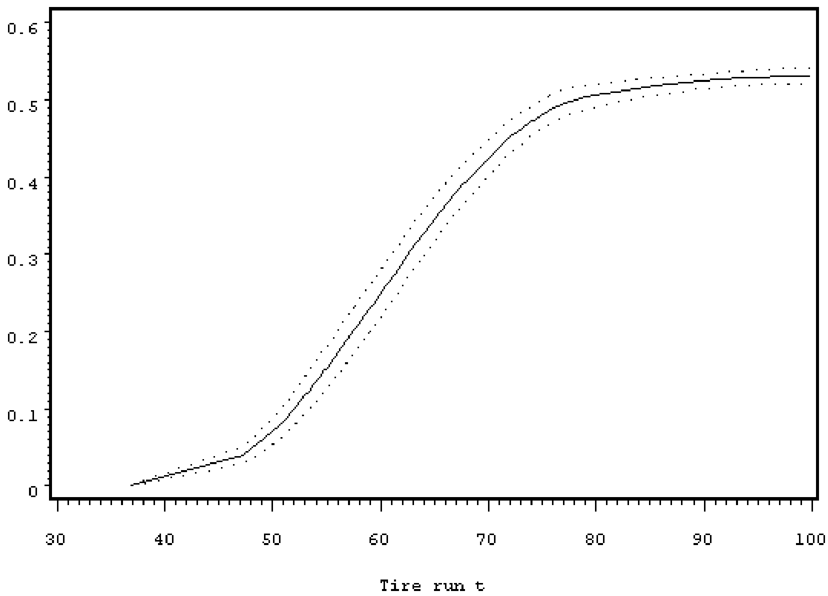

Figure 6: Semi-parametric estimator of $P^{(tr)}(t)$ (solid line) and its 95 % confidence interval (dotted line)

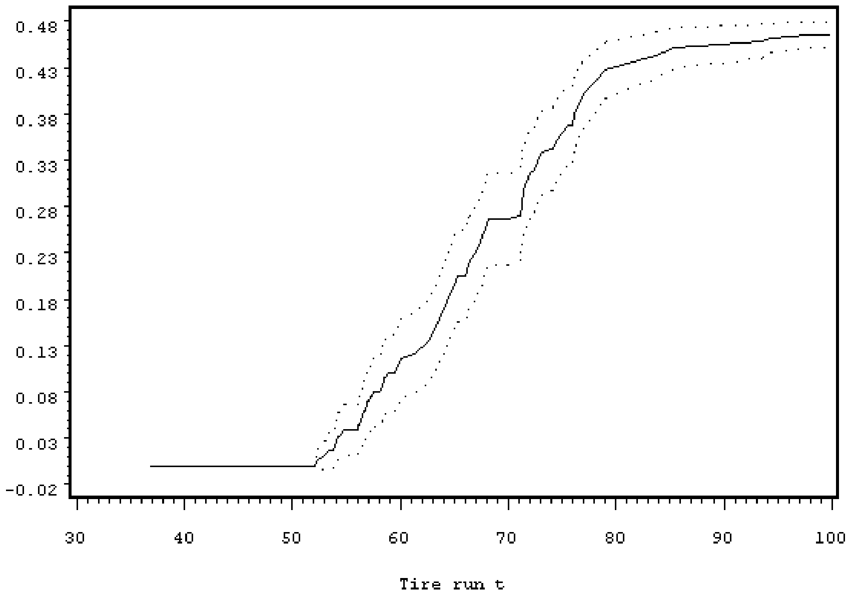

Figure 7: Semi-parametric estimator of $P^{(0)}(t)$ (solid line) and its 95 % confidence interval (dotted line)

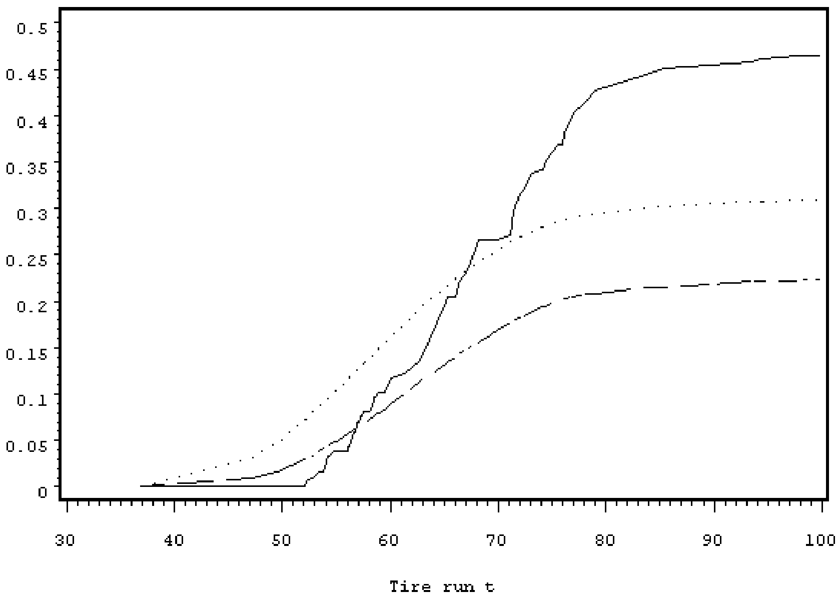

Figure 8: Dynamics of the proportions $\hat{P}^{(k)}(t)$ ($P^{(0)}(t)$ (solid line), $P^{(1)}(t)$ (dotted line), $P^{(2)}(t)$ (dashed line))