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Mathematical Models of Papermaking

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Abstract

The mathematical model of wood drying based on detailed transport phenomena considering both heat and moisture transfer have been offered in the article [1]. The adjustment of this model to the drying process of papermaking is carried out for the range of moisture content corresponding to the period of drying in which vapour movement and bound water diffusion in the web are possible. By averaging as the desired models are obtained sequence of the initial value problems for systems of two nonlinear first order ordinary differential equations.

Keywords: wood drying, mathematical models, heat and moisture transfer.

1 Introduction

The wet web from the press section is passed over a series of rotating steam–heated steel cylinders where water is evaporated and carried away by ventilation air. Paper drying can best be visualized as a repetitive two-phase process. In phase 1, the sheet picks up sensible heat while in contact with the steam cylinder. In phase 2, the sheet flashes off steam in the open draw between the top and bottom cylinders, thus causing the sheet to spontaneously cool and become ready to pick up sensible heat again.

The beginning into the drying section (**warm–up zone**) serve principally to raise the temperature of the paper sheet. Evaporation then quickly reaches a peak rate which is maintained as long as water is present on the fiber surfaces or within the large capillaries (**constant rate zone**). At the point where the remaining free moisture is concentrated in the smaller capillaries, the rate begins to decrease (**falling rate zone**). Finally, the residual rates within the sheet is more tightly held by physicochemical forces, and the evaporation rate is further reduced (**bound water zone**).

Let the moisture content M of web is defined by ratio

$$M = \frac{m_w}{m_f}. \tag{1}$$

Graphics of the function $M(t)$ during the drying process is convex in the narrow-band warm–up zone, linear in the constant rate zone, concave in the falling rate zone and very slowly diminishing in the bound water zone (see, [2]). On the other hand, the whole process of drying can be divided into three periods according to the moisture content level. The moisture content in paperbulk M is greater than the minimum value M_2 for liquid continuity in the first period. During this period **liquid movement and vapour flow occur**. Further, the moisture content M is less than the value M_2 but greater than the value M_1 of fiber saturation, in which period **only vapour movement is possible**. Finally, the moisture content M falls below the value M_1, when **vapour movement and bound water diffusion control thedrying**.

The mathematical model of paper drying for the range of moisture content $M \in [M_1, M_2)$ was outlined in our recent elaboration [3]. Now we are considering the more complicated case $M < M_1$. Simultaneously the boundary conditions for the governing differential equations having been given in this case in our opinion more adequately reflect reality.

Let us take into consideration that thetemperature depending value M_1 in the paper [4] isdefined by the following expression

$$M_1 = \frac{34.1 - 0.133(T - 273)}{100}.$$

2 Mathematical models

Since only bound water diffusion and vapour movement control the drying we have

$$j = j_b + j_v, \quad M < M_1,$$

$$j = j_v, \quad M_1 \le M < M_2.$$

For the moisture fluxes j_b, j_v in the paper [1] by virtue of Darcy'slaw are offered the expressions:

$$j_b = -D_b(1-\varepsilon)\frac{\partial \mu_b}{\partial x}, \tag{2}$$

$$j_v = -\frac{K_v \rho_v}{\mu_v}\frac{\partial p_v}{\partial x}. \tag{3}$$

According to paper [5] we can write

$$\frac{\partial \mu_b}{\partial x} = \xi \frac{\partial M}{\partial x} + \eta \frac{\partial T}{\partial x}, \tag{4}$$

where

$$h = \frac{p_v}{p_s}, \tag{5}$$

$$\xi = \frac{p_s}{\rho_v}\frac{\partial h}{\partial M}, \tag{6}$$

$$\eta = \frac{1}{\tilde{M}_v}\left\{\left[-187 - c_v \ln\left(\frac{T}{298.15}\right)\right] \times 10^{-3} + R\ln\left(\frac{p_v}{101325}\right) + \frac{RT}{p_v}\frac{\partial p_v}{\partial T}\right\} \tag{7}$$

Heat transfer and moisture mass balance equations have been derived within web (see, [1]). Since we are considering area $M < M_1$ these equations are respectively:

$$c_p \rho_w \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - H_{wv}\frac{\partial j_v}{\partial x}, \tag{8}$$

$$-\rho_w \frac{\partial M}{\partial t} = \frac{\partial (j_v + j_b)}{\partial x}. \tag{9}$$

Substituting (4) in (2) and after that (2) and(3) in the heat transfer and moisture balance equations (8) and(9) we obtain the following system of nonlinear differential equations:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(D_h \frac{\partial M}{\partial x} + E_h \frac{\partial T}{\partial x}\right), \tag{10}$$

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial x}\left(D_m \frac{\partial M}{\partial x} + E_m \frac{\partial T}{\partial x}\right), \tag{11}$$

where coefficients D_h, D_m, E_h, E_m are functions of the moisture content M and the temperature T, and are expressed in the following way:

$$D_h = \frac{H_{wv} K_v \rho_v}{c_p \rho_w \mu_v} p_s \frac{\partial h}{\partial M}, \tag{12}$$

$$D_m = \frac{c_p}{H_{wv}} D_h + \xi \frac{D_b(1-\varepsilon)}{\rho_w}, \tag{13}$$

$$E_h = \frac{1}{c_p \rho_w} \tilde{\lambda}, \tag{14}$$

$$E_m = \frac{c_p}{H_{wv}} \left(E_h - \frac{\lambda}{c_p \rho_w}\right) + \eta \frac{D_b(1-\varepsilon)}{\rho_w}, \tag{15}$$

and for the sake of convenience we introduce the designation

$$\tilde{\lambda} = \lambda + \frac{H_{wv} K_v \rho_v}{\mu_v} \left(h \frac{\partial p_s}{\partial T} + p_s \frac{\partial h}{\partial T}\right) \tag{16}$$

For the range of moisture content $M \in [M_1, M_2)$

$$j_b = 0, \quad h = 1, \quad \frac{\partial h}{\partial M} = 0,$$

therefore in this case we have

$$D_h = D_m = 0.$$

Let

$$\tau_i = \frac{d_i}{U_0}$$

is the duration of drying phase with number $i = 1, 2$; choose x–axis perpendicular to paper sheet and consider in time movement of fixed cross–section of paper sheet during two sequential drying phases.

Taking into account the convective heat and moisture transfer with external surroundings, we can define for $M < M_1$ in the concrete the boundary conditions:

$$\tilde{\lambda}\frac{\partial T}{\partial x} = \kappa(T - T_f), \quad \frac{\partial M}{\partial x} = 0, \quad x = 0, \tag{17}$$

$$\tilde{\lambda}\frac{\partial T}{\partial x} = \alpha_1(T_a - T), \quad D_m\frac{\partial M}{\partial x} + E_m\frac{\partial T}{\partial x} = \beta_1(M_a - M), \quad x = \delta, \tag{18}$$

(for the first phase of drying – $t \in [0, \tau_1]$);

$$\tilde{\lambda}\frac{\partial T}{\partial x} = \alpha_2(T - T_a), D_m\frac{\partial M}{\partial x} + E_m\frac{\partial T}{\partial x} = \beta_2(M - M_a), \quad x = 0, \tag{19}$$

$$\tilde{\lambda}\frac{\partial T}{\partial x} = \alpha_2(T_a - T), D_m\frac{\partial M}{\partial x} + E_m\frac{\partial T}{\partial x} = \beta_2(M_a - M), \quad x = \delta \tag{20}$$

(for the second phase of drying – $t \in [\tau_1, \tau_1 + \tau_2]$).

On the other hand, for $M \in [M_1, M_2)$ we must use only the first ones of the boundary conditions (17)–(20) like it was done in [3]. Let us introduce the averaged variables

$$m(t) = \frac{1}{\delta}\int_0^\delta M(t,x)\,dx, \quad \Theta(t) = \frac{1}{\delta}\int_0^\delta T(t,x)\,dx. \tag{21}$$

Using procedure of averaging (similarly as it was done in [10]) and considering the sequence of drying cylinders we obtain for $m < M_1$ systems of two first order nonlinear ordinary differential equations:

$$\frac{d\Theta}{dt} = \frac{1}{\delta}\left[\frac{D_h}{D_m}\beta_1(M_a - m) + \frac{\alpha_1}{c_p\rho_w}(1 - \frac{E_m D_h}{E_h D_m})(T_a - \Theta) - \frac{\kappa}{c_p\rho_w}(\Theta - T_f)\right],$$

$$\frac{dm}{dt} = \frac{1}{\delta}\left[\beta_1(M_a - m) - \frac{\kappa}{c_p\rho_w}\frac{E_m}{E_h}(\Theta - T_f)\right]$$

(for the first phase of drying – $t \in [n(\tau_1+\tau_2), n(\tau_1+\tau_2)+\tau_1]$, $n = 0, 1, 2, \ldots$);

$$\frac{d\Theta}{dt} = \frac{2}{\delta}\left[\frac{D_h}{D_m}\beta_2(M_a - m) + \frac{\alpha_2}{c_p\rho_w}(1 - \frac{E_m D_h}{E_h D_m})(T_a - \Theta)\right],$$

$$\frac{dm}{dt} = \frac{2\beta_2}{\delta} (M_a - m)$$

(for the second phase of drying – $t \in [n(\tau_1 + \tau_2) + \tau_1, (n + 1)(\tau_1 + \tau_2)]$, $n = 0, 1, 2, \ldots$).

For the range of moisture content $m \in [M_1, M_2)$ these equations using the modified boundary conditions receive shorter view:

$$\frac{d\Theta}{dt} = \frac{1}{\delta c_p \rho_w} \left[\alpha_1 (T_a - \Theta) - \kappa(\Theta - T_f)\right],$$

$$\frac{dm}{dt} = \frac{1}{\delta c_p \rho_w} \frac{E_m}{E_h} \left[\alpha_1 (T_a - \Theta) - \kappa(\Theta - T_f)\right]$$

(for the first phase of drying – $t \in [n(\tau_1+\tau_2), n(\tau_1+\tau_2)+\tau_1]$, $n = 0, 1, 2, \ldots$);

$$\frac{d\Theta}{dt} = \frac{2\alpha_2}{\delta c_p \rho_w} (T_a - \Theta),$$

$$\frac{dm}{dt} = \frac{2\alpha_2}{\delta c_p \rho_w} \frac{E_m}{E_h} (T_a - \Theta)$$

(for the second phase of drying – $t \in [n(\tau_1 + \tau_2) + \tau_1, (n + 1)(\tau_1 + \tau_2)]$, $n = 0, 1, 2, \ldots$).

So, the obtained mathematical models of paper drying process in papermaking consists of the sequence of initial value problems for the systems of two nonlinear first order ordinary differential equations. Moreover, these mathematical models are applicable for the rangeof values of moisture content $M < M_2$. Further, taking into account the behavior of the graphics $M(t)$ in the neighboring zones of drying (see, [2]), it is possible to continue the obtained curves.

3 Identification of the parameters

Following [6] we can use expressions

$$p_s = \frac{10133}{76} 10^{f(T)},$$

$$\frac{\partial p_s}{\partial T} = 307 \times 10^{f(T)} f'(T),$$

$$f(T) = 16.3737 - \frac{2818.6}{T} - 1.6908 lg(T) - 5.7546 \times 10^{-3} T + 4.00073 \times 10^{-6} T^2. \quad (22)$$

The thermal conductivity of web as it was done in [7] can be evaluated by expressions

$$\lambda = 4.19 \left[\left(\frac{1-\varepsilon}{\frac{2}{3}+M} \right) (5.18 + 9.6M) + 0.57\varepsilon \right] \times 10^{-2},$$

where the voidage of web ε delineated as

$$\varepsilon = 1 - \frac{\rho}{1000} \left(\frac{2}{3} + M \right)$$

and ρ according to classical assumptions is calculated following:

$$\rho = \frac{1 + 0.12K_\alpha}{1.12} \rho_w \frac{1+M}{1+K_\alpha M}.$$

The coefficient of fibers thickening K_α was chosen $K_\alpha = 0.6$. Furthermore, the specific heat of web c_p also following [**?**] can be evaluated by expression

$$c_p = (0.266 + 0.00116T + M) \times 4190.$$

The bound water diffusion coefficient D_b and the permeability to water vapour flow K_v in [1] were taken

$$D_b = 8.0 \times 10^{-13}, \quad K_v = 1.2 \times 10^{-15}.$$

It must be mentioned that K_v is one of most significant physical parameters for mathematical modelling and for the real paper processing we use$K_v < 1.2 \times 10^{-15}$. The density of water vapour ρ_v in external air having its relative humidity and temperature can be evaluated according to the universal gas law and expression (5)

$$\rho_v = \frac{p_v \tilde{M}_v}{RT} = \frac{h p_s \tilde{M}_v}{RT},$$

consequently

$$M_a = \frac{\rho_v}{\rho_a}.$$

For calculations of the partial vapour pressure p_v in the web is necessary to exploit the temperature dependent relation between the relative humidity h in the voidage of web and the corresponding moisture content M. To this end is possible to use isotherms offered in the classical work [8]:

$$h = \exp\left(K_2 K_1^{100M} - 0.0251\right), \tag{23}$$

with

$$K_1 = 1.0327 - 0.000674T, \quad K_2 = 17.884 - 0.1432T + 0.0002363T^2. \tag{24}$$

A number of later fitted expressions for these isotherms are describedand analyzed in the paper [9]. Unfortunately, when the relative humidity h tends to 1, then M in all offered expressions approaches infinity, in other words, the isotherm curve becomes asymptotic to the normal at point $h = 1$. The expression (23) allows also to obtain the derivatives $\frac{\partial h}{\partial T}, \quad \frac{\partial h}{\partial M}$. For evaluating of the coefficients of proportionality $\alpha_i, \quad i = 1, 2$; similarly that it was done in the article [10] we can use expressions

$$\alpha_i = 0.044 \left(\frac{d_i U_0}{\nu_a} \right)^{0.77} \frac{T}{T_a} \frac{k_a}{d_i} \frac{\tilde{\lambda}}{\lambda}, \quad i = 1, 2. \tag{25}$$

The relation between coefficients α_i, β_i is given in the article [11]

$$\beta_i = \frac{\alpha_i}{c_a \rho_a \rho_0} \frac{\tilde{M}_v p_s (1 + M) \rho_w}{RT\rho} \frac{\partial h}{\partial M}, \quad i = 1, 2. \tag{26}$$

Finally, the remaining parameters relating to physical properties of watervapour, water, air and drying cylinders steel is possible to find in-appropriate handbooks.

Notation

c_a – specific heat of air, $\frac{J}{kg\,deg}$,

c_p – specific heat of web, $\frac{J}{kg\,deg}$,

c_v – specific heat of water vapour, $\frac{J}{kg\,deg}$,

d_i – length of the drying phase with number $i = 1, 2; m$,

D_b – bound water diffusion coefficient, $\frac{kg\,s}{m^2}$,

D_h – heat transfer coefficient for moisture content gradient defined by expression (12),

D_m – moisture transfer coefficient for moisture content gradient defined by expression (13),

E_h – heat transfer coefficient for temperature gradient definedby expression (14),

E_m – moisture transfer coefficient for temperature gradient definedby expression (15),

f – function of temperature defined by expression (22),

h – relative humidity,

H_{wv} – heat of vaporization of water, $\frac{J}{kg}$

j – total moisture flux in the web, $\frac{kg}{m^2 s}$,

j_b – bound water flux in the web, $\frac{kg}{m^2 s}$,

j_v – water vapour flux in the web, $\frac{kg}{m^2 s}$,

k_a – thermal conductivity of air, $\frac{W\,deg}{m}$,

K_1, K_2 – temperature dependent parameters in the equation (23)defined by expressions (24),

K_v – web permeability to water vapour flow,m^2,

K_α – coefficient of web fibers thickening,

m – averaged moisture content in web defined by expressions (21),

m_f – mass of fibers in a given volume of web,kg,

m_w – mass of water (including watervapour) in a given volume of web, kg,

M – moisture content in the web defined by expression (1),$\frac{kg}{kg}$,

M_1 – fiber saturation point,$\frac{kg}{kg}$,

M_2 – minimum moisture content of wood for liquid continuity,$\frac{kg}{kg}$,

M_a – equilibrium moisture content in air with respect toits temperature and relative humidity,$\frac{kg}{kg}$,

$\tilde{M}_v = 18$ – molar weight of vapour,$\frac{kg}{kmol}$

p_s – water vapour pressure at saturation condition,$\frac{N}{m^2}$,

p_v – partial water vapour pressure, $\frac{N}{m^2}$,

$R = 8314$ – universal gas constant, $\frac{J}{kmol\,deg}$,

t – time, s,

T – temperature, K,

T_a – temperature of external air, K,

T_f – temperature of steam drying cylinder, K,

U_0 – velocity of paper sheet pulling, $\frac{m}{s}$,

x – one-dimensional spatial coordinate, m,

α_1, α_2 – coefficients of proportionality in the boundary conditions(18), (19), (20) defined by expressions (25),

β_1, β_2 – coefficients of proportionality in the boundary conditions(18), (19), (20) defined by expressions (26),

δ – thickness of paper sheet,m,

ε – voidage of web, $\frac{m^3}{m^3}$,

η – function of moisture content and temperature defined by expression(7),

Θ – averaged temperature in web defined by expressions (21),

κ – coefficient of proportionality in the boundary condition (17)

λ – thermal conductivity of web, $\frac{J}{m\, deg\, s}$,

$\tilde{\lambda}$ – modified coefficient of thermal conductivity of web defined by expression (16),

μ_b – chemical potential of bound water, $\frac{J}{kg}$,

μ_v – water vapour viscosity, $\frac{N\, s}{m^2}$,

ν_a – kinematic viscosity of air, $\frac{m^2}{s}$,

ρ – density of wet web, $\frac{kg}{m^3}$,

ρ_a – density of air, $\frac{kg}{m^3}$, ρ_0 – density of water, $\frac{kg}{m^3}$,

ρ_v – density of water vapour, $\frac{kg}{m^3}$,

ρ_w – basic density (see, [12]), $\frac{kg}{m^3}$,

τ_i – duration of the drying phase with number $i = 1, 2; s$,

ξ – function of moisture content and temperature defined by expression (6).

4 References

1. S. Pang. "Relationship between a diffusion model and a transport model for soft wood drying. *Wood and Fiber Science.* **29**, (1), 1997, 58 – 67

2. G. A. Smook. *Handbook for Pulp and Paper Technologists.* Angus Wilde Publications, Vancouver.1992

3. J. Cepītis. "A certain mathematical model of paper drying." *Mathematical Modelling and Analysis.* Technika, Vilnius. **5**, 2000, 26 – 31

4. G. Bramhall. "Mathematical model for lumber drying. Part I. Principles involved." *Wood Science.* **12**, (1), 1979, 14 – 21

5. M. A. Stanish. "The roles of bound water chemical potential and gas phase diffusion in moisture transport through wood." *Wood Sci.Technol.* **20**, 1986, 53 – 70

6. F. Kayihan. "Simultaneous heat and mass transfer with local three-phase equilibria in wood drying." *Proc. 3rd International Drying Symposium.* **1**, 1982, 123 – 134

7. J. F. Siau. *Transport Processes in Wood.* Springer-Verlag, New York. 1984

8. R. S. Bradley. "Polymolecular adsorbed films. II. The general theory of the condensation of vapors on finely divided solids." *J.Che.Soc.* 1936, 1799–1804

9. S. Avramidis. "Evaluation of "three-variable" models for the prediction of equilibrium moisture content in wood." *Wood Sci.Technol.* **23**, 1989, 251–258

10. A. Buikis, A. D. Fitt. "A mathematical model for the heat treatment of glass fabric sheets." *IMA Journal of Mathematics Applied in Business and Industry.* **9**, 1998, 1 – 32

11. S. Avramidis, G. Hatzikiriakos. "Convective heat and mass transfer in nonisothermal moisture desorption." *Holzforschung.* **49**, (2), 1995, 163 – 167

12. A. J. Stamm. *Wood and Cellulose Science.* Ronald Press, New York. 1964