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NONUNIFORM ESTIMATE OF MAXIMUM DENSITY CONVERGENCE RATE FOR INDEPENDENT RANDOM VARIABLES

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Abstract

The nonuniform estimate of convergence rate in the maximum density limit theorem of independent nonidentically distributed random variables is obtained. This result is generalization of the work presented in [1].

INTRODUCTION

Let $\{X_j, \ j \geq 1\}$ be a sequence of independent random variables (r.v.`s) with distribution functions

$$F_j(x) = P(X_j < x) \quad \forall_j \geq 1$$

and distribution densities

$$p_j(x) = F_j'(x).$$

Let us define a structure of nonlinear random variables

$$Z_n = \max(X_1, \ldots, X_n)$$

and denote

$$m_n(x) = \max_{1 \leq j \leq n} (1 - F_j(a_n + b_n x)),$$

$$u_n(x) = \sum_{j=1}^{n} (1 - F_j(a_n + b_n x))$$

here $\{a_n, \ n \geq 1\}$ and $\{b_n > 0, \ n \geq 1\}$ are sequences of centering and normalizing constants.

If

$$\lim_{n\to\infty} m_n(x) = 0 \quad \forall x,$$
$$\lim_{n\to\infty} u_n(x) = u(x) > 0$$

then (see work [1])

$$\lim_{n\to\infty} P(Z_n < a_n + b_n x) = H(x), \tag{1}$$

here $H(x)$ is nondegenerate distribution function and $H(x) = e^{-u(x)}$

Assume, that

$$\lim_{n\to\infty} p_n(x) = H'(x), \tag{2}$$

here

$$p_n(x) = \prod_{j=1}^{n} F_j(a_n + b_n x)\ b_n \sum_{j=1}^{n} \frac{p_j(a_n + b_n x)}{F_j(a_n + b_n x)}$$

is a distribution density of the random variable $(Z_n - a_n)/b_n$.

Classical maxima theorem (r.v.`s $X_j$ are independent and identically distributed) includes the necessary and sufficient conditions under which convergence in the integral maxima theorem implies convergence in the local maxima theorem (see works [3], [5], [6]). In maxima scheme of nonidentically distributed r.v.`s such conditions are not defined. Nevertheless the examples can be presented, when the convergence in the local distribution density theorem follows from the convergence in the integral theorem.

In the presented work we will get nonuniform estimate of the convergence rate in the relation (2).

Convergence rate in the local maximum distribution density theorem was investigated in the works [4] and [2]. In the first of them an uniform estimate of the convergence rate is obtained. The second work is devoted for nonuniform estimate. In both of there works the classical maxima scheme is investigated.

MAIN RESULTS

Let us denote

$$v_n(x) = u_n(x) - u(x),$$

$$h_n(x) = b_n \sum_{j=1}^{n} \frac{p_j(a_n + b_n x)}{F_j(a_n + b_n x)},$$

$$h(x) = -u'(x).$$

Theorem. *Let relations* (1) *and* (2) *are satisfied. Then for all x such that*

$$m_n(x) \le 1/2, \quad m_n(x)\, u_n(x) \le 1/4, \quad |v_n(x)| \le 1/2, \quad H(x) > 0$$

we have the following estimate of convergence rate

$$|p_n(x) - H'(x)| \le h_n(x)\Delta_n(x) + R_n(x).$$

Here

$$\Delta_n(x) = H(x)(q_n(x) + r_n(x) + q_n(x)\, r_n(x)),$$

$$q_n(x) = \frac{8}{3} m_n(x)\, u_n(x),$$

$$r_n(x) = \frac{4}{3} |v_n(x)|,$$

$$R_n(x) = H(x)|h_n(x) - h(x)|(1 + r_n(x)).$$

Remark. The quantity $\Delta_n(x)$ is an estimate of convergence rate in relation (1) (see [1]).

Proof. We have

$$|p_n(x) - H'(x)| \le \left| p_n(x) - h_n(x) e^{-u_n(x)} \right| + \left| h_n(x) e^{-u_n(x)} - H'(x) \right|. \qquad (3)$$

Let us estimate the first summand on the right hand side of the inequality (3). We have

$$\left| p_n(x) - h_n(x) e^{-u_n(x)} \right| = h_n(x) \left| \prod_{j=1}^{n} F_j(a_n + b_n x) - e^{-u_n(x)} \right|.$$

Applying ihe inequality

$$\exp\left\{ -\sum_{j=1}^{n} t_j \right\} - \prod_{j=1}^{n} (1 - t_j) \left(\exp\left\{ 2\sum_{j=1}^{n} t_j^2 \right\} - 1 \right) \le \prod_{j=1}^{n} (1 - t_j) \le \exp\left\{ -\sum_{j=1}^{n} t_j \right\} \quad \left(\max_{1 \le j \le n} t_j \le 1/2 \right)$$

we get

$$\left|\prod_{j=1}^{n} F_j\left(a_n + b_n x\right) - e^{-u_n(x)}\right| \le e^{-u_n(x)}\left(\exp\{2m_n(x)u_n(x)\} - 1\right),$$

if $m_n(x) \le 1/2$. From the latter relation and the inequality

$$\left|e^t - 1\right| \le \frac{4}{3}|t| \quad \left(|t| \le 1/2\right) \tag{4}$$

we obtain

$$\left|\prod_{j=1}^{n} F_j\left(a_n + b_n x\right) - e^{-u_n(x)}\right| \le \frac{8}{3} m_n(x) u_n(x) e^{-u_n(x)},$$

if $m_n(x) \le 1/2$ and $m_n(x)\, u_n(x) \le 1/4$.

Thus

$$\left|p_n(x) - h_n(x) e^{-u_n(x)}\right| \le h_n(x) e^{-u_n(x)} q_n(x). \tag{5}$$

Now we shall estimate the second summand on the right hand side of inequality (3). Using the equality

$$ab - cd = a(b - c) + c(a - d),$$

We get

$$\left|h_n(x) e^{-u_n(x)} - H'(x)\right| = \left|h_n(x) e^{-u_n(x)} - e^{-u(x)} h(x)\right| \le$$

$$h_n(x)\left|e^{-u_n(x)} - e^{-u(x)}\right| + e^{-u_n(x)}\left|h_n(x) - h(x)\right|.$$

Applying the inequality (4), we obtain

$$\left|e^{-u_n(x)} - e^{-u(x)}\right| = H(x)\left|e^{-v_n(x)} - 1\right| \le \frac{4}{3} H(x)\left|v_n(x)\right|,$$

if $|v_n(x)| \le 1/2$.

From the latter relation follows, that

$$\left|h_n(x) e^{-u_n(x)} - H'(x)\right| \le h_n(x) H(x) r_n(x) + e^{-u_n(x)}\left|h_n(x) - h(x)\right|. \tag{6}$$

Since

$$e^{-u_n(x)} \le \left|e^{-u(x)} - H(x)\right| + H(x) \le H(x)\left(1 + r_n(x)\right),$$

from the inequality (3) taking into account inequalities (5) and (6) we obtain an estimate of the convergence rate.

Theorem is proved.

Example. Let

$$F_j(x) = 1 - \frac{\alpha_j}{x}, \quad x \geq \alpha_j > 0, \quad \forall_j \geq 1, \quad p_j(x) = \frac{\alpha_j}{x^2}.$$

Setting the centering and normalizing constants

$$a_n = 0, \quad b_n = \sum_{j=1}^{n} \alpha_j,$$

we get

$$m_n(x) = \frac{\max\limits_{1 \leq j \leq n} \alpha_j}{\frac{1}{x} \sum\limits_{j=1}^{n} \alpha_j}, \qquad u_n(x) = \frac{1}{x}.$$

Thus

$$\lim_{n \to \infty} P(Z_n < b_n x) = e^{-\frac{1}{x}}, \quad x > 0,$$

and

$$\lim_{n \to \infty} P'(Z_n < b_n x) = \frac{1}{x^2} e^{-\frac{1}{x}}, \quad x > 0.$$

We further have

$$v_n(x) = 0,$$

$$h(x) = \frac{1}{x^2},$$

$$h_n(x) = \sum_{j=1}^{n} \frac{\alpha_j}{b_n x^2 - \alpha_j x}.$$

From the theorem follows, that

$$\left| P(Z_n < b_n x) - e^{-\frac{1}{x}} \right| \leq \Delta_n(x) = \frac{8}{3} \frac{\max\limits_{1 \leq j \leq n} \alpha_j}{b_n} \frac{e^{-\frac{1}{x}}}{x^2},$$

$$\left| P'(Z_n < b_n x) - \frac{1}{x^2} e^{-\frac{1}{x}} \right| \leq \Delta_n(x) \sum_{j=1}^{n} \frac{\alpha_j}{b_n x^2 - \alpha_j x} + e^{-\frac{1}{x}} \left| \sum_{j=1}^{n} \frac{\alpha_j}{b_n x^2 - \alpha_j x} - \frac{1}{x^2} \right|,$$

if

$$\frac{\max\limits_{1\le j\le n}\alpha_j}{b_n x}\le\frac{1}{2}$$

and

$$\frac{\max\limits_{1\le j\le n}\alpha_j}{b_n x^2}\le\frac{1}{4}\,.$$

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Netolygus maksimumo tankio funkcijos konvergavimo greièio ávertis nepriklausomiems atsitiktiniams dydž iams

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Straipsnyje gautas netolygus konvergavimo greièio ávertis maksimumo tankio funkcijos ribinëje teoremoje, nagrinëjamiems nepriklausomiems, nevienodai pasiskirsèiusiems atsitiktiniams dydþ iams. Ðis rezultatas yra tyrimø, pateiktø [1], apibendrinimas.

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