Nonlinear Analysis: Modelling and Control, Vilnius, IMI, 1998, No 2

MATHEMATICAL MODELLING OF MILITARY OPERATIONS

G.Misevičius, V.Pakalnytė, A.Pincevičius, R.Eidukevičius, R.-J.Rakauskas
The Military Academy of Lithuania, Šilo 5a, 2055 Vilnius, Lithuania

The aim of this paper is to show how the theory of systems of differential equations may be applied both to describe the battle actions and to control the trajectory of projectiles. We will restrict our attention to deterministic cases. Those cases are not well adapted to the real situation and must be treated as the first approach.

1. BATTLE BETWEEN THE REGULAR ARMY AND THE PARTISANS

The modeling of such problems takes origin in the works of F.Lanchester. During the 1st World War he had created models describing the air battle. Later on they have been adapted for different types of military operations. We will restrict our attention to the modelling of armed conflict between the regular army and the partisan detachment. The specific character of this kind of battle is conditioned by the notorious fact that partisans dislocation is unknown for the enemy.

This battle is approximated by the following system of differential equations:

$$\begin{cases} \dfrac{dx(t)}{dt} = -a\,x(t)\,y(t) - b\,x(t) + P(t), \\ \dfrac{dy(t)}{dt} = -c\,x(t) - d\,y(t) + Q(t). \end{cases} \tag{1}$$

Here $x(t)$ and $y(t)$ denote the quantities of adversaries, t stands for time.

We will restrict our attention to main factors which condition the loss of warriors.

1. Assume that the loss of the soldiers of the regular army is proportional to the number of ones of the enemy, that is

$$y(t) \Rightarrow -c\,x(t), \tag{2}$$

where c is the constant characterizing the efficiency of actions of a soldier ($c = r_x p_x$), r_x is power of fire, p_x is the probability of hitting the target. Let us make an assumption that the military unit of 100 soldiers defends the position of 1000 m long. The area under fire by one soldier is about 20 m^2 and the probability of hitting the man is $p_x \approx p_y \approx 0.05$ (the area of warrior is approximately $1.75 \times 0.6 \approx 1.0$ m^2). The power shots in one soldier is about 200 – 300 shots per day. Thus we get $c \approx 1.22$.

The loss of partisans in the battle may be expressed in the following way:

$$x(t) \Rightarrow a\,x(t)\,y(t). \tag{3}$$

Here

$$a \approx \frac{r_y A_{ry}}{A_x}, \tag{4}$$

where r_y stands for power of fire of a soldier of the regular army, $A_x = R$ is an area defended by partisans, A_{ry} is the fire efficiency of a soldier of the regular army, that is the area fired by one soldier (≈ 20 m^2). If partisans defend a town of area $A_x = 4 \cdot 10^8$ m^2, then $a \approx 1.25 \cdot 10^{-5}$.

The partisans defend the territory being invisible, thus the forces of the regular army cannot estimate the efficiency of their own actions.

2. The loss caused by shell, bombardment, epidemic also is taken into consideration:

$$x(t) \Rightarrow -bx(t), \quad y(t) \Rightarrow -dy(t). \tag{5}$$

3. We may consider the support coming from the rear:

$$x(t) \Rightarrow P(t) \text{ and } y(t) \Rightarrow Q(t). \tag{6}$$

Henceforth we make a restriction regarding the loss due to the struggle actions, only. Thus the system of equations becomes

$$\begin{cases} \dfrac{dx(t)}{dt} = -a\,x(t)\,y(t), \\ \dfrac{dy(t)}{dt} = -c\,x(t). \end{cases} \tag{7}$$

This simplification is based upon the fact that in the case of the partisan war the loss caused by shell is insignificant. Dividing the second equation of (7) by the first one we get

$$\frac{dy}{dx} = \frac{c}{ay} \tag{8}$$

Integrating this equation we get:

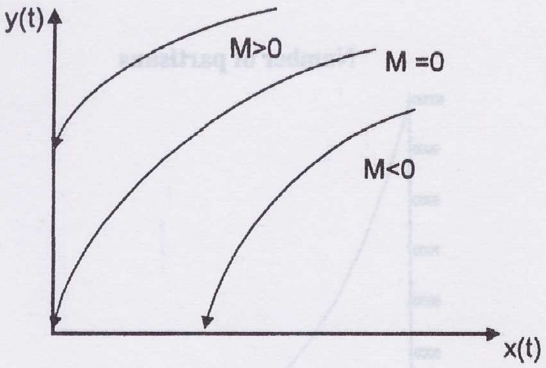
$$\begin{aligned} ay^2(t) &= 2cx(t) + M, \\ M &= ay_0^2 - 2cx_0. \end{aligned} \tag{9}$$

For the graphical interpretation of the results obtained, see Figure 1.

The integral lines of this equation are parabolas. Regular army wins in the case the inequality

$$y_0 > \sqrt{\frac{2cx_0}{a}} \tag{10}$$

holds. In our case ($c = 1{,}22$, $a = 1{,}25\cdot10^{-5}$), the regular forces win if they have $y_0 > 100000$ soldiers, the initial number of partisans being $x_0 = 10000$.



In case M>0, the regular forces win,
M=0 is the equilibrium $\left(y_0 = \sqrt{\frac{2cx_0}{a}}\right)$,
M<0 the partisans win.

Fig. 1

It would be interesting to discuss the time relationship, e.g. for how long the town can be defended if the number of partisans is ≈ 10000. The results of numerical solution of the system of equations (7) are graphically represented in figures 2 – 5 .

In Fig. 2 we see that the unit of the regular army loses half of the soldiers in twelve hours, therefore the regular army must find additional force or step back.

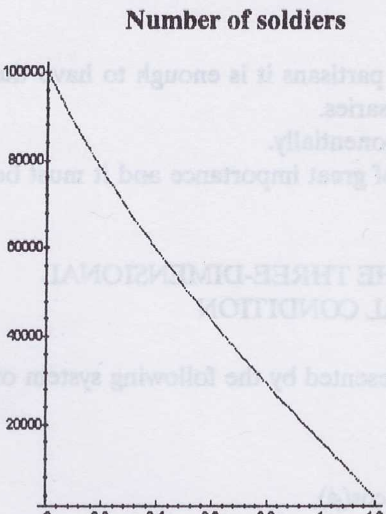


Fig. 2

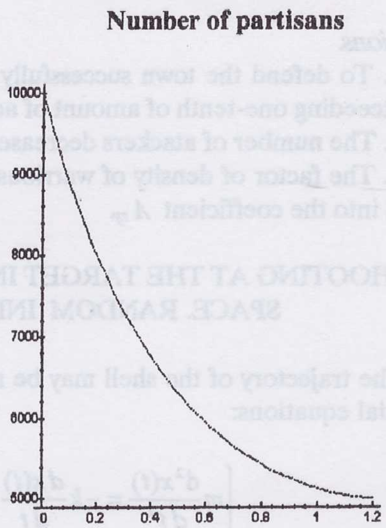


Fig. 3

Fig. 3 shows that the partisans lose only one third of their combatants.

If the superiority of the regular units is significant (130000 soldiers), then the combat for the partisans will be a loss (see Fig. 4, 5).

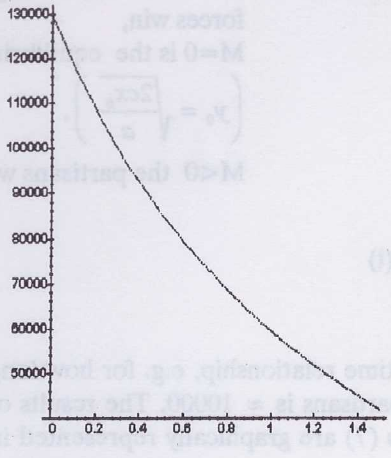


Fig. 4

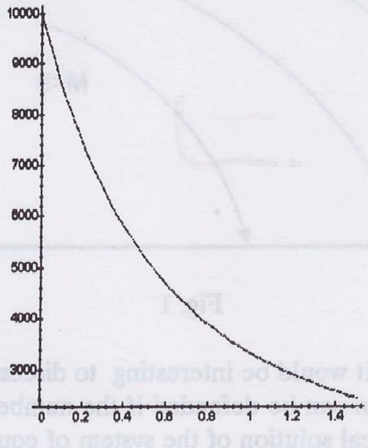


Fig. 5

Conclusions.

1. To defend the town successfully for partisans it is enough to have the forces exceeding one-tenth of amount of adversaries.

2. The number of atackers decrease exponentially.

3. The factor of density of warrious is of great importance and it must be included into the coefficient A_{ry}.

2. SHOOTING AT THE TARGET IN THE THREE-DIMENSIONAL SPACE. RANDOM INITIAL CONDITION

The trajectory of the shell may be represented by the following system of differential equations:

$$\begin{cases} m\dfrac{d^2x(t)}{dt^2} = -k\dfrac{dx(t)}{dt} + F\cos(\phi), \\ m\dfrac{d^2y(t)}{dt^2} = -k\dfrac{dy(t)}{dt} + F\sin(\phi), \\ m\dfrac{d^2z(t)}{dt^2} = -k\dfrac{dz(t)}{dt} - mg. \end{cases} \tag{11}$$

Here t denote time, k is the coefficient of aerodynamic resistance, m is the mass of the shell, g – is acceleration due to the force of gravity, $x(t)$ and $y(t)$ denote the horizontal coordinates of the moving body at time moment $t, z(t)$ – vertical coordinate, F– force, ϕ – the angle between the direction of the moving body and the x – axis, v_0 – initial velocity of the shell.

Denote the coordinates of the target as x_t, y_t, z_t. We suppose that the magnitude of v_0 does not vary and is constant on multiple shots. Denote $v_0 = \sqrt{x_{sp}^2 + y_{sp}^2 + z_{sp}^2}$. The projection of the velocity vector to xy-plane is

$$v_{0xy} = \sqrt{x_{sp}^2 + y_{sp}^2} .$$

We suppose that the direction of the initial velocity vector coincides with the direction of the gun tube. When aiming, the position of the gun tube is determined by two angles α_1 and α_2, i.e. α_1 is the angle between the velocity vector and xy-plane and α_2 is the angle between the projection of the velocity vector to xy-plane and x-axis. Evidently, both angles can be determined.

Let us take the following initial conditions :

$$\begin{cases} \text{Parameters: } g = 9.8; \quad k = 0.005; \quad v_0 = 200; \quad m = 1; \ \alpha = \dfrac{\pi}{3}; \quad \phi = \dfrac{\pi}{5}; \quad F = 15, \\ \text{Initial values of coordinates and speed: } x(0) = 0; \quad y(0) = 0; \quad z(0) = 0 \\ D(x)(0) = v_0 \cos\alpha_1 \cos\alpha_2, \ D(y)(0) = v_0 \cos\alpha_1 \sin\alpha_2; \ D(z)(0) = v_0 \sin\alpha_1 . \end{cases} \qquad (12)$$

The initial values of the velocity vector are supposed to be random. We used MAPLE software package to solve the system (11) in this case:

$$\begin{aligned} x(t) &= 40000\cos\alpha_1 \cos\alpha_2 (1 - e^{-11200t}) + (750t - 150000(1 - e^{-11200t}))(\sqrt{5} + 1); \\ y(t) &= 40000\cos\alpha_1 \sin\alpha_2 (1 - e^{-11200t}) + (750t - 150000(1 - e^{-11200t}))\sqrt{2}\sqrt{5 - \sqrt{5}}; \quad (13) \\ z(t) &= 40000\sin\alpha_1 (1 - e^{-11200t}) - 1960t + 392000(1 - e^{-11200t}) . \end{aligned}$$

In case $\alpha_1 = \pi/3$, $\alpha_2 = \pi/4$, the time moment when the shell falls on the ground (that is, $z(t_{det}) = 0$) is

$$t_{det} = 34.36436805$$

and the coordinates ($x_{det} = x(t_{det})$, $y_{det} = y(t_{det})$) of the point where the shell falls on the ground are:

$$x_{det} = 9004.6042,$$
$$y_{det} = 7152.7575.$$

For each pair of N (N=30) different random initial values of angles determining the orientation of the gun in space, we find:
1) the time moment of landing of the shell, that is the moment for $z(t) = 0$;
2) the coordinates of the point of landing, that is x_{fal} (t of falling), y_{fal} (t of falling).
Next we calculate the mean values of the coordinates of the point of landing:

$$x_{mean} = 9013.187732,$$
$$y_{mean} = 7151.707533$$

and their standart deviations :

$$x_{stand\_dev} = 53.26759814,$$
$$y_{stand\_dev} = 35.26392292.$$

Figure 6 represents the places of landing of N shells with the random initial conditions. The big cross indicates the place of landing of the shell with the deterministic initial condition. The horizontal and the vertical lines of the cross are the standart deviations of the coordinates Fig. 7-8 give the block-diagram of x_{mean} and y_{mean}.

The places of landing of N=30 shells with the random initial conditions

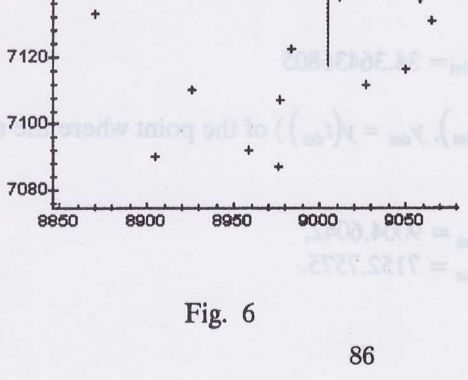


Fig. 6

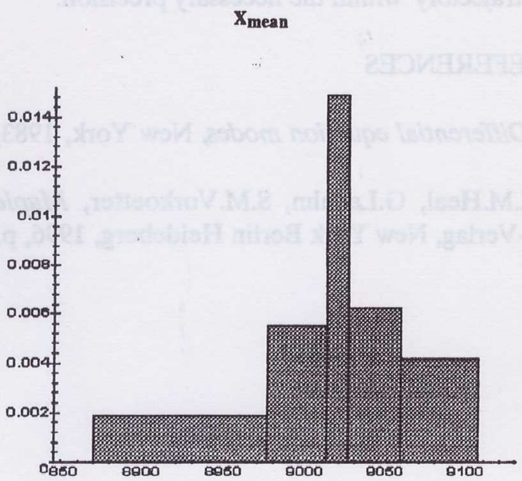


Fig. 7

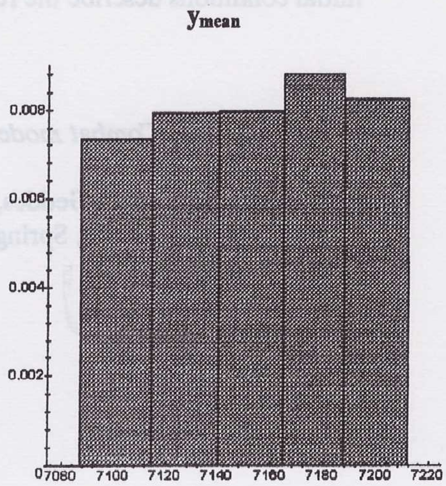


Fig. 8

All of N trajectories of the shell's flight are represented in Fig. 9:

Eikonal of trajectories

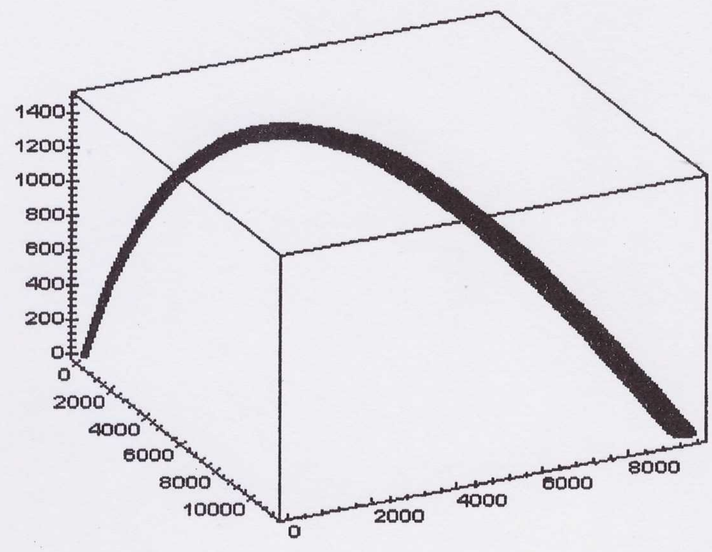


Fig.9

Conclusions. The trajectories of the flight of the shell calculated with the random initial conditions describe the real trajectory within the necessary precision.

REFERENCES

1. C.S.Coleman, *Combat model. Differential equation modes,* New York, 1983, p. 109-131.
2. M.B.Monagan, K.O.Geddes, K.M.Heal, G.Labahn, S.M.Vorkoetter, *Maple V. Programing Guide,* Springer-Verlag, New York Berlin Heideberg, 1996, p. 383.