Nonlinear Analysis: Modelling and Control, 2001, v. 6, No. 1, 57-65

Representation Formula for Solutions of Eikonal Type Equations

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Received: 14.02.2001

Accepted: 28.03.2001

Abstract

Equations of an eikonal type ones arise in many areas of applications, including optics, fluid mechanics, material sciences, and control theory. This article investigates the representation formula for semiconcave solutions of the boundary problem for eikonal type equation. The formula is given by forming envelope of some fundamental solutions of the equation. Using these fundamental solutions we obtain the representation formula for solutions of Cauchy problem for appropriate Hamilton – Jacobi equation.

Key words: Eikonal, Hamilton – Jacobi, semiconcave solution.

1 Introduction

In this paper we are concerned with the boundary problem

$$H(x,u,u_x) = 0, x \in \Omega, \tag{1}$$

$$u|_{\partial\Omega} = \varphi, \tag{2}$$

where $\Omega \subset R^n, H, \varphi$ are given. In the view of recent works this problem arise in the theory of propagating interfaces with occur in wide variety of applications and include fluid mechanics, material science, computer vision, burning flames, grid generation, optimization and control [8]. The same problems arise in optics too. For example, the eikonal equation

$$\sum_{i=1}^{3} u_{x_i}^2 = n^2(x) \tag{3}$$

describes the propagation of light. We investigate more general equation

$$(A(x)u_x, u_x) = 1 \tag{4}$$

with boundary condition (2). We are concerned with generalized solutions of (1), (2), because the classical solutions exist in general only in some neighborhood of the boundary $\partial\Omega$. We give a definition of the generalized solution of (1), (2) in the sense of work [4].

Definition. The Lipschits continuous function $u(x)$ in $\overline{\Omega}$ is called the generalized (semiconcave) solution of (1), (2) if $u(x)$solves (1) a.e. on Ω, satisfies (2) and the inequality

$$u(x+l) - 2u(x) + u(x-l) \leq C_\delta |l|^2,$$

holds for all points $x, x+l, x-l \in \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) \geq \delta\}$

In the case, when domain Ω is unbounded, we additionally require

$$\lim_{\substack{|x| \to +\infty \\ x \in \Omega}} u(x) = +\infty.$$

One of essential properties of equation (1) is that envelope of family solutions (1) is also the solution of (1). In order to give the solution of problem (1), (2) we define the family of fundamental solutions of (1) and then the solution of problem (1), (2) is presented as envelope of this family.

This article investigates the representation formula for semiconcave solutions of problem (4), (2). We begin our analysis by describing the fundamental solutions of (4). Next, by forming envelopes we give certain representation formula for semiconcave solution of (4), (2). Finally, we state the connection between the fundamental solutions of (4) and the appropriate Hamilton – Jacobi equation

$$u_t + (A(x)u_x, u_x) = 0 \tag{5}$$

The representation formulas for solutions of Cauchy problem for Hamilton – Jacobi equations are given in [1] – [7].

2 The fundamental solutions of (4)

Let $x \in \Omega$ and the matrix $A(x) \in C^{2,\alpha}(\Omega)$ is symmetric. Assume that uniformly on Ω

$$a_1|\xi|^2 \le (A(x)\xi,\xi) \le a_2|\xi|^2 \quad (6)$$

for any $\xi \in R^n$, where $a_1, a_2 \in R, a_1 \succ 0.$

Consider the generalized solution $u^r(x,\xi)$ of (4) on $R^n \setminus B_r(\xi), B_r(\xi) = \{x \in R^n : |x-\xi| \prec r\}$, satisfying boundary conditions

$$u^r \,|_{S_r(\xi)} = 0, \quad (7)$$

$$\lim_{|x| \to +\infty} u^r(x,\xi) = +\infty \quad (8)$$

where $S_r(\xi) = \{x \in R^n : |x-\xi| = r\}$ Fix any $\xi \in R^n$. Define the function

$$G(x,\xi) = \begin{cases} \lim\limits_{\substack{r \to 0 \\ r \prec r_0}} u^r(x,\xi), when |x-\xi| \succ r_0, \forall r_0 \succ 0 \\ 0, when x = \xi, \end{cases} \quad (9)$$

where $\xi \in R^n$.

Theorem 1. The function $G(x,\xi)$ is semiconcave solution of (4) on $R^n \setminus \{0\}$ such that

$$\frac{1}{\sqrt{a_2}}|x-\xi| \le G(x,\xi) \le \frac{1}{\sqrt{a_1}}|x-\xi| \quad (10)$$

for every $x,\xi \in R^n$.

The proof of this theorem based on results of work [4]. In view of the theorem 5.1 in [4], and the properties of $A(x)$ there exists the Lipschits continuous function $u(x)$ in $\overline{\Omega}$ (Ω is unbounded domain) witch solves (4) a.e. and satisfies

$$v \,|_{\partial\Omega} = \chi(x),$$

$$\lim_{|x| \to +\infty} v(x) = -\infty$$

$$v(x+l) - 2v(x) + v(x-l) \ge -C_\delta, \text{ for any } x, x+l, x-l \in \Omega_\delta,$$

if and only if there exists the Lipschits continuous continuation of the function $\chi(x)$ on $\Omega - \varphi(x)$ so that it satisfies on Ω a.e.

$$(A(x)\varphi_x, \varphi_x) \le 1,$$

and

$$\lim_{|x| \to +\infty} \varphi(x) = -\infty$$

The prove of theorem 5.1 also implies

$$v(x) \le \varphi(x)$$

and the constant C_δ depends only on $A(x)$ locally in Ω. Let $\Omega = R^n \setminus B_r(\xi)$ and $\chi(x) = 0$

From (6) we deduce that the function

$$\varphi = -\frac{1}{\sqrt{a_2}}|x-\xi| + \frac{r}{\sqrt{a_2}}|x-\xi|$$

is the continuation of the function $\chi(x) = 0$ on $S_r(\xi)$ into $R^n \setminus B_r(\xi)$ and

$$(A(x)\varphi_x, \varphi_x) \le a_2|\varphi_x|^2 = 1.$$

Consequently, there exists the function $v_r(x,\xi)$ witch solves (4) a.e. in $R^n \setminus B_r(\xi)$ and satisfies

$$v_r\,|_{S_{r(\xi)}} = 0,$$

$$\lim_{|x|\to+\infty} v_r(x,\xi) = -\infty,$$

$$v_r(x+l,\xi) - 2v_r(x,\xi) + v_r(x-l,\xi) \ge -C_\delta(N)|l|^2,$$

where $C_\delta(N) \ge 0$ depends only on values of $A(x)$ in $\Omega(\delta,N) = \{x : r+\delta \le |x-\xi| \le N\}$, for any $x, x+l, x-l \in \Omega(\delta,N)$. Additionally on $R^n \setminus B_r(\xi)$ we have

$$v_r(x,\xi) \le -\frac{1}{\sqrt{a_2}}|x-\xi| + \frac{r}{\sqrt{a_2}}$$

It is obviously that $u^r(x,\xi) = -v_r(x,\xi)$ is a semiconcave solution of problem (4), (7), (8) and

$$u^r(x,\xi) \ge \frac{1}{\sqrt{a_2}}|x-\xi| - \frac{r}{\sqrt{a_2}} \tag{11}$$

$$u^r(x+l,\xi) - 2u^r(x,\xi) + u^r(x-l,\xi) \le C_\delta(N)|l|^2, \tag{12}$$

Now we prove

$$u^r(x,\xi) \le \frac{1}{\sqrt{a_1}}|x-\xi| - \frac{r}{\sqrt{a_1}} \tag{13}$$

when $x \in R^n \setminus B_r(\xi)$. On the other hand, (6) implies

$$1 = (A(x)v_{rx}, v_{rx}) \ge a_1|v_{rx}|^2.$$

Since the function

$$-\frac{1}{\sqrt{a_1}}|x-\xi|+\frac{r}{\sqrt{a_1}}$$

is the solution of problem

$$a_1|u_x|^2=1,$$
$$u\,|_{S_{r(\xi)}}=0,$$

in $R^n \setminus B_r(\xi)$, then owing to the remark from theorem 5.1, as above, we prove the inequality

$$-\frac{1}{\sqrt{a_1}}|x-\xi|+\frac{r}{\sqrt{a_1}}\le v_r(x,\xi)$$

witch implies (13).

Next, we show that the fundament solution $G(x,\xi)$ is bounded on R^n for any fixed $\xi\in R^n$. Let $r_1 \prec r_2$, then (11) implies

$$u^{r_1}\,|_{S_{r_2}} \succ u^{r_2}\,|_{S_{r_2}}.$$

From theorem 4.2 in [4] we can deduce

$$u^{r_1}(x,\xi)\ge u^{r_2}(x,\xi) \qquad (14)$$

on $R^n \setminus B_r(\xi)$, for $r_1, r_2 \prec r_0$. So $\{u^r\}$ is the monotonically increasing sequence and (13) implies the existence of $\lim\limits_{r\to 0} u^r(x,\xi)$, when $|x-\xi|\succ r_0$, for any $r_0 \succ 0$.

Prove now the function $G(x,\xi)$ is semoconcave solution of (4) on $R^n \setminus \{0\}$ satisfying (10). Let $r\to 0$, then inequalities (11), (12) implies (10). Using the equation (4) and (6) we compute

$$|u_x^r(x,\xi)|\le \frac{1}{\sqrt{a_1}}=L,$$

a.e. in $R^n \setminus B_{r_0}(\xi)$, $r \prec r_0$, and consequently

$$|\,u^r(x+l,\xi)-u^r(x,\xi)|\le L|l|,$$

for any $x, x+l \in R^n \setminus B_{r_0}(\xi)$. So, when $r\to 0$, from (10) we have Lipschits continuity of $G(x,\xi)$ with the same constant L on R^n. Next, according to (12) we obtain the estimate

$$u^r(x+l,\xi)-2u^r(x,\xi)+u^r(x-l,\xi)\le C_\delta(N)|l|^2$$

on $B_N(\xi)\setminus B_{r_0}(\xi)$, when $r+\delta\le r_0$, and thus, when $r\to 0$,

$$G(x+l,\xi)-2G(x,\xi)+G(x-l,\xi)\le C_\delta(N)|l|^2 \qquad (15)$$

on $B_N(\xi)\setminus B_\delta(\xi)$. Finally, we prove that the function $G(x,\xi)$ solves (4) a.e. in R^n. From the Lemma 3.1, Theorem 5.1 in [4] it follows: if the family $\{u^\mu(x)\}$ of solutions of (1) on Ω is satisfying

$$|u^\mu(x+l,\xi)-u^\mu(x,\xi)|\le L|l|,\ \ x,x+l\in\overline{\Omega},$$

$$u^\mu(x+l,\xi)-2u^\mu(x,\xi)+u^\mu(x-l,\xi)\ge -C_\delta(N)|l|^2,$$

$$x,x+l,x-l\in\Omega_\delta\cap B_N(0),$$

where , L C_δ does not depend on μ, then there exists the sequence $\{u^\mu(x)\}$ coming uniformly to solution $u(x)$ of (1) witch is satisfying the same inequalities. Obviously the family $\{-u^r(x)\}$satisfies on $B_N(\xi)\setminus B_{r_0}(\xi)$ $r\le r_0$, the conditions specified above, thus (14) implies the $G(x,\xi)$ a.e. solves (4).

Remark. The function $G(x,\xi)$ is continuous with respect to ξ on R^n.

Fix $x_0\in R^n,\forall\varepsilon\succ 0$. The Dini Theorem ensures the uniformly convergence

$$\{u^r(x_0,\xi)\}\to G(x_0,\xi)$$

on compact sets $K\subset R^n$. Thus we can find $\delta_0\succ 0$ such that

$$G(x_0,\xi)-u^r(x_0,\xi)\prec\varepsilon,$$

when $r\prec\delta_0,\xi\in K$. Suppose $\xi_1,\xi_2\in K,$ $|\xi_1-\xi_2|=r_0\prec\delta_0,$ and $\xi_0=\dfrac{\xi_1-\xi_2}{2}$. Then

$$B_{\frac{r_0}{2}}(\xi_0)\subset B_{r_0}(\xi_1)\cap B_{r_0}(\xi_2).$$

From definition of $u^r(x,\xi)$ and (13) we obtain

$$u^{\frac{r_0}{2}}(x,\xi_0)-u^{r_0}(x,\xi_1)=u^{\frac{r_0}{2}}(x,\xi_0)\le\frac{2r_0}{\sqrt{a_1}}$$

when $x\in S_{r_0}(\xi_1)$. Therefore the Theorem 2.2 in [4] ensures that

$$|u^{\frac{r_0}{2}}(x_0,\xi_0)-u^{r_0}(x_0,\xi_1)|\le\frac{2r_0}{\sqrt{a_1}}.$$

Using a similar arguments we have the estimate

$$|u^{\frac{r_0}{2}}(x_0,\xi_0)-u^{r_0}(x_0,\xi_2)|\leq \frac{2r_0}{\sqrt{a_1}}.$$

Thus

$$\left|G(x_0,\xi_1)-G(x_0,\xi_2)\right|\leq G(x_0,\xi_1)-u^{r_0}(x_0,\xi_1)+|u^{\frac{r_0}{2}}(x_0,\xi_0)-u^{r_0}(x_0,\xi_1)|+$$
$$|u^{\frac{r_0}{2}}(x_0,\xi_0)-u^{r_0}(x_0,\xi_2)|+G(x_0,\xi_1)-u^{r_0}(x_0,\xi_1)\leq 2\varepsilon+\frac{4\delta_0}{\sqrt{a_1}}$$

,

wherefrom follows the continuity of $G(x,\xi)$ whit respect to ξ on R^n.

3 Representation formula for solutions of problem (4), (2).

Now we use the fundamental solutions to derive certain global representation formula for semiconcave solutions of (4), (2).

Theorem2. Let the matrix $A(x)$ satisfy the above conditions and Ω is bounded with smooth boundary δΩ. Suppose also that

$$\left|\varphi(x)-\varphi(y)\right|\leq \frac{1}{\sqrt{a_2}}\left|x-y\right|, \tag{16}$$

for any $x,y\in\partial\Omega$. Then there exists the unique solution of (4), (2) which can be represented by formula

$$u(x)=\min_{\xi\in\partial\Omega}\{\varphi(\xi)+G(x,\xi)\}, \tag{17}$$

The remark implies the function $\varphi(\xi)+G(x,\xi)$ is continuous on R^{2n}. Since δΩ is compact the function is bounded on R^n. We prove that $u(x)$ satisfies (2). Assume that $x_0\in\partial\Omega$, and ξ_0 gives the minimum in formula (17). Then (10) implies

$$u(x_0)=\varphi(\xi_0)+G(x_0,\xi_0)\leq\varphi(x_0),$$

that is

$$G(x_0,\xi_0)\leq\varphi(x_0)-\varphi(\xi_0).$$

According this inequality and (10), (16) we deduce

$$\frac{1}{\sqrt{a_1}}\left|x_0-\xi_0\right|\leq\frac{1}{\sqrt{a_2}}\left|x_0-\xi_0\right|.$$

Thus $x_0=\xi_0$ and $u(x_0)=\varphi(x_0)$ holds. The other properties of general solution can be proved in a similar way as in Lemma 1 in [2].

Next, we demonstrate how to use fundamental solution of (4) in odder to define fundamental solution of appropriate Hamilton-Jacobi equation (5) and how to build the generalized solution $u(t,x)$ of (5) on $S=\{(0,+\infty)\times R^n\}$satisfying the initial condition

$$u(0,x)=\varphi(x) \qquad (18)$$

Definition. The locally Lipschits continuous function $u(t,x)$ on S is called the generalized (semiconcave) solution of (5), (18) if $u(t,x)$ solves (5) a.e. in S and satisfies:

$$u(t,x+l)-2u(t,x)+u(t,x-l)\le C_\delta |l|^2$$

for any $(t,x+l),(t,x),(t,x-l)\in S_\delta$, where $S_\delta=\{(\delta,+\infty)\times R^n\}$,

$$\lim_{t\to 0} u(t,x)=\varphi(x), \forall x\in R^n .$$

Theorem 3. Let the matrix $A(x)$ satisfy the above conditions and $\varphi(x)$ is bounded and lower semicontinous on R^n. Then there exists the unique semiconcave solution of (5), (18) on S, which can represented by formula

$$u(t,x)=\min_{\xi\in R^N}\left\{\varphi(\xi)+\frac{G^2(x,\xi)}{4t}\right\}.$$

The scheme of the proof is similar as applied to Theorem 1.2 in [2].

Remark. Thus the fundamental solution of (5) is the function $\frac{G^2(x,\xi)}{4t}$. The fundamental solutions for the more general Hamilton-Jacobi equations in the another way are defined in [2].

4 References

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