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On the Existence of Functions being Univalent in Half-plane together with their Derivatives

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Abstract

In articles [1]-[3] a class of functions being univalent in unit disc with all their derivatives was investigated. It was proved such functions exist and must be the entire functions of exponential type. The example of such a function is an exponential function $f(z) = e^z$. In this work the question on existence of functions, beeing univalent in half-plane with all their derivatives was raised and the negative answer to such question was given.

Keywords: class of univalent functions, unit disc, half-plane, existence.

1 Introduction

In the articles [1] – [3] a class of functions being univalent in a single circle E with a center at the origin of coordinates was considered. It was supposed every function being univalent in E has got univalent derivatives of any order. It turned out that such functions exist and they are the entire functions of exponential type. The example of such function is function $f(z) = e^z$. In this article we consider a class of functions being univalent in a half-plane $\Pi = \{\operatorname{Re} z > 0\}$. The interest for such functions is stipulated, for instance, by the relation of such functions to various tasks of continuum mechanics [4], [5]. It should be noted that in many cases there is no mere analogy in methods, when studying the features of a class of univalent functions in disk E and a class of functions being univalent in a half-plane. The example of this is a matter of existence of functions being univalent in half-plane together with all their derivatives. The main result of the article presented is the following theorem:

Theorem. *There is no function being univalent in half-plane, and having non-vanishing univalent derivative of n-th order in this half-plane, if* $n > 4$.

2 Univalent Functions

In order to prove this theorem we need several lemmas. Let denote by U a class of functions being analytical and univalent in half-plane $\Pi = \{\mathrm{Re}\, z > 0\}$ which are normalized by conditions $F(1) = 0$, $F'(1) = 1$, but by means of S a class of functions $g(\omega)$ being univalent in a unit disk $E = \{|z| < 1\}$ and normalized by conditions $g(0) = 0$, $g'(0) = 1$.

Lemma 1. *Let* $F(z) \in U$. *Then the following estimates take place:*

$$\frac{2|z^2 - 1|}{(|z+1| + |z-1|)^2} \le |F(z)| \le \frac{2|z^2 - 1|}{(|z+1| - |z-1|)^2}, \quad z \in \Pi \tag{1}$$

Proof. If $F(z) \in U$, then function

$$g(\omega) = \frac{1}{2} F\left(\frac{1+\omega}{1-\omega}\right) \in S,$$

i.e. is univalent in a single circle $E = \{|\omega| < 1\}$, and normalized by conditions $g(0) = 0$ and $g'(0) = 1$ ([6]). For any function of *S,* estimates of module take place

$$\frac{|\omega|}{(1+|\omega|)^2} \le |g(\omega)| \le \frac{|\omega|}{(1-|\omega|)^2}, \qquad \omega \in E,$$

[5, p. 53], thus we get estimates

$$\frac{2\left|\dfrac{z-1}{z+1}\right|}{\left(1+\left|\dfrac{z-1}{z+1}\right|\right)^2} \le |F(z)| \le \frac{2\left|\dfrac{z-1}{z+1}\right|}{\left(1-\left|\dfrac{z-1}{z+1}\right|\right)^2}, \qquad z \in \Pi$$

being equivalent to estimates (1).

To proof the following lemma we need a combinatorial identity

$$C_m^{k-1}(k-1+r) + C_m^k(k+m+r) = C_{m+1}^k(m+r), \tag{2}$$

where $C_i^j = \dfrac{i!}{j!(i-j)!}$ - binomial coefficients.

Then

$$C_m^{k-1}(k-1+r) + C_m^k(k+m+r) = \frac{m!(k+r-1)}{(k-1)!(m-k+1)!} + \frac{m!(m+r+k)}{k!(m-k)!} =$$

$$\frac{m!\left[(k+r-1)k + ((m+r)+k)((m+1)-k)\right]}{k!(m+1-k)!} = \frac{m!(m+r)(m+1)}{k!(m+1-k)!} = C_{m+1}^k(m+r)$$

Lemma 2. *If analytical functions $F(z)$ and $g(\omega)$ are connected by the equality*

$$F(z) = 2g\left(\frac{z-1}{z+1}\right),$$

in domains Π and E respectively, then their derivatives are connected by the following equality

$$\frac{F^{(n+1)}(z)}{(n+1)!} = \sum_{k=0}^{n} (-1)^{n+k} C_n^k \frac{g^{(k+1)}(\omega)}{(k+1)!} \frac{2^{k+2}}{(z+1)^{n+k+2}}, \tag{3}$$

where $\omega = \dfrac{z-1}{z+1}$, $z \in \Pi$, $\omega \in E$.

In the proof we procede by induction. For $n=0$ the formula (3) is justified. (essentially, $F'(z) = 2g'(\omega)\left(\dfrac{z-1}{z+1}\right)' = 2g'(\omega)\cdot\dfrac{2}{(z+1)^2}$). Let formula (3) be justified for $n=m$, i.e. the following formula takes place

$$\frac{F^{(m+1)}(z)}{(m+1)!} = \sum_{k=0}^{m} (-1)^{m+k} C_m^k \frac{g^{(k+1)}(\omega)}{(k+1)!} \frac{2^{k+2}}{(z+1)^{m+k+2}}. \tag{4}$$

We will take derivatives with respect to *z* from both parts of equality (4), and then we get

$$\frac{F^{(m+2)}(z)}{(m+1)!} = \sum_{k=0}^{m} (-1)^{m+k} C_m^k \frac{2^{k+2}}{(k+1)!}\left(\frac{2g^{(k+2)}(\omega)}{(z+1)^{m+k+2}} - \frac{g^{(k+1)}(\omega)(m+k+2)}{(z+1)^{m+k+3}}\right) =$$

$$\sum_{k=0}^{m} (-1)^{m+k} C_m^k \frac{2^{k+3}}{(k+1)!} \frac{g^{(k+2)}(\omega)}{(z+1)^{m+k+4}} - \sum_{k=0}^{m} (-1)^{m+k} C_m^k \frac{2^{k+2}}{(k+1)!} \frac{g^{(k+2)}(\omega)(m+k+2)}{(z+1)^{m+k+3}}$$

$$=$$

$$\sum_{k=1}^{m} (-1)^{m+k-1} C_m^{k-1} \frac{2^{k+2}}{k!} \frac{g^{(k+1)}(\omega)}{(z+1)^{m+k+3}} - \sum_{k=1}^{m} (-1)^{m+k} C_m^k \frac{2^{k+2}}{(k+1)!} \frac{g^{(k+1)}(\omega)(m+k+2)}{(z+1)^{m+k+3}} +$$

$$(-1)^{2m+k} C_m^m \frac{2^{m+3}}{(m+1)!} \frac{g^{(m+2)}(\omega)}{(z+1)^{2m+4}} - (-1)^m C_m^0 \frac{2^2}{1!} \frac{g^{(0)}(\omega)(m+2)}{(z+1)^{m+3}} =$$

$$\sum_{k=1}^{m} (-1)^{m+k+1} \frac{2^{k+2}}{(k+1)!} \frac{g^{(k+1)}(\omega)}{(z+1)^{m+k+3}} - \left(C_m^{k-1}(k+1) + C_m^k(m+k+2)\right) +$$

$$(-1)^{2m+k+2} C_{m+1}^{m+1} \frac{2^{m+3}}{(m+1)!} \frac{g^{(m+2)}(\omega)}{(z+1)^{2m+4}} + (-1)^{m+1} C_{m+1}^0 \frac{2^2}{1!} \frac{g^{(0)}(\omega)(m+2)}{(z+1)^{m+3}}.$$

By using combinatorial identity (2) for $r = 2$, i.e. that

$$C_m^{k-1}(k+1) + C_m^k(m+k+2) = C_{m+1}^k(m+2)$$

we get

$$\frac{F^{(m+2)}(z)}{(m+1)!} = \sum_{k=0}^{m+1} (-1)^{m+k+1} C_{m+1}^k \frac{g^{(k+1)}(\omega)}{(k+1)!} \frac{2^{k+2}(m+2)}{(z+1)^{m+k+3}}.$$

Now the formula (3) is obvious to be correct for $n = m+1$ too. This means the formula (3) is established for any $n = 1,2,...$ and the lemma is proved.

Lemma 3. *For any* $|z| > 1$ *provided that* $|\arg z| \le \beta = \frac{\pi}{2} - \varepsilon$, $\varepsilon > 0$, the following estimate takes place:

$$\frac{1}{|z+1| - |z-1|} < \frac{1}{\cos\beta}. \tag{5}$$

Proof. Let's denote $\alpha = \arg z$. Then

$$\frac{1}{|z+1| - |z-1|} = \frac{1}{\sqrt{1+|z|^2 + 2|z|\cos\alpha} - \sqrt{1+|z|^2 - 2|z|\cos\alpha}} =$$

$$\frac{\sqrt{\frac{1}{|z|^2} + 1 + \frac{2}{|z|}\cos\alpha} + \sqrt{\frac{1}{|z|^2} + 1 - \frac{2}{|z|}\cos\alpha}}{4\cos\alpha} < \frac{2\left(\frac{1}{|z|^2} + 1\right)}{4\cos\alpha} < \frac{1}{\cos\beta}$$

for any $|z| > 1$, $|\arg z| \le \beta = \frac{\pi}{2} - \varepsilon$, $\varepsilon > 0$.

Lemma 4. *If* $F(z)$ *being univalent function in* Π *half-plane, then*

$F^{(n)}(z) = O\left(|z|^{2-n}\right)$ *when* $|z| \to \infty$, *where* $|\arg z| \le \beta = \dfrac{\pi}{2} - \varepsilon$, $\varepsilon > 0$, $n = 0,1,2,...$

Evidently, it is enough to give a proof for functions of class U (i.e. univalent in Π and normalized by conditions $F(1) = 0$ and $F'(1) = 1$). For $n = 0$ a statement of lemma arises from the upper estimate of module function $F(z)$ (lemma 1) with account of lemma 3. If $F(z) \in U$, then

$$g(\omega) = \frac{1}{2} F\left(\frac{1+\omega}{1-\omega}\right) \in S$$

and for its derivatives the following estimates are justified [4]

$$\left|\frac{g^{(n)}(\omega)}{n!}\right| \le \frac{n + |\omega|}{(1 - |\omega|)^{n+2}}, \qquad \omega \in E.$$

With account of this estimate and formula (3) of lemma 2 we have

$$\left|\frac{F^{(n+1)}(z)}{(n+1)!}\right| = \left|\sum_{k=0}^{n} (-1)^{n+k} C_n^k \frac{g^{(k+1)}(\omega)}{(k+1)!} \frac{2^{k+2}}{(z+1)^{n+k+2}}\right| \le$$

$$\le \sum_{k=0}^{n} C_n^k \frac{k + 1 + |\omega|}{(1 - |\omega|)^{k+3}} \frac{2^{k+2}}{|z+1|^{n+k+2}} = \sum_{k=0}^{n} C_n^k \frac{k + 1 + \left|\dfrac{z-1}{z+1}\right|}{\left(1 - \left|\dfrac{z-1}{z+1}\right|\right)^{k+3}} \frac{2^{k+2}}{|z+1|^{n+k+2}} =$$

$$= \sum_{k=0}^{n} C_n^k \frac{k + 1 + \left|\dfrac{z-1}{z+1}\right|}{(|z+1| - |z-1|)^{k+3}} \frac{2^{k+2}}{|z+1|^{n-1}}.$$

Hence it appears the stated matter with account of lemma 3.

3 Proof of the main theorem

Let the function $F(z)$ is univalent in the half-plane Π. Then according to lemma 4

$F^{(n)}(z) = O(|z|^{2-n})$, when $|z| \to \infty$,

where $|\arg z| \le \beta = \frac{\pi}{2} - \varepsilon$, $\varepsilon > 0$, $n = 0,1,2,...$

From here

$$\left|F^{(n)}(z)\right| \le \frac{K}{|z|^{n-2}} \tag{6}$$

for some real K and $|z| > R$ for sufficiently big R. If $F^{(n)}(z)$ is univalent and non-vanishing in Π, then function $1/F^{(n)}(z)$ is univalent in Π too, and according to lemma 4 $1/F^{(n)}(z) = O(|z|^2)$. This means, that

$$\left|F^{(n)}(z)\right| \ge \frac{K_1}{|z|^2} \tag{7}$$

for some real K_1 and $|z| > R_1$ for sufficiently big R_1. Thus, if $n > 4$ for sufficiently big $|z|$ we get contradictory estimates (6) and (7), and that proves the theorem.

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