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ON THE GENERATION AND ORIGIN OF $1/f$ NOISE

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Abstract

Simple analytically solvable model exhibiting $1/f$ spectrum in any desirably wide range of frequency is analysed. The model consists of pulses (point process) whose interevent times obey an autoregressive process with small damping. Analysis and generalizations of the model indicate to the possible origin of $1/f$ noise, i.e. random increments between the occurrence times of particles or pulses resulting in the clustering of the pulses.

INTRODUCTION

Fluctuations of signals and physical variables exhibiting behavior characterized by a power spectral density $S(f)$ diverging at low frequencies like $1/f^{\delta}$ ($\delta \simeq 1$) have been discovered in large diversity of uncorrelated systems, such as processes in condensed matter, traffic flow, quasar emissions, music, biological, evolution and artificial systems, human cognition and even distribution of prime numbers (see [1–3] and references herein). $1/f$ noise is an intermediate between the well understood white noise with no correlation in time and the random walk (Brownian motion) noise with no correlation between increments.

The widespread occurrence of signals exhibiting power spectral density with $1/f$ behavior suggests that a general mathematical explanation of such an effect might exist. However, except for some formal mathematical treatments like “fractional Brownian motion” or half-integral of a white noise signal [4] no generally recognized physical explanation of the ubiquity of $1/f$ noise is still proposed.

Physical models of $1/f$ noise are usually very specialized and they do not explain the omnipresence of the processes with $1/f^{\delta}$ spectrum [5, 6].

Here we present the simplest analytically solvable model of $1/f$ noise which can influence on the understanding of the origin, main properties and parameter dependences of the intensity of the flicker noise. Our model is a result of the search for necessary and sufficient conditions for the appearance of $1/f$ fluctuations in simple systems affected by random external perturbations initiated in [7] and originated from the observation of a transition from chaotic to nonchaotic behavior in an ensemble of randomly driven systems [8]. Contrary to the McWhorter model [9] based on the superposition of large number of Lorentzian spectra and requiring a very wide distribution of relaxation times, our model contains only one relaxation rate and can have an exact $1/f$ spectrum in any desirably wide range of frequency.

MODEL AND SOLUTION

The simplest version of our model consists of one particle moving along some orbit. The period of this motion fluctuates (due to external random perturbations of the system's parameters) about some average value $\bar{\tau}$. So, a sequence of transit times t_k at which the particle crosses some Poincaré section L_m is described by the recurrence equations

$$\begin{cases} t_k = & t_{k-1} + \tau_k, \\ \tau_k = & \tau_{k-1} - \gamma\left(\tau_{k-1} - \bar{\tau}\right) + \sigma\varepsilon_k. \end{cases} \tag{1}$$

Here $\gamma \ll 1$ is the period's relaxation rate, $\{\varepsilon_k\}$ denotes a sequence of uncorrelated normally distributed random variables with zero expectation and unit variance (the white noise source) and σ is the standard deviation of the white noise.

Note that the recurrence times $\tau_k = t_k - t_{k-1}$ follow an autoregressive $AR(1)$ process with offset $\bar{\tau} > 0$, regression coefficient $1 - \gamma$ and noise variance σ^2. So, introducing new variables $\theta_k = \tau_k - \bar{\tau}$ and $a = 1 - \gamma$ we can rewrite the second relation of Eqs. (1) in the form

$$\theta_k = a\theta_{k-1} + \sigma\varepsilon_k \tag{1a}$$

The intensity of the current of particles through the section L_m is

$$I\left(t\right) = \sum_k \delta\left(t - t_k\right) \tag{2}$$

where $\delta(t)$ is the Dirac delta function. The power spectral density of the current (2) is

$$S(f) = \lim_{T\to\infty} \left\langle \frac{2}{T} \left| \sum_{k=k_{\min}}^{k_{\max}} e^{-i2\pi f t_k} \right|^2 \right\rangle = \lim_{T\to\infty} \left\langle \frac{2}{T} \sum_k \sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i2\pi f(t_{k+q}-t_k)} \right\rangle . \tag{3}$$

Here T is the whole observation time interval, $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the interval of observation and the brackets $\langle\cdots\rangle$ denote the averaging over realizations of the process.

From Eqs. (1) follows an expression for the period

$$\tau_k = \bar{\tau} + (\tau_0 - \bar{\tau})(1-\gamma)^k + \sigma \sum_{j=1}^{k} (1-\gamma)^{k-j} \varepsilon_j, \tag{4}$$

where τ_0 is the initial period. We also can rewrite Eq. (1a) explicitly in the form of autoregressive $AR(1)$ process

$$\theta_k = \theta_0 a^k + \sigma \sum_{j=1}^{k} a^{k-j} \varepsilon_j. \tag{4a}$$

After some algebra we easily obtain an explicit expression for the transit times t_k,

$$t_k = t_0' + k\bar{\tau} + \frac{\sigma}{\gamma} \sum_{l=1}^{k} \left[1 - (1-\gamma)^{k+1-l}\right] \varepsilon_l. \tag{5}$$

Here t_0' is some constant for $k \gg \gamma^{-1}$ or $\tau_0 = \bar{\tau}$. In the later case t_0' is the initial time t_0. At $k \gg \gamma^{-1}$ Eq. (5) generates a stationary time series and the difference of transit times t_{k+q} and t_k in Eq. (3) is

$$t_{k+q} - t_k = q\bar{\tau} + \frac{\sigma}{\gamma} \left\{ [1-(1-\gamma)^q] \sum_{l=1}^{k} (1-\gamma)^{k+1-l} \varepsilon_l + \sum_{l=k+1}^{k+q} \left[1-(1-\gamma)^{k+q+1-l}\right] \varepsilon_l \right\}, \quad q \geq 0. \tag{6}$$

Substitution of Eq. (6) into Eq. (3) and averaging over realizations of the process or over the random variables ε_l yield [10]

$$S(f) = \lim_{T\to\infty} \frac{2}{T} \sum_k \sum_q e^{i2\pi f \bar{\tau} q - \pi^2 f^2 \sigma^2 g(q)}, \tag{7}$$

where

$$g(q)=\frac{2}{\gamma^2}\left\{\left[1-(1-\gamma)^q\right]^2\sum_{l=1}^{k}(1-\gamma)^{2l}+\sum_{l=1}^{q}\left[1-(1-\gamma)^l\right]^2\right\}, \qquad q\geq 0. \tag{8}$$

Summations in Eq. (8) for $k\gg\gamma^{-1}$ result in

$$g(q)=\frac{2}{\gamma^2}\left\{q-2\frac{(1-\gamma)\left[1-(1-\gamma)^q\right]}{1-(1-\gamma)^2}\right\}. \tag{9}$$

Expansion of expression (9) in powers of $\gamma q\ll 1$ is

$$g(q)=\frac{1}{\gamma}q^2-\frac{1}{3}q^3+\frac{1}{2}q^2+\ldots,\quad q\geq 0. \tag{10}$$

Note that the function $g(q)$ is even, i.e., $g(-q)=g(q)$. This follows from Eqs. (6) – (8) at $k-|q|\gg\gamma^{-1}$.

For $f\ll f_{\bar\tau}=(2\pi\bar\tau)^{-1}$ and $f<f_2=2\sqrt{\gamma}/\pi\sigma$ we can replace the summation in Eq. (7) by the integration

$$S(f)=2\bar I\int_{-\infty}^{+\infty}e^{i2\pi f\bar\tau q-\pi^2f^2\sigma^2g(q)}\,dq. \tag{11}$$

Here $\bar I=\lim\limits_{T\to\infty}(k_{\max}-k_{\min}+1)/T=(\bar\tau)^{-1}$ is the averaged current. Furthermore, at $f\gg f_1=\gamma^{3/2}/\pi\sigma$ it is sufficient to take into account only the first term of expansion (10). Integration in Eq. (11) yields to $1/f$ spectrum

$$S(f)=\left(\bar I\right)^2\frac{\alpha_H}{f},\quad f_1<f<\min(f_2,\,f_{\bar\tau}), \tag{12}$$

where α_H is a dimensionless constant (the Hooge parameter)

$$\alpha_H=\frac{2}{\sqrt\pi}Ke^{-K^2},\quad K=\frac{\bar\tau\sqrt\gamma}{\sigma}. \tag{13}$$

Using an expansion of the function $g(q)$ according to Eq. (9) at $\gamma q\gg 1$, $g(q)=2q/\gamma^2$, we obtain from Eq. (11) the Lorentzian power spectrum density for $f<f_1$

$$S(f)=2\bar I\frac{\sigma^2}{\bar\tau^2\gamma^2}\frac{1}{1+\left(\pi f\sigma^2/\bar\tau\gamma^2\right)^2}. \tag{14}$$

Therefore, the model containing only one relaxation time γ^{-1} can for sufficiently small parameter γ produce an exact $1/f$-like spectrum in wide range of frequency. Furthermore, due to the contribution to the transit times t_k of the large number of the random variables ε_l $(l = 1, 2, \ldots, k)$, our model represents a "long-memory" random process. As a result of the nonzero relaxation rate $(\gamma \neq 0)$ and, consequently, due to the finite dispersion of the τ period, $\sigma_\tau^2 \equiv \left\langle \tau_k^2 \right\rangle - \left\langle \tau_k \right\rangle^2 = \sigma^2/2\gamma(1-\gamma/2)$, $2k\gamma \gg 1$, the model is free from the unphysical divergency of the spectrum at $f \to 0$. So, we obtain from Eq. (14) the spectrum density $S(f) = (\bar{I})^2 \left(2\sigma^2/\tau\gamma^2\right)$ for $f \ll f_1$, $f_0 = \tau\gamma^2/\pi\sigma^2$.

GENERALIZATIONS AND NUMERICAL ANALYSIS

This simple exactly solvable model can easily be generalized in different directions: for large number of particles moving in similar orbits with coherent (identical for all particles) or independent (uncorrelated for different particles) fluctuations of the periods, for non-Gaussian or continuous perturbations of the systems' parameters and for spatially extended systems. So, when an ensemble of N particles moves on closed orbits and the period of each particle fluctuates independently (due to the perturbations by uncorrelated sequences of random variables $\{\varepsilon_k^v\}$, different for each particle v) the power spectral density of the collective current I of all particles can be calculated by the above method too and is expressed as the Hooge formula [1]

$$S(f) = (\bar{I})^2 \frac{\alpha_H}{Nf}. \tag{15}$$

It should be noticed that in many cases the intensity of signals or currents can be expressed in the form (2). This expression represent exactly the flow of identical objects: cars, electrons, photons and so on. More generally, in Eq. (2) instead of the Dirac delta function one should introduce time dependent pulse amplitudes $A_k(t - t_k)$. However, the low frequency power spectral density depends weakly on the shapes of the pulses, while fluctuations of the pulses amplitudes result, as a rule, in white or Lorentzian, but not $1/f$, noise [11]. The model (1) in such cases represents fluctuations of the averaged interevent time τ_k between the subsequent occurrence times of the pulses.

The model may also be generalized for the nonlinear relaxation of the interevent time τ_k. In such a case Eq. (1) can be written in the form

$$\begin{cases} t_k = & t_{k-1} + \tau_k, \\ \tau_k = & \tau_{k-1} - \dfrac{dV(\tau_{k-1})}{d\tau_{k-1}} + \sigma\varepsilon_k. \end{cases} \tag{16}$$

Here the function $V(\tau_k)$ represent the effective "potential well" for the Brownian motion of the interevent time τ_k. The steady state distribution density of the interevent time τ_k described by Eq. (16) is of the form

$$\psi(\tau_k) = C \exp\left[-\frac{2V(\tau_k)}{\sigma^2}\right], \tag{17}$$

where a constant C may be obtained from the normalization. For the power-law "potential well"

$$V(\tau_k) = \frac{1}{2}\gamma\left(\tau_k - \bar{\tau}\right)^{2n} \tag{18}$$

with integer n we have a generalization of Eq. (1)

$$\begin{cases} t_k = & t_{k-1} + \tau_k, \\ \tau_k = & \tau_{k-1} - \gamma n \left(\tau_{k-1} - \bar{\tau}\right)^{2n-1} + \sigma\varepsilon_k. \end{cases} \tag{19}$$

For sufficiently large $n \gg 1$ Eqs. (18) and (19) represent Brownian motion of the interevent time τ_k in the almost rectangle "potential well" restricting movement of τ_k mostly in the interval $(\bar{\tau} - h, \bar{\tau} + h)$ with $h \simeq \left(\sigma^2/\gamma\right)^{1/2n}$.

We can generate, of course, the stationary time series of the occurrence times t_k also for other restrictions for the interevent time τ_k, e.g., with the reflecting boundary conditions at some values of the interevent time $\tau_{\min}$ and $\tau_{\max}$.

Numerical analysis of the models like (1), (16) and (19) shows that power spectral density of the current (2) is $1/f$ in large interval of frequency only when the distribution density of the interevent times τ_k in the point $\tau_k = 0$ is nonzero, i.e., $\psi(0) \neq 0$ according to Eq. (17). In such a case the power spectral density may be expressed as (see also [10])

$$S(f) = 2(\bar{I})^2 \bar{\tau} \frac{\psi(0)}{f}. \tag{20}$$

For models with $\psi(0) = 0$ or $\psi(0)$ very close to zero we observe in numerical simulations the power spectral density $S(f) \propto 1/f^{3/2}$ (see Fig. 1).

It should be noticed that in general analytical results are in good agreement with the numerical simulations and Eqs. (11) – (14) describe quite well the power spectrum of the random process (1). Theoretical results predict not only the slope and intensity of $1/f$ noise but the frequency range $f_1 \div f_2, f_{\bar{\tau}}$ of $1/f$ noise and intensity of the very low frequency, $f \ll f_0$, white noise as well. As an illustrative example in Fig. 1 the numerically calculated power spectral density averaged over nine realizations of the process is presented for different parameters and conditions. We see in Fig. 1(a) the power spectral density $S(f) \propto 1/f$ when $\psi(0) \neq 0$ and $S(f) \propto 1/f^{3/2}$ if variation of τ_k is restricted in the interval with $\tau_{\min} > 0$ and, consequently, $\psi(0) = 0$. Fig. 1(b) illustrates transition from $1/f$ to

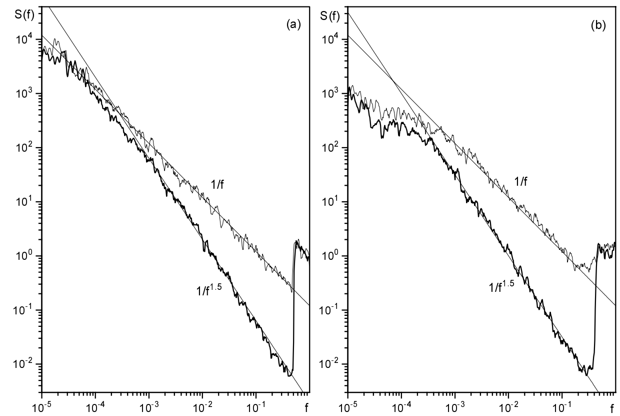

Fig. 1. Power spectral density vs frequency of the current generated by Eqs. (1) – (3) with the Gaussian distribution of the random increments $\{\varepsilon_k\}$. The sinuous curves represent the averaged over nine realizations results of numerical simulations (a) according to Eq. (1) with parameters $\sigma = 0.01$, $\bar{\tau} = 1$, $\gamma = 0$ and with reflecting boundary conditions for τ_k at $\tau_k = 0$ and $\tau_k = 2$ (fine curve) or at $\tau_k = 0.1$ and $\tau_k = 1.9$ (heavy curve) and (b) according to Eq. (19) with parameters $\sigma = 0.04$, $\bar{\tau} = 1.3$, $\gamma = 0.0016$ and $n = 1$ (fine curve) or $n = 10$ (heavy curve).

$1/f^{3/2}$ spectrum with increasing of the power n of the potential (18) and, as a result, due to the natural restriction of the variation of τ_k and obstruction of the pulses clustering.

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Apie $1/f$ triukšmo gavimą ir kilmę

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Ištirtas paprastas analiziškai sprendžiamas dinaminis modelis, kurio signalo spektras plačiame dažnių intervale yra $1/f$ pobūdžio. Signalas modeliuojamas taškiniu procesu sudarytu iš trumpų impulsų. Impulsų atsikartojimo laikų seką aprašo autoregresinis procesas su lėta relaksacija. Modelio analizė ir apibendrinimai atskleidžia galimą vidinę $1/f$ triukšmo kilmę – atsikartojimo laikų atsitiktinių pokyčių sąlygotą impulsų klasterizaciją.

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